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REMARKS CONCERNING J. WITTE'S THEOREM AND ITS APPLICATIONS

Józef BANAŚ x) and Jesus RIVERO

Abstract: In this paper some variant of the theorem due to Jürgen Witte [13] is discussed. Theorems on uniqueness for the Cauchy's problem of ordinary differential equations are derived. Moreover an application to differential equations in the case of Banach spaces is shown.

Key words: Ordinary differential equation, uniqueness criterion, Kamke comparison function, measure of noncompactness.

Classification: 34A10, 34G20

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1. Introduction. It is well known that the so-called uniqueness criteria for the initial value problem

(1) $$x' = f(t,x),\ x(0) = x_0$$

can be obtained via various types of Kamke comparison functions, among others (cf. for example [2, 6, 10, 12]).

Let us recall that a function $\omega : \langle 0,T\rangle \times R_+ \longrightarrow R_+$ (or $\omega : (0,T\rangle \times R_+ \longrightarrow R_+$) is called a Kamke comparison function (cf. [2]) if the inequality

$$\| f(t,x) - f(t,y) \| \le \omega(t, \|x-y\|)$$

together with some assumptions concerning the function $\omega(t,u)$ guarantee that the initial value problem (1) has at most one solution. Examples of Kamke comparison functions can be provided by the criteria of Lipschitz, Nagumo or Osgood [12], for instance.

In the paper [9] Rogers has used the function $\omega(t,u)=u/t^2$ which is no longer of Kamke type because the above mentioned inequality does not imply that (1) has at most one solution. In order to obtain a uniqueness criterion for (1) Rogers had to put an extra assumption concerning the behavior of a function f(t,x).

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x) This paper was done while the first author visited the Universidad de Los Andes, Venezuela

The result of Rogers was next generalized by Witte in [13] who considered the function of the form $\omega(t,u)=a(t)u$ with $a(t)$ being generally discontinuous at $t=0$.

The aim of this paper is to give some generalizations and applications of the result due to Witte. We formulate the uniqueness theorem for (1) being more general than that of Witte. Moreover, we consider also the so-called one-sided condition guaranteeing the uniqueness for (1). The applicability of Witte's type theorem to the case of a Banach space is also indicated.

2. Some lemma of Witte's type. In this section we remind the result of J. Witte [13] concerning some integral inequality. We point out also to some consequences of this result.
The mentioned result is formulated in a little more general form than in [13]. Apart from that we give a simple proof of this result because that given in [13] seems to be unnecessarily complicated.

Lemma 1. Let $u:\langle 0,1\rangle \longrightarrow \langle 0,+\infty)$ be a continuous function and let the following assumptions be satisfied:

(i) $a:(0,1\rangle \longrightarrow (0,+\infty)$ is a given continuous function,

(ii) there exists a function $A:(0,1\rangle \longrightarrow R$ such that $A'(t)=a(t)$ for almost all $t\in(0,1\rangle$ and there exists the limit $\lim_{t\to 0+} A(t)$ (finite or not),

(iii) $u(t) \leq \int_0^t a(s)u(s)ds$, $t\in\langle 0,t\rangle$,

(iv) $u(t)=o(\exp(A(t)))$ as $t\longrightarrow 0+$.

Then $u(t)\equiv 0$ on the interval $\langle 0,1\rangle$.

Proof. Let us put $F(t)=\int_0^t a(s)u(s)ds$, $t\in\langle 0,1\rangle$. We have

$$F'(t) = a(t)u(t)\leq a(t)F(t).$$

Hence

$$F'(t)\exp(-A(t)) - a(t)F(t)\exp(-A(t))\leq 0$$

which can be written in the form

$$d/dt\,[\exp(-A(t))F(t)]\leq 0$$

for almost all $t\in\langle 0,1\rangle$. This allows us to deduce that the function $t\longrightarrow \exp(-A(t))F(t)$ is nonincreasing (cf. [7]). Hence, choosing $\varepsilon > 0$ and taking t sufficiently small, in view of (iv) we get

$$\exp(-A(t))F(t) = \exp(-A(t)) \int_0^t a(s)u(s)ds \leq$$

$$\leq \varepsilon \exp(-A(t)) \int_0^t a(s)\exp(A(s))ds \leq \varepsilon \exp(-A(t))\exp(A(t)) = \varepsilon$$

so that $\lim_{t \to 0+} \exp(-A(t))F(t)=0$. Consequently

$$\exp(-A(t))F(t) \leq 0$$

for $t > 0$ which implies that

$$\int_0^t a(s)u(s)ds \leq 0.$$

Hence $u(t) \equiv 0$ and the proof is complete.

