Commentationes Mathematicae Universitatis Carolinae

Athanossios Tzouvaras

Some remarks on revealness

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 1, 63--69

Persistent URL: http://dml.cz/dmlcz/106509

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,1 (1987)

SOME REMARKS ON REVEALNESS

A. TZOUVARAS

Abstract: We present a further classification of classes according to their degree of revealness, which is uniformly induced by schemas of normal formulas.

Key words: Alternative Set Theory, revealed class, fully revealed class, normal formula.

Classification: 03E70

Revealed and, especially, fully revealed classes are, in a sense, good approximations of set-definable classes. Every such class includes an abundance of infinite sets and behaves well with respect to prolongation, countable meets, set-definable mappings etc.

In an attempt to explore deeper the concept, we define some forms being either between simple and full revealness, or weaker than simple revealness. These forms, as well as possibly others, arise naturally from the restriction of a general schema which describes full revealness (see Proposition 1).

Terminology and notation are the usual ones. Basic reference book on Alternative Set Theory is [V]. m,n,..., denote finite natural numbers, a,b,..., denote arbitrary natural numbers, while lower greek letters α, β, γ, ..., are used exclusively to denote ordinals. Thus, $\alpha < \beta$ means $\alpha \in \beta \cap \Omega$.

Recall that a class X is revealed if for every countable $Y \subseteq X$ there is a set u such that $Y \subseteq u \subseteq X$. X is fully revealed (f. revealed) if for every normal formula $\varphi(x,Z)$ of FL_V, the class $\{x; \varphi(x,X)\}$ is revealed. Clearly the latter condition is much stronger than the former.

The following is a rather well-known, easily proved characterization which shows the close connection between full revealness and saturation (see [S-V], § 1, or [P-S], Theorem 5.1).

Proposition 1. A class X is f. revealed iff the following schema holds:

$$A_{\varphi}^{\varphi_1,\varphi_2,\ldots}(X):(\forall n)(\exists x)(\varphi(x,X)\&\varphi_1(x)\&\ldots\&\varphi_n(x))\rightarrow$$
$$\rightarrow(\exists x)(\forall n)(\varphi(x,X)\&\varphi_n(x)),$$

for all normal formulas $\varphi(x,Z)$ and all sequences of set-formulas $(\varphi_n)_n$ of FL_V. □

By restricting the class of normal formulas for which the above schema is true, we get various weaker forms of full revealness.

In this paper we consider three such restrictions, namely the schemas:

$A_{x\in Z}^{\varphi_1,\varphi_2,\ldots}(X)$, $A_{x\subseteq Z}^{\varphi_1,\varphi_2,\ldots}(X)$, $A_{\varphi}^{\varphi_1,\varphi_2,\ldots}(X)$ for all positive φ.

(Recall that the formula $\varphi(x,Z)$ is positive in Z, or simply positive, if it belongs to the smallest class of formulas containing the set-formulas, the formula $x\in Z$ and closed under the positive operations &, , , .)

For simplicity, we drop the superscripts φ_1, $\varphi_2,\ldots$ from the above symbols of schemas, as well as the subscript $x\in Z$. Thus, the first of the foregoing three schemas is now written A(X). Every normal formula $\varphi(x,Z)$ yields an operator Γ_φ which transforms the class X to the class $\Gamma_\varphi(X)=\{x;\ \varphi(x,X)\}$. Then, it is easy to see that for every normal φ, every X and every $(\varphi_n)_n$,

$$A_\varphi(X) \text{ holds iff } A(\Gamma_\varphi(X)) \text{ holds.}$$

Thus, the forementioned schemas take the forms:

$$A(X):(\forall n)(\exists x\in X)(\varphi_1\&\ldots\&\varphi_n)\rightarrow(\exists x\in X)(\forall n)\varphi_n,$$
$$A(P(X)):(\forall n)(\exists x\in P(X))(\varphi_1\&\ldots\&\varphi_n)\rightarrow(\exists x\in P(X))(\forall n)\varphi_n,$$
$$A(\Gamma_\varphi(X)):(\forall n)(\exists x\in\Gamma_\varphi(X))(\varphi_1\&\ldots\&\varphi_n)\rightarrow$$
$$\rightarrow(\exists x\in\Gamma_\varphi(X))(\forall n)\varphi_n,\ \varphi \text{ positive},$$

where P is the power-class operator.

