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SOME FACTORIZATION THEOREMS FOR PARACOMPACT σ-SPACES

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Abstract: For closed mappings between paracompact σ-spaces and for continuous mappings between regular spaces with countable network there are proved factorization theorems by weight and dimension.

Key words: Paracompact σ-space, countable network, factorization.

Classification: 54F45

The method of factorization theorems plays an important role in dimension theory. Factorization theorems by weight and dimension are well-known in the classes of compacta and metric spaces (see e.g. [5, Theorems 3.3.2 and 4.2.5]), B.A. Pasynkov has proved a factorization theorem for p-paracompacta [7].

In [3] we have proposed a method of rigid systems for the study of dimensional properties of paracompact σ-spaces (i.e. paracompact spaces with a σ-discrete network). Here we develop this method and use it to prove some factorization theorems for paracompact σ-spaces, in particular, for closed mappings betweem paracompact σ-spaces (Corollary 1) and for continuous mappings between regular spaces with a countable network (Corollary 2). However, we do not know whether the general factorization theorem by weight and dimension for paracompact σ-spaces is true. To be precise, is it true that for a continuous mapping $f:X \to Y$ where X is normal and Y is a paracompact σ-space, there exists a paracompact σ-space Z and continuous mappings $g:X \to Z$ and $h:Z \to Y$ such that $f=h \circ g$, $\dim Z \leq \dim X$ and $w(Z) \leq w(Y)$.

The starting idea for our work was the concept of a weak bijection introduced by A.V. Archangelskii [1]. A continuous bijection $f:X \to Y$ is called weak if X is regular, Y is paracompact and there exists a σ-discrete family $\mathcal{H}$ in Y such that

$f^{-1}(\mathcal{K})$ is a network in X. In [1] it was proved that a space, from which there exists a weak bijection , is a paracompact σ-space. Besides, every paracompact σ-space can be mapped by a weak bijection onto a metric space.

The next definition generalizes the idea of a weak bijection.

Definition 1. A continuous mapping $f:X \longrightarrow Y$ of a regular space X onto a paracompact space Y is called σ-discrete if there exists a σ-discrete network $\mathcal{K}$ in X such that $f(\mathcal{K})$ is a σ-discrete network in Y.

The next result is analogous to a theorem of A.V. Archangelskii [1].

Proposition 1. If there exists a σ-discrete mapping from a regular space X then X is a paracompact σ-space.

Proof. Let $f:X \longrightarrow Y$ be a σ-discrete mapping and let $\mathcal{K}$ be a σ-discrete network in X such that the network $f(\mathcal{K})$ is σ-discrete in Y. We shall prove that X is paracompact , i.e. that every open cover $\mathcal{U}$ has a σ-discrete open refinement $\mathcal{V}$. Notice first that without loss of generality one can assume that for each $K \in \mathcal{K}$ there exists $U(K) \in \mathcal{U}$ such that $K \subset U(K)$. Since Y is collectionwise normal, there exists a σ-discrete open family $\{O_K : K \in \mathcal{K}\}$ in Y such that $f(K) \subset O_K$ for each $K \in \mathcal{K}$. Hence the family $\mathcal{V} = \{f^{-1}(O_K) \cap U(K) : K \in \mathcal{K}\}$ is a σ-discrete open refinement of $\mathcal{U}$.

Discrete sets and paracompactness are preserved by closed (continuous) mappings. Hence, we have

Proposition 2. If X is a paracompact σ-space and a mapping $f:K \longrightarrow Y$ is a closed continuous one, then f is σ-discrete.

Let us note

Proposition 3. Every continuous mapping of a space with a countable network onto a regular space is σ-discrete.

Definition 2. By a quasi- rigid system we call an inverse system $\{X_\alpha, \mathcal{F}_\alpha, \pi^\alpha_\beta, \alpha, \beta \in A\}$ such that all the spaces X_α are paracompact with a σ-discrete network $\mathcal{F}_\alpha$ and $\pi^\alpha_\beta(\mathcal{F}_\alpha) = \mathcal{F}_\beta$ for all $\beta \leq \alpha$. A quasi-rigid system is rigid if all the projections π^α_β are continuous bijections.

