Commentationes Mathematicae Universitatis Carolinae

Petr Savický

Boolean functions represented by random formulas

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 2, 397--398

Persistent URL: http://dml.cz/dmlcz/106550

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1987

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

ANNOUNCEMENT OF NEW RESULTS

(of authors having an address in Czechoslovakia)

BOOLEAN FUNCTIONS REPRESENTED BY RANDOM FORMULAS

P. Savický (Katedra kybernetiky a informatiky, MFF UK, Malostranské nám. 25, 118 00 Praha 1, Czechoslovakia), received 30.3. 1987

Let $n \geq 1$ be a fixed natural number. A Boolean function of n variables is any function $f:\{0,1\}^n \rightarrow \{0,1\}$, cf. [1].

We study the representation of Boolean functions by Boolean formulas of the following type.

Definition 1. Let $x_1, x_2, \ldots, x_n$ be variables. Let B be a set of function symbols all of which have the same number of arguments $k \geq 2$. For all $i \geq 0$ let H_i be the set of Boolean formulas defined in the following way.

$H_0 = \{x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n\}$

$H_{i+1} = \{\alpha(\varphi_1, \varphi_2, \ldots, \varphi_k); \alpha \in B,\ \varphi_j \in H_i$ for $j=1,2,\ldots,k\}$.

Elements of H_i for $i=0,1,2,\ldots$ are formal expressions of increasing complexity. Given an interpretation of all symbols $\alpha \in B$ as functions $\alpha:\{0,1\}^k \rightarrow \{0,1\}$, any formula of H_i for all $i \geq 0$ represents a Boolean function in a natural way.

Definition 2. Let F_i be a random variable whose values are formulas from H_i and the distribution of which is the uniform distribution on H_i.

Our aim is to present some results on the distribution of Boolean functions represented by the formula F_i. For an arbitrary set $A \subseteq \{0,1\}^n$ and $i \geq 0$ let $F_i \upharpoonright A$ denote the restriction of the Boolean function represented by F_i to the set A.

Theorem 1. Let $B=\{\&, \vee\}$ (i.e. k=2) and F_i be as in Definition 2. Then for every $A \subseteq \{0,1\}^n$ and every $f:A \rightarrow \{0,1\}$ we have

$$\lim_{i \to \infty} P(F_i \upharpoonright A=f) = \begin{cases} 0 & \text{if } f \text{ is not a constant function} \\ 1/2 & \text{if } f \equiv 0 \text{ or } f \equiv 1 \end{cases}$$

The following theorem deals with the selection function $s(x,y,z)$ defined as $s(0,y,z)=y$ and $s(1,y,z)=z$ for all $y,z \in \{0,1\}$.

Theorem 2. Let $B=\{s\}$ (k=3) and F_i be as in Definition 2. Then for every $A \subseteq \{0,1\}^n$ and every $f:A \rightarrow \{0,1\}$ we have

$$\lim_{i \to \infty} P(F_i \upharpoonright A=f) = (1/2)^{|A|}$$

The convergence in Theorem 1 is of the order of magnitude $O(1/i)$ and in Theorem 2 $O(c^i)$ where $c<1$.

Outline of the proofs. Note that $F_{i+1} = \alpha(F_i^1, F_i^2, \ldots, F_i^k)$

where F_i^j are independent realizations of the random formula F_i and α is a random element of B independent of F_i^j. Using this we can establish a recurrent relation for $P(F_i \upharpoonright A=f)$. We prove Theorems 1 and 2 for one and two element sets A by a direct computation using this relation. Theorem 1 in the general case is a simple consequence. Theorem 2 for $|A| \geq 3$ can be proved by induction on $|A|$.

An analogous type of probability distribution on formulas was used by Valiant ([2]) in a probabilistic construction of monotone formulas of the size $O(n^{5.3})$, for the majority function.

References:

[1] Savage J.E.: The Complexity of Computing, Wiley-Interscience, New York, 1976.

[2] Valiant L.G.: Short monotone formulae for the majority function, Journal of Algorithms 5(1984), 363-366.