Definition 2. The class X is called: a) weakly revealed (w. revealed) if A(X) is true, b) strongly revealed (s. revealed) if A(P(X)) is true and c) weak fully revealed (w.f. revealed) if $A(\Gamma_\varphi(X))$ is true for all positive φ.

The following contains some trivial facts.

Proposition 3. a) X is w. revealed iff for every sequence $(X_n)_n$ of set-definable classes such that $X \cap (X_1 \cap \ldots \cap X_n) \neq \emptyset$ for every $n \in FN$, we have $X \cap (\bigcap_n X_n) \neq \emptyset$.

b) Every revealed class is w. revealed.

c) Every s. revealed class is revealed.

d) Every w.f. revealed class is s. revealed.

Proof. a) is immediate from the explicit formulation of $A(X)$, and it is well-known that every revealed class satisfies a). c) Let X be s. revealed and $Y = \{y_1, y_2, \ldots\} \subseteq X$. Put $\varphi_n(x) \equiv \{y_1, \ldots, y_n\} \subseteq x$. Then apply the schema $A(P(X))$ with those φ_n to get an $x \in P(X)$ such that $Y \subseteq x$. d) is obvious since $x \subseteq Z$ is positive. □

Proposition 4. X is w.f. revealed iff for every positive φ, $\Gamma_\varphi(X)$ is w.f. revealed.

Proof. Immediate from the fact that if φ, ψ, are positive formulas, then the formula σ such that $\Gamma_\sigma = \Gamma_\varphi \circ \Gamma_\psi$ is positive. □

Proposition 5. The following are equivalent: a) X is s. revealed, b) for any sequence $(u_n)_n$ of subsets of X, there is a set u such that $\bigcup\{u_n; n \in FN\} \subseteq u \subseteq X$, c) $P(X)$ is revealed.

Proof. a) $\to$ b). Let $\{u_1, u_2, \ldots\} \subseteq P(X)$, and put $\varphi_n(x) \equiv u_1 \cup \ldots \cup u_n \subseteq x$. By $A(P(X))$, we get easily a set $u \in P(X)$ such that $\bigcup\{u_n; n \in FN\} \subseteq u$.
b) $\to$ c). Let again $\{u_1, u_2, \ldots\} \subseteq P(X)$. Then there is some u such that $\bigcup\{u_n; n \in FN\} \subseteq u \subseteq X$. Therefore $\{u_1, u_2, \ldots\} \subseteq P(u) \subseteq P(X)$.
c) $\to$ a). This is immediate from the fact that, by definition, X is s. revealed iff $P(X)$ is w. revealed, and Prop. 3(b). □

Corollary 6. a) Every Π-class is s.revealed. b) Every Π-semiset is w.f. revealed.

Proof. a) If $X = \bigcap\{X_n; n \in FN\}$, then $P(X) = \bigcap\{P(X_n); n \in FN\}$, thus $P(X)$ is revealed. b) If $X = \bigcap\{u_n; n \in FN\}$, then for every positive φ, $\Gamma_\varphi(X) = \bigcap\{\Gamma_\varphi(u_n); n \in FN\}$ (cf. [T], 1.6). □

The following shows that strong revealness does not become stronger if we replace $P(X)$ by $P^2(X)$, $P^3(X)$ etc.

Proposition 7. If X is s. revealed, then $P^n(X)$ is s. revealed for all $n \in FN$.

Proof. It suffices to prove that if X is s. revealed, then so is $P(X)$. Let X be s. revealed and let $\bigcup\{u_n; n\in FN\}\subseteq P(X)$. Then $u_n\subseteq P(X)$ and $\bigcup u_n\subseteq X$ for every n, hence $\bigcup\{\bigcup u_n; n\in FN\}\subseteq X$. By assumption, there is u such that $\bigcup\{\bigcup u_n; n\in FN\}\subseteq u\subseteq X$. It follows that $\bigcup u_n\subseteq u$ for every n, whence $u_n\subseteq P(u)$. Therefore $\bigcup\{u_n; n\in FN\}\subseteq$ $\subseteq P(u)$ and $P(u)\subseteq P(X)$. □

Let us denote by WR,R,SR,WFR,FR the (uncodable) classes of w. revealed, revealed, s. revealed, w.f. revealed and f. revealed classes respectively. Then, by Prop. 3,

$$FR\subseteq WFR\subseteq SR\subseteq R\subseteq WR.$$

We are going to show that all these inclusions are proper.

