COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

27,4 (1986)

ON THE SOLUTION OF TRANSONIC FLOWS WITH WEAK SHOCKS

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Abstract

We prove that the solution of a compressible (generally transonic) flow of an ideal fluid can be obtained as a limit of viscous solutions, if the viscosity and heat conductivity tend to zero. To obtain an isentropic irrotational flow it is necessary to control the entropy and temperature on the boundary in a convenient way.

AMS classification: 35M05, 35Q10, 35F30, 76A02, 76H05, 76N15

Key words: viscous flow, transonic flow, Navier-Stokes equations, entropy, weak solution, conservation law equations, method of characteristics

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1. Introduction

Irrotational isentropic transonic flow is described by the boundary value problem for a velocity potential u:

$$(1.1)\qquad a)\qquad -\frac{\partial}{\partial x_i}\left(\rho(|\nabla u|^2)\frac{\partial u}{\partial x_i}\right) = 0 \qquad \text{in } \Omega,$$

$$b)\qquad \rho(|\nabla u|^2)\,\frac{\partial u}{\partial n} = g \qquad \text{on } \partial\Omega,$$

where

$$(1.2)\qquad \rho(|\nabla u|^2) = \rho_0\left(1 - \frac{\kappa-1}{2a_0^2}|\nabla u|^2\right)^{\frac{1}{\kappa-1}}.$$

The constants ρ_0 and a_0 are the density and speed of sound respectively at zero velocity, $\kappa > 1$ is the adiabatic constant.

From physical reasons it is necessary to control the entropy in an appropriate way since the entropy information is not contained in equation (1.1,a). Bristeau, Glowinski, Pironneau, Perriaux, Perrier, Poirier propose in their papers (see e. g. [1]) the entropy condition in the form

$$\Delta u \le K. \tag{1.3}$$

Problem (1.1) - (1.3) was studied theoretically in [3,4] where it was proved that the condition (1.3) together with the assumption of the bounded velocity

$$|\nabla u|^2 \le s_0 < \frac{2a_0^2}{\kappa-1} \tag{1.4}$$

have compactification properties.

In this paper we try to give theoretical foundations of the viscosity method used in the transcnic flows. (For some numerical approaches see e. g. [8].) We start from the fact that the entropy flux is automatically governed by the conservation law equations for small parameters of the viscosity μ and heat conductivity k and prove the existence of a weak nonviscous solution as a limit of viscous flow fields, if μ, k $\to$ 0+.

Similar results were derived by Di Perna [2] for a nonstationary hyperbolic system. In [6] C. Morawetz applied artificial viscosity and hodograph approach.

Here we give a brief survey of our fundamental results which will appear in detail in the forthcoming paper [5].

2. Formulation

Let $\Omega \subset R^N$ (N = 2 or 3) be a simply connected domain with a Lipschitz-continuous boundary $\partial\Omega$. We shall use the following notation: ρ - density, p - pressure, T - temperature, T_0 - temperature at zero velocity, $v = (v_1,\ldots,v_N)$ - velocity, S - entropy, c_p and c_v specific heats at constant pressure and volume, respectively, μ - viscosity, k-thermal conductivity, $R = c_p - c_v$, $\kappa = c_p/c_v$. R, c_p, c_v, μ, k are positive constants. Hence, $\kappa > 1$. $n = (n_1,\ldots,n_N)$ denotes a unit outer normal to $\partial\Omega$.