The notion of a rigid system was introduced and considered in [2],[3]. It was proved there that the limit of a rigid system is a paracompact σ-space and that every paracompact σ-space is homeomorphic to the limit of a rigid system consisting of metric spaces. Notice that in [3] we have constructed an example of a rigid system consisting of complete metric spaces, the limit of which is not metrizable. Furthermore, in [4], we have constructed an example of such a system the limit of which is even not stratifiable.

The notion of a quasi-rigid system can be defined in categorical terms. Consider a category $\mathcal{C}$ with objects $(X,\mathcal{F})$ and morphisms $f:(X,\mathcal{F})\rightarrow(Y,\mathcal{K})$, where X is paracompact, $\mathcal{F}$ is a σ-discrete network in X, f is continuous and $f(\mathcal{F})=\mathcal{K}$. Then quasi-rigid systems are exactly the inverse systems in the category $\mathcal{C}$.

It is obvious that the limit projections of a quasi-rigid system are σ-discrete. Hence the proposition 1 implies

Proposition 4. The limit space of a quasi-rigid system is a paracompact σ-space.

In [2],[3] we have proved the next

Theorem 1. The following conditions for a space X are equivalent:

1) X is a paracompact σ-space and $\dim X\leq n$.

2) X is a limit of a rigid system consisting of spaces of dimension $\dim\leq n$.

3) X is a limit of a rigid system consisting of metric spaces of dimension $\dim\leq n$.

The next theorem slightly strengthens the previous one:

Theorem 2. X is a paracompact σ-space and $\dim X\leq n$ if it is homeomorphic to a limit of a quasi-rigid system of metric spaces X_α such that $\dim X_\alpha\leq n$.

Proof. Let $S=\{X_\alpha, \mathcal{F}_\alpha, \pi^\alpha_\beta, \alpha,\beta\in A\}$ be a quasi-rigid system, $X=\varprojlim S$, $\pi_\alpha: X\rightarrow X_\alpha$ be limit projections, X_α be metrizable and $\dim X_\alpha\leq n$ for each $\alpha\in A$. We shall prove that the quasi-rigid system S is cylindrical in the sense of Yajima [8] that each finite cozero cover of X has a σ-locally finite refinement consisting of sets of the form $\pi_\alpha^{-1}(U)$, where U is a cozero-set in X_α.

Let $\mathcal{F} = \pi_\alpha^{-1}(\mathcal{F}_\alpha)$ and ω be a finite open (cozero) cover of X. Notice that without loss of generality one can assume that $\mathcal{F}$ is a refinement of ω. Let $\mathcal{B}$ be a standard base in X consisting of the sets of the form $\pi_\alpha^{-1}(U)$, where U is open in X_α ($\alpha \in A$) and let $\mathcal{U}$ be a maximal subfamily of $\mathcal{B}$ which refines ω. Moreover, without loss of generality one can assume that $\mathcal{F}$ is a refinement of $\mathcal{U}$. Then for every $F \in \mathcal{F}$ there exists an open set $O(F) \in \mathcal{U}$ and an element $\alpha(F) \in A$ such that $F \subset O(F)$, $\omega' = \{O(F) : F \in \mathcal{F}\}$ is a σ-discrete family and $O(F) = \pi_{\alpha(F)}^{-1}(U)$ for an open set $U \subset X_{\alpha(F)}$. Thus the inverse system S is cylindrical. Hence by a theorem of Yajima [8] $\dim X \le \sup\{\dim X_\alpha : \alpha \in A\} \le n$ (it actually follows from a result of B.A. Pasynkov [7, Proposition 10]).

Theorem 3. For every σ-discrete mapping $f: X \to Y$ there exists a paracompact σ-space Z and σ-discrete mappings $g: X \to Z$ and $h: Z \to Y$ such that $\dim Z \le \dim X$, $w(Z) \le w(Y)$ and $f = h \circ g$.