Proposition 8. $FR\subsetneq WFR$.

Proof. Let X be a proper Π-semiset. By Corollary 6, $X\in WFR$. Clearly $X\notin FR$, since $V\setminus X$ cannot be revealed. □

Proposition 9. $WFR\subsetneq SR$.

Proof. Put $R=\bigcup\{a\times\{a\}; a\in N\setminus FN\}$. R is s. revealed. Indeed, if $u\subseteq R$, then there are $c,d\in N$ such that $FN<c<\mathrm{dom}(u)<d$, hence $u\subseteq\bigcup\{a\times\{a\}; c<a<d\}$. It follows from the revealness of $N\setminus FN$ that, given a sequence of subsets of R, $(u_n)_n$, there are c,d of N such that $\bigcup\{u_n; n\in FN\}\subseteq\bigcup\{a\times\{a\}; c<a<d\}$.
Let now $\{b_0,\dots,b_e\}$ be an infinite set of infinite natural numbers in their natural ordering. Put $w=\bigcup\{b_d\times\{d\}; d\leq e\}$. Let $X=R\cup w$. Clearly X is s. revealed. Consider the formula $\varphi(x,Z)\equiv$ $\equiv(\forall y\in N)(x\in Z''\{y\})$. Then, φ is positive and it is easy to verify that $\Gamma_\varphi(X)=FN$. Since FN is not revealed, X is not w.f. revealed according to Prop. 4. □

Proposition 10. $SR\subsetneq R$.

Proof. Consider an infinite set $\{a_0,\dots,a_d\}$ of infinite natural numbers in their natural ordering and such that $a_{x+1}-a_x$ is infinite for all $x\leq d$. Let $w_{x+1}=[a_x,a_{x+1})$ and enumerate all w_n, $n\in FN$, as follows : $w_n=\{y_{\alpha n}; \alpha\in\Omega\}$. Put $X_\alpha=\{y_{\alpha n}; n\in FN\}$ for every $\alpha\in\Omega$. We shall construct two sequences $(u_\alpha)_{\alpha\in\Omega}$, $(z_\alpha)_{\alpha\in\Omega}$ with the following properties:

i) $u_\alpha\subseteq u_\beta$ for $\alpha<\beta$,

ii) $X_\alpha\subseteq u_\alpha$ for all $\alpha\in\Omega$,

iii) $(z_\alpha)_{\alpha\in\Omega}$ is a decreasing sequence of natural numbers coinitial to $\bigcup\{w_x; x > FN\}$,

iv) $\{z_\beta; \beta < \alpha\} \cap u_\alpha = \emptyset$, for all $\alpha \in \Omega$.

v) For every $\alpha \in \Omega$ there is some $c > FN$ such that $|w_x \setminus u_\beta| > c$ for all $x \leq d$ and all $\beta < \alpha$.

Suppose the sequences have been constructed and put $X = \bigcup\{u_\alpha; \alpha \in \Omega\}$. Since $(u_\alpha)_{\alpha \in \Omega}$ is increasing, it is easy to see that X is revealed. Further, by (ii), $\bigcup\{w_n; n \in FN\} = \bigcup\{X_\alpha; \alpha \in \Omega\} \subseteq X$ and, by (iv), if $Z = \{z_\alpha; \alpha \in \Omega\}$, then $Z \cap X = \emptyset$. Suppose $\bigcup\{w_n; n \in FN\} \subseteq u \subseteq X$ for some set u. Then, there is some $e > FN$ such that $\bigcup\{w_x; x \leq e\} \subseteq u \subseteq X$. But this contradicts the fact that Z is coinitial to $\bigcup\{w_x; x > FN\}$ and $Z \cap X = \emptyset$. Therefore there is no u such that $\bigcup\{w_n; n \in FN\} \subseteq u \subseteq X$ which, by Prop. 5(b), implies that X is not s. revealed.