Stationary flow of a compressible, perfect, viscous, conductive gas in the domain Ω is governed by the following system:

(2.1) $$p = R\rho T \qquad \text{(state equation)}$$

(2.2) a) $$\frac{\partial}{\partial x_i}(\rho v_i) = 0 \text{ in } \Omega \text{ (continuity equation)}$$

b) $$\rho v_i n_i = g \text{ on } \partial\Omega,$$

c) $$\int_{\partial\Omega} g\,ds = 0,$$

(2.3) a) $$\rho v_j \frac{\partial v_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = -\frac{2}{3}\mu\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) + 2\mu\frac{\partial}{\partial x_j} e_{ij}(v),$$

$$e_{ij}(v) = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \text{ in } \Omega,\ i = 1,\ldots,N$$

(Navier-Stokes equations),

b) $$v = v^0 \quad \text{on } \partial\Omega,$$

(2.4) a) $$T\frac{\partial}{\partial x_i}(\rho S v_i) = k\,\Delta T + 2\mu e_{ij}(v)\, e_{ij}(v) - \frac{2}{3}\mu\left(\frac{\partial v_i}{\partial x_i}\right)^2 \quad \text{in } \Omega$$

(energy equation),

b) $$\frac{\partial T}{\partial n} = h \quad \text{on } \partial\Omega,$$

c) $$S = c_v \ln \frac{T}{\rho^{\kappa-1}}.$$

(We use the summation convention over repeated indices.)

g, h, v^0 are given functions, μ, k, κ, c_p, c_v, R given constants, ρ, p, T, S, v are unknown functions.

Let us use the usual notation $W^{1,p}(\Omega)$, $W_o^{1,p}(\Omega)$ and $L^p(\Omega)$, $L^p(\partial\Omega)$ $(1 \le p \le +\infty)$ for the Sobolev and Lebesgue spaces, respectively. Further, we put $W^{1,p}(\Omega,R^N) = W^{1,p}(\Omega) \times \ldots \times W^{1,p}(\Omega)$ (N-times), $L^p(\Omega,R^N) = L^p(\Omega) \times \ldots \times L^p(\Omega)$ etc.

We shall assume that for each $\mu > 0$, $k > 0$ the above problem has a weak solution satisfying the conditions

(2.5) $\quad \rho \in W^{1,2}(\Omega),\ 0 < \tilde{\rho}_0 \le \rho(x) \le \tilde{\rho}_1 < +\infty,$

(2.6) $\quad v \in W^{1,2}(\Omega,R^N),\ |v| \le K,$

(2.7) $\quad g \in L^\infty(\partial\Omega),\ h \in L^1(\partial\Omega),\ |g|,\ \|h\|_{L^1(\partial\Omega)} \le K,$

(2.8) $\quad T \in W^{1,2}(\Omega),\ 0 < \tilde{T}_0 \le T(x),$

(2.9) $\quad \left|\int_\Omega p\,dx\right| \le K$

with constants $\tilde{\rho}_0$, $\tilde{\rho}_1$, K, $\tilde{T}_0$ independent of μ, k,

and the equations

(2.10) $\quad \int_\Omega \rho v_i \frac{\partial\phi}{\partial x_i}\,dx = \int_{\partial\Omega} g\phi\,ds \quad \forall\phi \in W^{1,2}(\Omega);$

(2.11) $\quad \int_\Omega \rho v_j \frac{\partial v_i}{\partial x_j}\phi_i\,dx = \int_\Omega p\frac{\partial\phi_i}{\partial x_i}\,dx + \frac{2}{3}\mu\int_\Omega \frac{\partial v_j}{\partial x_j}\frac{\partial\phi_i}{\partial x_i}dx - 2\mu\int_\Omega e_{ij}(v)e_{ij}(\phi)\,dx$

$$\forall\ \phi = (\phi_1,\ldots,\phi_N) \in W_o^{1,2}(\Omega,R^N),$$

$$v^o \in W^{1,2}(\Omega,R^N),\ v = v^o \text{ on } \partial\Omega;$$

(2.12) $\quad -\int_\Omega T\rho S v_i \frac{\partial\phi}{\partial x_i}\,dx + \int_{\partial\Omega} TSg\phi\,ds - \int_\Omega \frac{\partial T}{\partial x_i}\rho v_i S\phi\,dx =$

$$= -k\int_\Omega \nabla T.\nabla\phi\,dx + k\int_{\partial\Omega} h\phi\,ds + \int_\Omega E(v)\phi\,dx \quad \forall\phi \in W^{1,2}(\Omega) \cap L^\infty(\Omega),$$

where

(2.13) $\quad E(v) = 2\mu e_{ij}(v)e_{ij}(v) - \frac{2}{3}\mu\left(\frac{\partial v_i}{\partial x_i}\right)^2.$