Proof. Consider a σ-discrete network $\mathcal{K}$ in the space X such that $\mathcal{F} = f(\mathcal{K})$ is a σ-discrete network in the space Y. By Definition 1 and Proposition 1 the spaces X and Y are paracompact. The space Y can be represented as a limit of a rigid system $\{Y_\alpha, \mathcal{F}_\alpha, \pi_\beta^\alpha, \alpha, \beta \in A\}$ such that all the spaces Y_α are metrizable, $|A| = w(Y)$, each element of the index set A has only a finite number of predecessors and $\pi_\alpha(\mathcal{F}) = \mathcal{F}_\alpha$ for all $\alpha \in A$. We denote by A_k the set of all elements of A with exactly k predecessors ($k = 0, 1, \dots$) and let $B_n = \bigcup_{k=0}^{n} A_k$. For each $\alpha \in A$ we define $f_\alpha = \pi_\alpha \circ f$. Hence $f_\alpha(\mathcal{K}) = \mathcal{F}_\alpha$, $f_\beta = \pi_\beta^\alpha \circ f_\alpha$ for each $\beta \le \alpha$.

By induction we shall construct a quasi-rigid system $S = \{Z_\alpha, \mathcal{L}_\alpha, p_\beta^\alpha, \alpha, \beta \in A\}$ and systems of σ-discrete mappings $\{(g_\alpha : X \to Z_\alpha) : \alpha \in A\}$ and $\{(h_\alpha : Z_\alpha \to Y_\alpha) : \alpha \in A\}$ with the following properties for each $\alpha \in A$ and each $\beta < \alpha$: 1) Z_α is a metrizable space; 2) $\dim Z_\alpha \le \dim X$; 3) $w(Z_\alpha) \le w(Y)$; 4) $f_\alpha = h_\alpha \circ g_\alpha$; 5) $h_\alpha(\mathcal{L}_\alpha) = \mathcal{F}_\alpha$; 6) $g_\alpha(\mathcal{K}) = \mathcal{L}_\alpha$; 7) $g_\beta = p_\beta^\alpha \circ g_\alpha$; 8) $p_\beta^\alpha \circ h_\beta = \pi_\beta^\alpha \circ h_\alpha$.

By Pasynkov's factorization theorem for metric spaces for each $\alpha \in A_0$ there exist a space Z_α and mappings g_α and h_α. Assume that a quasi-rigid system $S_m = \{Z_\alpha, \mathcal{L}_\alpha, p_\beta^\alpha, \alpha, \beta \in B_m\}$ and systems of mappings $\{(g_\alpha : X \to Z_\alpha) : \alpha \in B_m\}$ and $\{(h_\alpha : Z_\alpha \to Y_\alpha) : \alpha \in B_m\}$ satisfying the conditions 1) - 8) are already constructed. We shall construct a quasi-rigid system S_{m+1} and mappings g_α and h_α, $\alpha \in A_{m+1}$.