<u>Construction of the sequences</u>. Assume u_β, z_β have been defined for all $\beta < \alpha$ and satisfy properties (i)-(v). Then $\bigcup\{X_\beta; \beta < \alpha\} \subseteq \bigcup\{u_\beta; \beta < \alpha\}$, $\{z_\beta; \beta < \alpha\} \cap (\bigcup\{u_\beta; \beta < \alpha\}) = \emptyset$ and there is some $c > FN$ such that $|w_x \setminus u_\beta| > c$ for all $x \leq d$ and all $\beta < \alpha$. Then, clearly, using the prolongation axiom, we can extend $\bigcup\{u_\beta; \beta < \alpha\}$ to a set u such that $u \cap \{z_\beta; \beta < \alpha\} = \emptyset$ and $|w_x \setminus u| > c$ for all $x \leq d$. Choose, besides, v such that $X_\alpha \subseteq v \cap \{z_\beta; \beta < \alpha\} = \emptyset$ and $|v \cap w_x| \leq 1$ for every $x \leq d$. This is certainly possible since each X_α meets every interval w_n in exactly one point. Put $u_\alpha = u \cup v$. Then $|w_x \setminus u_\alpha| > c-1$ for all $x \leq d$. Suppose $\{r_\alpha; \alpha \in \Omega\}$ is a fixed enumeration of the class $\bigcup\{w_x; x > FN\}$. Choose z_α such that $z_\alpha < r_\alpha$, $z_\alpha < z_\beta$ for every $\beta < \alpha$ and $z_\alpha \in \bigcup\{w_x; x > FN\} \setminus u_\alpha$. This is possible because of condition (v). It is obvious that the defined u_α, z_α conform with all requirements and the construction is complete. $\square$

<u>Lemma 11</u>. a) A class X is non-revealed iff there is a function f such that $f''FN \subseteq X$ and the class $\{a; f(a) \notin X\}$ is coinitial to $N \setminus FN$.

b) A class X is w. revealed iff for every function f such that $f''FN \subseteq X$, there is some $a > FN$ such that $f(a) \in X$.

<u>Proof</u>. Both claims are easily proved using the prolongation axiom. $\square$

<u>Proposition 12</u>. $R \subsetneq WR$.

<u>Proof</u>. Take a decreasing Ω-sequence $(x_\alpha)_{\alpha \in \Omega}$ of natural

numbers coinitial to $N \setminus FN$, and consider the class $X = N \setminus \{x_\alpha; \alpha \in \Omega\}$. By Lemma 11(a), where f is the identity, we get that X is not revealed. On the other hand, let $f''FN \subseteq X$. It suffices to show that $f''(N \setminus FN) \cap X \neq \emptyset$. Suppose the contrary. Then $f''(N \setminus FN) \subseteq \{x_\alpha; \alpha \in \Omega\}$. Clearly, for non-trivial f (i.e. f such that $f''FN$ is countable), there is $u \subseteq N \setminus FN$ such that $f''u$ is infinite. Therefore $\{x_\alpha; \alpha \in \Omega\}$ contains an infinite subset v. But then v must be coinitial to $N \setminus FN$, a contradiction. $\square$

Let us remark that in proving Proposition 10, we constructed a class X of the form $X = \bigcup\{u_\alpha; \alpha \in \Omega\}$ where $(u_\alpha)_{\alpha \in \Omega}$ is increasing, which failed to be s. revealed. Clearly, every such class is revealed but the converse is open for us. Let us call these classes <u>completely revealed</u> (c. revealed) and let CR denote the class of all c. revealed classes. Then,

<u>Proposition 13</u>. $SR \subsetneq CR$.