It is easy to find out that

(2.14) $\quad E(v) \ge 0.$

Let us remark that from (2.4,a), (2.8) and (2.14) we derive the entropy condition, i. e. the second law of thermodynamics, which is postulated in the form

$$(2.15) \qquad T\,\mathrm{div}(\rho S\, v) - k\, T\, \mathrm{div}\left(\frac{\mathrm{grad}\, T}{T}\right) \geq 0.$$

3. Fundamental estimates

We shall derive the estimates of the solutions to problems (2.10) - (2.12) for μ, $k > 0$, $k = \beta\mu$, where $\beta > 0$ is a constant independent of μ and k. Our considerations will be carried out under (2.5) - (2.9) and the following fundamental assumption:

$$(3.1) \qquad \left|\frac{1}{\mu}\int_{\partial\Omega} Sg\,ds\right| \leq K \qquad \forall\mu > 0.$$

It holds e. g., if

$$(3.2) \qquad \int_{\partial\Omega} |S - S_0|\,ds \leq \mu\bar{K},\ \forall\mu > 0,$$

where $S_0 = c_v \ln(T_0/\rho_0^{\kappa-1})$ and $p_0\rho_0 = RT_0$. The constants K, $\bar{K}$ are independent of μ, k.

By c we shall denote a positive generic constant independent of μ, k, which can have different values at different places.

3.3. Theorem. We have

$$\int_\Omega \frac{|\nabla T|^2}{T^2}\,dx + \frac{1}{k}\int_\Omega \frac{E(v)}{T}\,dx \leq c.$$

Proof follows from (2.12), where we put $\phi = \frac{1}{T}$ and from (2.7), (2.8), (3.1). ☐

3.4. Theorem. We have

a) $\int_\Omega T^2 dx \leq c$, b) $\int_\Omega E(v)\,dx \leq c$, c) $\int_\Omega |\nabla T|\,dx \leq c$.

Sketch of the proof. Substituting $\phi = 1$ in (2.12), and using the Cauchy inequality we get

$$(3.5) \qquad \int_\Omega E(v)dx \le c + c\left(\int_\Omega T^2 dx\right)^{\frac{1}{2}}.$$

Similarly as in [7] we prove that

$$(3.6) \qquad \|p\|_{L^2(\Omega)} \le c\{\textstyle\sum_{i=1}^N \|\frac{\partial p}{\partial x_i}\|_{W^{-1,2}(\Omega)} + |\int_\Omega p dx|\}.$$

Since

$$(3.7) \qquad [v,\phi] = 2\int_\Omega e_{ij}(v)e_{ij}(\phi)dx - \frac{2}{3}\int_\Omega \frac{\partial v_j}{\partial x_j} \frac{\partial \phi_i}{\partial x_i} dx$$

is a bilinear form on $W^{1,2}(\Omega,R^N) \times W^{1,2}(\Omega,R^N)$ and $[v,v] \ge 0$, the Cauchy inequality holds. From (2.12) and (3.6) we derive the estimate

$$\|p\|_{L^2(\Omega)} \le c(1 + \mu^{\frac{1}{2}}(\int_\Omega E(v)dx)^{\frac{1}{2}}).$$

This, the equation $p = R\rho T$ and (3.5) imply

$$(3.8) \qquad \|T\|_{L^2(\Omega)} \le c(1 + \mu^{\frac{1}{2}}\|T\|^{\frac{1}{2}}_{L^2(\Omega)}),$$

which already gives assertion a). Assertions b),c) immediately follow by applying (2.8), (3.5), Theorem 3.3 and the Cauchy inequality. ◻