Consider $\alpha \in A_{m+1}$ and a mapping $\tilde{f}_\alpha = f \Delta (\Delta\{g_\beta : \beta < \alpha, \beta \in A_m\}) : X \to Y_\alpha \times \Pi\{Z_\beta : \beta < \alpha, \beta \in A_m\}$. By Pasynkov factorization theorem for metric spaces there exist a metric space Z_α and σ-discrete mappings $\tilde{h}_\alpha : Z_\alpha \to Y_\alpha \times \Pi\{Z_\beta : \beta < \alpha, \beta \in A_m\}$ and $g_\alpha : X \to Z_\alpha$ such that $\tilde{f}_\alpha = \tilde{h}_\alpha \circ g_\alpha$, $\dim Z_\alpha \leq \dim X$ and $w(Z_\alpha) \leq w(Y \times \Pi\{Z_\beta : \beta < \alpha, \beta \in A_m\}) \leq w(Y)$. Define a mapping h_α as the composition of $\tilde{h}_\alpha$ with the projection of $\tilde{h}_\alpha(Z_\alpha)$ onto Y_α in the product $Y_\alpha \times \Pi\{Z_\beta : \beta < \alpha, \beta \in A_m\}$. Then $f_\alpha = h_\alpha \circ g_\alpha$. Let $\mathcal{L}_\alpha = g_\alpha(\mathcal{K})$. It is easy to notice that the family $\mathcal{L}_\alpha$ is a σ-discrete network in the space Z_α. For $\beta < \alpha$ and $\beta \in A_m$ we define p^α_β as the composition of $\tilde{h}_\alpha$ with the projection of $\tilde{h}_\alpha(Z_\alpha)$ onto Z_β in the product $Y_\alpha \times \Pi\{Z_\beta : \beta < \alpha, \beta \in A_m\}$. It is easy to construct other projections p^α_β and to notice that $h_\alpha(\mathcal{L}_\alpha) = \mathcal{F}_\alpha$, $g_\alpha(\mathcal{K}) = \mathcal{L}_\alpha$ and $p^\alpha_\beta(\mathcal{L}_\alpha) = \mathcal{L}_\beta$ for each $\beta \leq \alpha$. In the same manner we can verify the conditions 7) and 8).

If we make such a construction for every $\alpha \in A_m$, we shall get a quasi-rigid system S_{m+1}. Thus we get quasi-rigid systems $\{S_m : m \in \mathbb{N}\}$ satisfying the conditions 1) - 8) and such that $S_m \subset S_{m+1}$. It is obvious that the quasi-rigid system $S = \cup\{S_m : m \in \mathbb{N}\}$ satisfies the conditions 1) - 8).

Define the space Z as the inverse limit of the quasi-rigid system S and let $p_\alpha : Z \to Z_\alpha$ be the limit projections ($\alpha \in A$). By Theorem 2, Z is a paracompact σ-space and $\dim Z \leq \dim X$.. Using the condition 3) and the inequality $|A| \leq w(Y)$ we get $w(Z) \leq |A| \cdot W(Y) \leq w(Y)$. Moreover, the system $\mathcal{L} = \pi^{-1}_\alpha(\mathcal{L}_\alpha)$ ($\alpha \in A$) is a σ-discrete network in Z. From the condition 7) and the definition of limits of inverse systems it follows that there exists a unique mapping $g : X \to Z$ which is defined by the system $\{(g_\alpha : X \to Z_\alpha) : \alpha \in A\}$ and such that $g_\alpha = p_\alpha \circ g$ for every $\alpha \in A$. By the condition 8 the system $\{(h_\alpha : Z_\alpha \to Y_\alpha) : \alpha \in A\}$ is a morphism in the category of quasi-rigid systems from the system S to the system $\{Y_\alpha, \mathcal{F}_\alpha, \pi^\alpha_\beta, \alpha, \beta \in A\}$. Hence there exists a mapping $h : Z \to Y$ such that $\pi_\alpha \circ h = h_\alpha \circ p_\alpha$ for every $\alpha \in A$. Hence $h(\mathcal{L}) = \mathcal{F}$ and analogously $g(\mathcal{K}) = \mathcal{L}$. Thus the mappings g and h are σ-discrete. To complete the proof one has only to notice that $f=h \circ g$.

Theorem 3 and Proposition 2 imply immediately the following

Corollary 1. For every closed mapping $f:X \to Y$ of a paracompact σ-space X there exist a paracompact σ-space Z and continuous mappings $g:X \to Z$ and $h:Z \to Y$ such that $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$ and $f=h \circ g$.

Theorem 3 together with Proposition 3 immediately imply the following

Corollary 2. For every continuous mapping $f:X \to Y$ of a regular space X with a countable network onto a regular space Y there exist a regular space Z with a countable network and continuous mappings $g:X \to Z$ and $h:Z \to Y$ such that $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$ and $f=h \circ g$.

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