<u>Proof</u>. Let X be s. revealed and let $X = \{x_\alpha; \alpha \in \Omega\}$ be an enumeration of X. Since for every countable sequence $\{u_1, u_2, \dots\} \subseteq \subseteq P(X)$ there is some u such that $\bigcup\{u_n; n \in FN\} \subseteq u \subseteq X$, it is clear that we can define inductively an increasing sequence $(u_\alpha)_{\alpha \in \Omega}$ such that $x_\alpha \in u_\alpha$ for every $\alpha \in \Omega$ and $u_\alpha \subseteq X$. Hence $X = \bigcup\{u_\alpha; \alpha \in \Omega\}$. Therefore $SR \subseteq CR$ and combining this with the proof of Prop. 10, we get $SR \subsetneq CR$. $\square$

<u>Proposition 14</u>. X is s. revealed iff there is an increasing Ω-sequence $(u_\alpha)_{\alpha \in \Omega}$ such that $X = \bigcup\{u_\alpha; \alpha \in \Omega\}$ and $P(X) = \bigcup\{P(u_\alpha); \alpha \in \Omega\}$.

<u>Proof</u>. Suppose X satisfies the conditions and $\{v_1, v_2, \dots\} \subseteq \subseteq P(X)$. Let $v_n \in P(u_{\alpha_n})$ for every $n \in FN$. If α is some ordinal greater than all α_n, then clearly $\bigcup\{v_n; n \in FN\} \subseteq u_\alpha \subseteq X$. Conversely, suppose X is s. revealed. By Prop. 7, P(X) is s. revealed, hence c. revealed according to Prop. 13. Let $P(X) = \bigcup\{r_\alpha; \alpha \in \Omega\}$ where $(r_\alpha)_{\alpha \in \Omega}$ is increasing. Then $X = \bigcup P(X) = \bigcup\{\bigcup r_\alpha; \alpha \in \Omega\}$. Put $u_\alpha = \bigcup r_\alpha$. Clearly $(u_\alpha)_{\alpha \in \Omega}$ is increasing and $X = \bigcup\{u_\alpha; \alpha \in \Omega\}$. Let $v \in P(X)$. Then $v \in r_\alpha$ for some α, hence $v \subseteq \bigcup r_\alpha = u_\alpha$ and this proves the claim. $\square$

One could see the notions of revealness defined in this paper in a more general context as follows: Let O denote the class of operators induced by normal formulas. O together with the usual law of composition of transformations, form a semigroup. The operator I induced by the formula $x \in Z$ is the unit element of the semigroup. By subsemigroup of O we mean any subclass of O closed under composition and containing I. For example the subclass O_p of positive operators is a subsemigroup of O. Let us denote by $\langle \Gamma_1, \dots, \Gamma_n \rangle$ the subsemigroup generated by the elements $\Gamma_1, \dots, \Gamma_n$ of O. Then, clearly,

a) X is w. revealed iff $A(\Gamma(X))$ holds for every $\Gamma \in \langle I \rangle$.

b) X is s. revealed iff $A(\Gamma(X))$ holds for every $\Gamma \in \langle P \rangle$.

(See Prop. 7.)

Generalizing, one could say that every subsemigroup S of O defines a reasonable notion of revealness, say S-revealness, in the obvious way, that is,

X is S-revealed iff $A(\Gamma(X))$ holds for every $\Gamma \in S$.

Of course not every such notion is expected to be non-trivial, interesting and useful. Some of them, however, might be. For example the notion corresponding to the subsemigroup $\langle P, \sim \rangle$, where $\sim$ is the operator of the formula $x \notin Z$, much stronger than strong revealness, seems to be interesting.

We finish with some questions:

1) Is every revealed class completely revealed?

2) Does there exist any subsemigroup S of O such that revealness be equivalent to S-revealness?

3) Does there exist any "small" subsemigroup T of O such that full revealness be equivalent to T-revealness?

References

[P-S] P. PUDLÁK, A. SOCHOR: Models of the Alternative Set Theory, J. Symb.Logic 49(1984), 570-585.

[S-V] A. SOCHOR, P. VOPĚNKA: Revealments, Comment.Math .Univ.Carolinae 21(1980), 97-119.

[T] A. TZOUVARAS: Definability degrees for classes in AST, submitted to Comment.Math.Univ.Carolinae.

[V] P. VOPĚNKA: Mathematics in the Alternative Set Theory, Teubner-Texte, Leipzig 1979.

Department of Mathematics, University of Thessaloniki, Thessaloniki, Greece

(Oblatum 5.3. 1986)