3.9. Theorem. Let $\|v^0\|_{W^{1,2}(\Omega,R^N)} + \|v^0\|_{L^\infty(\Omega,R^N)} \le K$. Then

$$(3.10) \qquad a) \int_\Omega \frac{|\nabla v|^2}{T} dx \le c, \qquad b) \int_\Omega |\nabla v|^{4/3} dx \le c.$$

Proof. We have $v - v^0 \in W_0^{1,2}(\Omega,R^N)$; thus, in virtue of the regularization process, $v - v^0$ can be approximated by $\phi \in C^\infty(\Omega,R^N)$ with a compact support in Ω (i. e., $\phi \in D(\Omega;R^N)$) such that $\|\phi\|_{L^\infty(\Omega,R^N)} \le 2K$. For these ϕ, repeating the use of Green's theorem (similarly as in the proof of Korn's inequality in [7]), we get the inequality

$$(3.11)\qquad 2\int_\Omega \frac{e_{ij}(\phi)e_{ij}(\phi)}{T}dx \geq \int_\Omega \left(\frac{|\nabla\phi|^2}{T} + \frac{(\operatorname{div}\phi)^2}{T}\right)dx - $$
$$- c\left(\int_\Omega \frac{|\nabla\phi|^2}{T}dx\right)^{\frac{1}{2}} \left(\int_\Omega \frac{|\nabla T|^2}{T^2}dx\right)^{\frac{1}{2}} \|\phi\|_{L^\infty(\Omega,R^N)}.$$

Further, the inequality

$$\frac{1}{\mu} E(\phi) \geq 2\, e_{ij}(\phi)\, e_{ij}(\phi) - (\operatorname{div}\phi)^2,$$

combined with (3.11) and Theorem 3.1 implies the estimate

$$\frac{1}{\mu}\int_\Omega \frac{E(v-v^o)}{T}dx \geq \int_\Omega \frac{|\nabla(v-v^o)|^2}{T}dx - c\left(\int_\Omega \frac{|\nabla(v-v^o)|^2}{T}dx\right)^{\frac{1}{2}}.$$

By Theorem 3.1 and the assumption of Theorem 3.9,

$$\frac{1}{\mu}\int_\Omega \frac{E(v-v^o)}{T}dx \leq c.$$

If we put $a = \left(\int_\Omega \frac{|\nabla(v-v^o)|^2}{T}dx\right)^{\frac{1}{2}}$, we see that $a^2 - \hat{c}a - \tilde{c} \leq 0$ with constants $\hat{c}$, $\tilde{c} > 0$ independent of a, μ, k. This implies the existence of a constant $a_1 > 0$ independent of μ, k such that $a \in [0,a_1]$. Now we already easily derive (3.10,a).

Assertion (3.10,b) will be obtained from (3.10,a), Theorem 3.4 and the repeated application of the Cauchy inequality:

$$\int_\Omega |\nabla v|^{4/3}dx \leq \left(\int_\Omega \frac{|\nabla v|^2}{T}dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^{2/3}T\,dx\right)^{\frac{1}{2}} \leq$$
$$\leq \left(\int_\Omega \frac{|\nabla v|^2}{T}dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^{4/3}dx\right)^{\frac{1}{4}} \left(\int_\Omega T^2dx\right)^{\frac{1}{4}}. \qquad \square$$

4. Limit for $\mu \to 0+$

On the basis of the above results we can consider a sequence $\{\mu_n\}$, $\mu_n > 0$, $\mu_n \to 0$ for $n \to \infty$ and a sequence of solutions $\{\rho_n, T_n, p_n, S_n, v^n\}$ of problems (2.10) - (2.12) with $\mu := \mu_n$, $k := \beta\mu_n$ satisfying the conditions

(4.1) $\rho_n \rightharpoonup \rho$ (weakly) in $L^2(\Omega)$,

$T_n \to T$ in $L^p(\Omega)$ $\forall p \in [1,2)$, $T_n \to T$ almost everywhere in Ω,

$v^n \rightharpoonup v$ (weakly) in $W^{1,4/3}(\Omega,R^N)$, $v^n \to v$ (strongly) in

$L^p(\Omega,R^N)$ $\forall p \in [1,\frac{4N}{3N-4})$,

$v^n \to v$ almost everywhere in Ω.