Remark. Taking $a(t)=1/t$ and $A(t)=\ln t$ we obtain the well known Nagumo's criterion [8]. The condition (iv) has now the form

$$u(t) = o(t) \text{ as } t \to 0+.$$

Similarly, assuming that $a(t)$ is continuous on the interval $\langle 0,1\rangle$ and putting $A(t)= \int_0^t a(s)ds$ we can derive the Gronwall's lemma [12]. Further notice that in the case $a(t)=1/t^{1+\alpha}$, $\alpha > 0$ and $A(t)=-1/t^{\alpha}$ the condition (iv) has the form

$$u(t) = o(\exp(-1/\alpha t^{\alpha})) \text{ as } t \to 0+.$$

Particularly for $\alpha=1$ we obtain the Rogers's lemma [9].
Finally, consider the situation $a(t)=c/t$, when $c>1$, $c=$const. This case is not covered by Nagumo's criterion [12]. Note that here the assumption (iv) may have the form

$$u(t) = o(t^c) \text{ as } t \to 0+.$$

3. Theorems on uniqueness. Now we give the theorems on uniqueness of solutions of ordinary differential equations.

Theorem 1. Let $f(t,x)$ be a continuous function on the set $(0,1\rangle \times R$ and satisfy the conditions:

(2) $|f(t,x)-f(t,y)| \leq a(t)|x-y|$

(3) $|f(t,x)-f(t,y)| = o(a(t)\exp(A(t)))$ as $t \to 0+$ uniformly with respect to $x,y \in \langle x_0-\delta, x_0-\delta\rangle$, $\delta > 0$ - arbitrary,

where $a(t)$, $A(t)$ are such as in Lemma 1.

Then the initial value problem (1) has at most one solution.

Proof. Let us suppose that x, y are solutions of our problem. Then by (2) we have

$$|x(t) - y(t)| \leq \int_0^t a(s)|x(s) - y(s)|ds.$$

Further, for an arbitrary $\varepsilon > 0$ and t sufficiently small, by virtue of (3) we get

$$|x(t)-y(t)| \leq \int_0^t |f(s,x(s))-f(s,y(s))|ds \leq \varepsilon \int_0^t a(s)\exp(A(s))ds \leq$$
$$\leq \varepsilon \exp(A(t))$$

which in view of Lemma 1 completes the proof.

Remark. J. Witte [13] instead of (3) assumed that

$$(3') \qquad f(t,x) = o(a(t)\exp(\int_0^t a(s)ds)) \text{ as } t \longrightarrow 0+,$$

uniformly with respect to $x \in \langle -\delta, \delta \rangle$. An analogous assumption was made also by Rogers [9], namely he assumed that

$$(3") \qquad f(t,x) = o(\exp(-1/t)/t^2) \text{ as } t \longrightarrow 0+.$$

Actually the assumptions like to those of (3') and (3") imply the assumption (3) but not conversely. We show this with the following example.
Let

$$f(t,x) = \begin{cases} \exp(-1/t)/t+\exp(-1/t)+1, & \text{for } x \geq t\exp(-1/t), \\ x/t^2+\exp(-1/t)+1, & \text{for } 0 \leq x \leq t\exp(-1/t), \\ \exp(-1/t)+1, & \text{for } x \leq 0. \end{cases}$$

This function is continuous on the set $(0,1\rangle \times R$ and satisfies the assumptions of Theorem 1. The unique solution of the problem $x' = f(t,x)$, $x(0)=0$ is $x(t)=t\exp(-1/t)+t$. But on the other hand it is easy to check that f(t,x) does not satisfy the assumption (3') or (3").
This example shows that our theorem is more general than those given in [9, 13].