<u>4.2. Theorem.</u> $\rho_n \to \rho$ in $L^2(\Omega)$.

<u>Proof.</u> From the properties of the form [.,.] defined in (3.7) it follows that

$$(4.3)\qquad \left|2\mu_n\int_\Omega e_{ij}(v^n)e_{ij}(\phi^n)dx - \frac{2}{3}\mu_n\int_\Omega \frac{\partial v_i^n}{\partial x_i}\frac{\partial \phi_j^n}{\partial x_j}dx\right| \le$$

$$c\mu_n^{\frac{1}{2}}[\|\psi\|_{W^{1,2}(\Omega,R^N)} + \|\psi\|_{L^\infty(\Omega,R^N)}],$$

$$\phi^n = \frac{\psi}{T_n},\qquad \psi \in W_0^{1,2}(\Omega,R^N) \cap L^\infty(\Omega,R^N).$$

On the basis of the estimates from Section 3 we find out that the sequence $\{h^n\}$, where

$$(4.4)\qquad h_i^n = -\rho_n v_j^n \frac{\partial v_i^n}{\partial x_j}\frac{1}{T_n} - R\rho_n \frac{\partial T_n}{\partial x_i}\frac{1}{T_n},$$

is bounded in $L^2(\Omega)$ and hence, we can assume that $h^n \rightharpoonup h$ in $L^2(\Omega)$. Let $p > N$, $1/q = 1 - 1/p$. From the compact imbedding $W_0^{1,p}(\Omega;R^N) \subset\subset L^2(\Omega;R^N)$ and continuous imbedding $W_0^{1,p}(\Omega;R^N) \subset W_0^{1,2}(\Omega,R^N) \cap L^\infty(\Omega,R^N)$ we prove that

$$(4.5)\qquad h^n \to h \text{ in } W^{-1,q}(\Omega,R^N).$$

Now, let us use equation (2.11), where we substitute $\phi := \phi^n$ and apply the theorem on "negative norms":

$$(4.6)\qquad \|\rho\|_{L^q(\Omega)} \le c\{\sum_{i=1}^N \|\frac{\partial \rho}{\partial x_i}\|_{W^{-1,q}(\Omega)} + |\int_\Omega \rho dx|\}.$$

Then, taking into account (4.3), (4.5) and $\rho_n \to \rho$ in $L^1(\Omega)$, we find that

$$\lim_{m,n\to\infty} \|\rho_n-\rho_m\|_{L^q(\Omega)} = 0.$$

Finally, by interpolation we have $\|\rho_n-\rho_m\|_{L^2(\Omega)} \to 0$ and hence, $\rho_n \to \rho$ in $L^2(\Omega)$. ∎

Theorem 4.2 implies that we can consider the following additional assumption

$$(4.1)^* \qquad \rho_n \to \rho \text{ almost everywhere in } \Omega.$$

Now we shall prove that by the limit process $\mu \to 0+$, $k = \beta\mu \to 0+$ we get a solution of the conservation law equations for a nonviscous fluid.

4.7. Theorem. Let v, T, ρ be the limits from (4.1), $S = c_v \ln \frac{T}{\rho^{\kappa-1}}$ and let $S_n = c_v \ln \frac{T_n}{\rho_n^{\kappa-1}} \to S$ in $L^1(\partial\Omega)$. Then $v \in W^{1,4/3}(\Omega,R^N)$, $T \in W^{1,1}(\Omega)$, $\rho \in L^\infty(\Omega)$, $|v| \le K$, $\tilde{\rho}_0 \le \rho \le \tilde{\rho}_1$, $T \ge \tilde{T}_0$ and

$$(4.8) \qquad \int_\Omega \rho v_i \frac{\partial \phi}{\partial x_i}\, dx = \int_{\partial\Omega} g\phi ds \qquad \forall \phi \in W^{1,2}(\Omega),$$

$$(4.9) \quad \int_\Omega \rho v_j \frac{\partial v_i}{\partial x_j} \phi_i \, dx = R\int_\Omega \rho T \frac{\partial \phi_i}{\partial x_i}\, dx \qquad \forall \phi \in W_0^{1,2}(\Omega,R^N),$$

$$(4.10) \int_\Omega \rho v_i S \frac{\partial \psi}{\partial x_i}\, dx = \int_{\partial\Omega} Sg\psi ds \qquad \forall \psi \in W^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Proof. The limit process in the continuity equation is an easy consequence of Lebesgue's theorem. Let us prove (4.9). If we put in (2.11) $p := R\rho_n T_n$, then by the Hölder inequality, properties of v^n, ρ_n, T_n and Lebesgue's theorem we show that

$$(4.11) \quad \int_\Omega \rho_n v_j^n \frac{\partial v_i^n}{\partial x_j} \phi_i \, dx \to \int_\Omega \rho v_j \frac{\partial v_i}{\partial x_j} \phi_i \, dx \qquad \forall \phi \in D(\Omega,R^N).$$

Concerning the viscous terms, we have

$$(4.12) \qquad \left|\frac{2}{3}\mu_n\int_\Omega \frac{\partial v_i^n}{\partial x_i}\frac{\partial \phi_i}{\partial x_i}\,dx - 2\mu_n\int_\Omega e_{ij}(v^n)e_{ij}(\phi)\,dx\right| \leq$$

$$\leq c\mu_n^{\frac{1}{2}}\|\phi\|^2_{W^{1,2}(\Omega,R^N)}.$$

Hence, by (4.11) - (4.12)

$$\int_\Omega \rho v_j \frac{\partial v_i}{\partial x_j}\phi_i\,dx = R\int_\Omega \rho T\frac{\partial \phi_i}{\partial x_i}dx \qquad \forall\phi\in D(\Omega,R^N).$$

Now, due to $T\in L^2(\Omega)$, $\rho\in L^\infty(\Omega)$ and the density of $D(\Omega,R^N)$ in $W_0^{1,2}(\Omega,R^N)$ we get (4.9).

Finally, we prove (4.10). It is evident that $\ln T_n \to \ln T$ and $\ln \rho_n \to \ln \rho$ in $L^2(\Omega)$ and thus, $S_n \to S$ in $L^2(\Omega)$. If we use the assumption that $S_n \to S$ in $L^1(\partial\Omega)$ and put $\phi := \frac{\psi}{T_n}$ in (2.12), we can pass to the limit for $n \to \infty$. ⌑

5. Potential isentropic flow

Let $s_0 < \frac{2a_0^2}{\kappa-1}$. We define the set

$$(5.1) \qquad N_{s_0} = \{\nabla u;\ u\in W^{1,\infty}(\Omega),\ |\nabla u|^2 \leq s_0,\ \int_\Omega u\,dx = 0\}$$

and denote by P the projector of the space $L^2(\Omega,R^N)$ onto N_{s_0}.

5.2. Definition. Let $\{v^n\}$ be a sequence of velocities from (4.1). We say that v^n converges to a potential flow, if

$$(5.2)^* \qquad \|v^n - Pv^n\|_{L^2(\Omega,R^N)} \to 0, \quad \text{if } n\to\infty.$$

Let us assume that ρ_n, v^n satisfy the continuity equation in $\Omega\cup\Omega'$, where Ω' is a (sufficiently large) domain lying in the upwind direction to Ω and all fluid particles travel from Ω' into Ω through a common part $B\subset\partial\Omega'\cap\partial\Omega$, $B\neq\emptyset$. I. e., B is the outlet of Ω' and the inlet of Ω. Let $\mathcal{O} = \Omega\cup\Omega'\cup B$, $\nabla u_n = Pv^n$, (5.3) $$|v^n|^2 \leq s_0 \quad \forall n.$$