Now we give a uniqueness theorem involving one-sided conditions (see e.g. [4, 12]). Similarly, as before, let us assume that $a:(0,1\rangle \longrightarrow (0,+\infty)$ is a given continuous function and $A:(0,1\rangle \longrightarrow R$ is such that $A'(t)=a(t)$ for almost all $t \in (0,1\rangle$, and the limit $\lim_{t\to 0+} A(t)$ exists. Furthermore, we will assume that $f:(0,1\rangle \times R^n \longrightarrow R^n$ is continuous and

$$(f(t,x) - f(t,y))(x-y) \leq a(t)(x-y)^2$$

for $t \in (0,1\rangle$ and $x,y \in R^n$ (the above multiplication is understood as the scalar product).
Moreover,

$$f(t,x) - f(t,y) = o(a(t)\exp(A(t))) \text{ as } t \longrightarrow 0+,$$

uniformly with respect to $x,y \in \langle x_0-\delta, x_0+\delta\rangle$ where $\delta > 0$ is arbitrary. Then we have the following theorem.

Theorem 2. Under the above assumptions the initial value problem (1) has at most one solution on the interval $\langle 0,1\rangle$.

Proof. Let $x_1(t)$, $x_2(t)$ be solutions of our problem. Denote $\delta(t)=(x_1(t)-x_2(t))^2$. Using the first assumed inequality we get

$$\delta'(t)=((x_1(t)-x_2(t))^2)' = 2(x_1(t)-x_2(t))(x_1'(t)-x_2'(t)) =$$
$$= 2(f(t,x_1(t))-f(t,x_2(t)))(x_1(t)-x_2(t)) \leq$$
$$\leq 2a(t)(x_1(t)-x_2(t))^2 = 2a(t)\delta(t).$$

Hence

$$\delta'(t) - 2a(t)\delta(t) \leq 0$$

and consequently

$$\delta'(t)\exp(-2A(t)) - 2a(t)\exp(-2A(t))\delta(t) \leq 0.$$

Thus

$$d/dt\,[\delta(t)\exp(-2A(t))] \leq 0$$

for almost all $t \in \langle 0,1\rangle$. The above inequality implies that the function $t \longrightarrow \delta(t)\exp(-2A(t))$ is nonincreasing. On the other hand, taking $\varepsilon > 0$ arbitrary and t sufficiently small and using our assumptions we derive

$$\delta(t)\exp(-2A(t)) = \exp(-2A(t))(x_1(t)-x_2(t))^2 =$$

$$= \exp(-2A(t))\left(\int_0^t (f(s,x_1(s))-f(s,x_2(s)))ds\right)^2 \leq$$

$$\leq \exp(-2A(t))\, \varepsilon^2 \left(\int_0^t a(s)\exp(A(s))ds\right)^2 \leq \varepsilon^2 \exp(-2A(t))\exp(2A(t)) = \varepsilon^2$$

and further we get

$$\lim_{t \to 0_+} \sigma(t)\exp(-2A(t)) = 0.$$

Finally we deduce that $\sigma(t) \leq 0$ which gives $x_1(t) \equiv x_2(t)$. This assertion finishes the proof.

4. Application to the case of Banach spaces. In this section we give an application of Theorem 1 to the existence problem for ordinary differential equations in Banach spaces (cf. [4]). In our considerations we will use the notion of the so-called measure of noncompactness defined in the axiomatic way in the work [1]. We recall shortly some basic facts.

Let us assume that E is a given real Banach space. Denote by $\mathfrak{M}_E$, $\mathfrak{N}_E$ the families of all bounded and nonempty subsets or nonempty and relatively compact subsets of E, respectively.

Definition [1]. A function $\mu : \mathfrak{M}_E \longrightarrow \langle 0,+\infty)$ will be called a measure of noncompactness in the space E provided the following conditions are satisfied:

(i) the family $\ker \mu = [X \in \mathfrak{M}_E : \mu(X)=0]$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$,

(ii) $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$,

(iii) $\mu(\overline{X}) = \mu(\operatorname{Conv} X) = \mu(X)$,

(iv) $\mu(\lambda X+(1-\lambda)Y) \leq \lambda\mu(X)+(1-\lambda)\mu(Y)$ for $\lambda \in \langle 0,1\rangle$,

(v) if X_n are closed and $X_{n+1} \subset X_n$ for n=1,2,... and if $\lim_{n\to\infty} \mu(X_n)=0$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$ and $X_\infty \in \ker \mu$.

For the properties of measures of noncompactness we refer to [1].