We consider a nondegeneracy of the velocity fields. I. e., either

(5.4,a) $$v_i^n \cdot x_i \geq \alpha > 0 \quad \forall n$$

or

(5.4,b) $$v_i^n \frac{\partial u_n}{\partial x_i} \geq \alpha > 0 \quad \forall n.$$

(α is a constant independent of n.) Further, let $v^n \in C^2(\overline{\mathcal{O}}, R^N)$, $\rho_n \in C^1(\overline{\mathcal{O}})$, $T_n \in C^2(\overline{\Omega})$ and let the velocity field "conserves the entropy information in the limit":

(5.5) $$\mu_n \| \nabla v^n \|_{C(\overline{\mathcal{O}}, R^{N^2})} \to 0, \text{ if } n \to \infty.$$

If $x \in \overline{\Omega}$, then there exists exactly one trajectory (i. e. characteristic) $x^n(t) = x^n(x;t)$ passing through x:

(5.6) $$\frac{dx^n}{dt} = v^n(x^n), \quad x^n(0) = x.$$

Let each such trajectory enter the domain Ω at a point $\tilde{x}^n(x) \in B$ at a time $\tilde{t}^n(x) < 0$.

On the basis of (5.4,a) or (5.4,b) it is easy to prove the existence of $t_0 \in (-\infty, 0)$ such that $\tilde{t}^n(x) \geq t_0$ for all $x \in \overline{\Omega}$ and all n. Hence, if $x \in \overline{\Omega}$, $t \leq t_0$, then $x^n(x;t) \notin \Omega$. For $t \in (t_0, 0]$ we denote $\overline{\Omega}_t^n = \{y = x^n(x;t);\ x \in \overline{\Omega}\}$. Now we demand that Ω' is so large that $\overline{\Omega}_t^n \subset \overline{\mathcal{O}} \quad \forall t \in (t_0, 0]$.

If we put

(5.7) $$F_n = k_n \frac{\Delta T_n}{\rho_n T_n} + \frac{E(v^n)}{\rho_n T_n} \quad \text{in } \overline{\Omega},$$
$$F_n = 0 \quad \text{in } \overline{\Omega}' - B,$$

then (2.4,a) can be written as

(5.8) $$\frac{dS_n}{dt} = F_n.$$

(d/dt is the total time derivative, i. e. $d/dt = \partial/\partial t + v_i\, \partial/\partial x_i$.)

Integrating (5.8) we get

$$(5.9) \qquad S_n(x) - S_n(\bar{x}^n(x)) = \int_{t_0}^{0} F_n(x^n(x;t))dt.$$

The main result of this section is the following

5.10. Theorem. Let

$$S_n \rightrightarrows c_v \ln \frac{T_0}{\rho_0^{\kappa-1}} \text{ (uniformly) on } B.$$

Then

$$S_n \to c_v \ln \frac{T_0}{\rho_0^{\kappa-1}} \text{ in } L^2(\Omega).$$

Sketch of the proof by the method of characteristics: We already know that the sequence $\{S_n\}$ is convergent in $L^2(\Omega)$. Let us prove that its limit is $S_0 = c_v \ln(T_0/\rho_0^{\kappa-1})$. Let us consider an arbitrary $\theta \in D(\Omega)$ and extend it onto R^N by zero. Then

$$\int_\Omega [S_n(x) - S_n(\bar{x}^n(x))]\rho_n(x)\theta(x)dx =$$
$$= \int_{t_0}^{0} dt \int_\Omega F_n(x^n(x,t))\rho_n(x)\theta(x)dx.$$