Now let us consider an ordinary differential equation

$$x' = f(t,x) \tag{4}$$

with the initial condition

$$x(0) = x_0. \tag{5}$$

In what follows we will assume that f is a function defined on the

Cartesian product of the interval $\langle 0,T\rangle$ and the closed ball $K(x_0,r)$ in the space E, with values in E. Moreover, the function f is uniformly continuous and bounded, $\|f(t,x)\| \le A$.
Further, let μ be a given measure of noncompactness in the space E such that $\{x_0\} \in \ker\mu$ and a(t), A(t) be given functions of the same type as in Theorem 1. We assume that f satisfies the following comparison oondition

(6) $$\mu(x_0+f(t,X)) \le a(t)\,\mu(X), \text{ for } t\in\langle 0,T\rangle,\ X\subset K(x_0,r)$$

and

(7) $$\mu(x_0+f(t,X)) = o(a(t)\exp(A(t))), \text{ as } t \to 0+$$

uniformly with respect to $X\subset K(x_0,r)$.

Under the above assumptions we have the following theorem which generalizes those given in [3, 11].

Theorem 3. Let $T\le 1$, $A\le r$. Then the equation (4) has at least one solution x which satisfies the condition (5). Moreover, $x(t)\in E_\mu = [x\in E: \mu(\{x\})=0]$ for all $t\in\langle 0,T\rangle$

Proof. Let us consider the set X_0 consisting of all functions $x:\langle 0,T\rangle \to E$ such that $x(0)=x_0$ and $\|x(t)-x(s)\| \le A|t-s|$. Actually X_0 is nonempty, closed, convex and equicontinuous in the space $C(\langle 0,T\rangle,E)$ with the usual maximum norm. The transformation F defined by the formula

$$(Fx)(t) = x_0 + \int_0^t f(s,x(s))ds$$

maps continuously the set X_0 into itself and our problem is equivalent to the existence of a fixed point of F. Further, consider the sets $X_{n+1}=\mathrm{Conv}\,FX_n$, n=0,1,2,... . All these sets are of the same type as X_0 and $X_{n+1}\subset X_n$. Putting

$$u_n(t) = \mu(X_n(t)),\ t\in\langle 0,T\rangle$$

we have $0\le u_{n+1}(t)\le u_n(t)$ and moreover, in view of the properties of measures of noncompactness [1] we deduce that the sequence $u_n(t)$ converges uniformly to a function $u_\infty(t)=\lim_{m\to\infty} u_n(t)$. Furthermore, using (6) and Lemma 5 from [3] we get

(8) $$u_n(t)=\mu(x_0+\int_0^t f(s,X_n(s))ds) \le \int_0^t \mu(x_0+f(s,X_n(s)))ds \le$$

$$\leq \int_0^t a(s)\mu(X_n(s))ds \leq \int_0^t a(s)u_n(s)ds.$$

Next, let us fix an arbitrary $\varepsilon > 0$. Then from (7) we infer that there exists $\delta > 0$ such that

$$\mu(x_0+f(t,X)) \leq \varepsilon\, a(t)\exp(A(t))$$

for $t \in (0,\delta\rangle$, $X \subset K(x_0,r)$. Hence we get

$$u_n(t) \leq \int_0^t \mu(x_0+f(s,X_{n-1}(s)))ds \leq \varepsilon\int_0^t a(s)\exp(A(s))ds \leq \varepsilon \exp(A(t))$$

for $t \in (0,\delta\rangle$, so that $u_n(t)=o(\exp(A(t)))$ as $t \to 0+$ and consequently

(9) $$u_\infty(t) = o(\exp(A(t))) \text{ as } t \to 0+.$$

Moreover, the functions $t \to a(t)u_n(t)$ are integrable on the interval $\langle 0,T\rangle$ and the sequence $a(t)u_n(t)$ converges uniformly to a function $a(t)u_\infty(t)$ so that via (8) we obtain

$$u_\infty(t) \leq \int_0^t a(s)u_\infty(s)ds.$$

Combining the above inequality and (9) and applying Lemma 1 we conclude that $u_\infty(t) \equiv 0$.

Finally notice that

$$\lim_{n\to\infty} \{\max [u_n(t): t \in \langle 0,T\rangle]\} = 0$$

so that the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty, closed, convex and contained in $\ker \mu$. Now using the Schauder fixed point principle and some properties of measures of noncompactness [1] we obtain the desired assertion. Thus the proof is complete.

<u>Remark</u>. In the proof of Theorem 3 we have used the ideas of the proof of an existence theorem given in [5].

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