Let us study e. g. the term

$$(5.11) \qquad Q_n = k_n \int_\Omega \frac{\Delta T_n(x^n(x;t))}{\rho_n(x^n(x;t)T_n(x^n(x;t))}\rho_n(x)\theta(x)dx =$$
$$= k_n \int_{\Omega\cap\Omega_t^n} \frac{\Delta T_n(y)}{\rho_n(y)T_n(y)}\rho_n(y^n(y;t))\theta(y^n(y;t))\left|\frac{Dy^n}{Dy}(y;t)\right|dy,$$

where $y^n(y;t) = y^n(t)$ and

$$(5.12) \qquad \frac{dy^n(t)}{dt} = -v^n(y^n(t)), \quad y^n(0) = y.$$

From the mass conservation law it follows:

$$(5.13) \qquad \rho_n(x)\left|\frac{Dy^n(y;t)}{Dy}\right| = \rho_n(y), \quad y = x^n(x;t), \quad x \in \Omega.$$

Hence,

$$(5.14) \qquad Q_n = k_n \int_{\Omega \cap \Omega_t^n} \frac{\Delta T_n(y)}{T_n(y)} \, \theta(y^n(y;t))dy.$$

Since θ has a compact support in Ω, we can apply Green's theorem to (5.14). Then using condition (5.5), the assumption of Theorem 5.10 and estimates from Section 3, we derive the relation

$$(5.15) \qquad \lim_{n\to\infty} \int_{\Omega} \rho_n(x) [S_n(x) - S_o] \theta(x) dx = 0.$$

Finally, from $\rho_n \to \rho$ in $L^2(\Omega)$ and the density of the set $\{\rho\theta;\ \theta \in D(\Omega)\}$ in $L^2(\Omega)$ we prove the assertion of our theorem. ¤

Similarly we get

5.16. Theorem. If

$$T_n - T_o(1 - \frac{\kappa-1}{2a_o^2}|v^n|^2) \rightleftarrows 0 \text{ on } B,$$

then

$$T_n - T_o(1 - \frac{\kappa-1}{2a_o^2}|v^n|^2) \to 0 \text{ in } L^1(\Omega).$$

5.17. Corollary. Under the assumptions of Theorems 5.10 and 5.16 we have

$$\rho = \rho_o(1 - \frac{\kappa-1}{2a_o^2} |v|^2)^{\frac{1}{\kappa-1}}.$$

Moreover, if (5.2)* holds, then $v = \nabla u$, $u \in W^{2,4/3}(\Omega)$ and u is a weak solution of the transonic potential flow problem

$$\int_{\Omega} \rho(|\nabla u|^2) \nabla u \cdot \nabla\phi dx = \int_{\partial\Omega} g\phi ds \qquad \forall \phi \in W^{1,2}(\Omega). \qquad ¤$$

References

[1] M. O. Bristeau, R. Glowinski, J. Periaux, P. Perrier, O. Pironneau: On the numerical solution of nonlinear problems in fluid dynamics by least squares and finite element methods.

(I) Least square formulation and conjugate gradient solution of the continuous problem. Comp. Meth. Appl. Mech. Eng. 17/18 (1979), 619-657.

[2] R. Di Perna: Compensated compactness and general systems of conservation laws. Transactions of the American Mathematical Society, Vol. 292, No. 2, Dec. 1985, 383-419.

[3] M. Feistauer, J. Mandel, J. Nečas: Entropy regularization of the transonic potential flow problem. CMUC, 25(3) 1984, 431-443.

[4] M. Feistauer, J. Nečas: On the solvability of transonic potential flow problems. Z. für Analysis und ihre Anwendungen. Bd 4(4)1985, 305-329.

[5] M. Feistauer, J. Nečas: Viscosity method in a transonic flow (to appear).

[6] C. Moravetz: On a weak solution for a transonic flow problem. Comm. Pure Appl. Math. 38(1985), 797-818.

[7] J. Nečas, I. Hlaváček: Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction. Elsevier Scientific Publishing Comp., Amsterdam - Oxford - New York, 1981.

[8] G. Poirier: Traitement numérique en éléments finis de la condition d'entropie des équations transsoniques (Thesis). L'Université Pierre et Marie Curie, Paris VI, 1981.

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(Oblatum 16. 9. 1986)