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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
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β-STRUCTURES

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Abstract: This article is dedicated to the investigation of a structure of the form $\langle A, B, \mathcal{R}, \mathcal{S} \rangle$, where $\mathcal{R} \subseteq A \times A$ is reflexive and symmetric relation, $\mathcal{S} \subseteq A \times B$ and some further presumptions are satisfied. Problems of this structure can be seen as a generalization of the concept of a study of shapes, founded on an analysis of the structure $\langle V, \mathcal{R} \rangle$, where $\mathcal{R}$ is a relation of indiscernibility on V.

Key words: β-structure, realization w.r.t. a binary relation, shut class, overt class, reduction.

Classification: 03K10, 03K99

Introduction. This article is devoted to a study of a constitution of an object as a collection, which grows up on a frame of a relevant intention, directed into a universe, continuance of which is established by elementary domains of conectedness, specified by a hitherto merely looming up attention.

Formally, let A be the universe in question, B a collection of looming attentions and let $\mathcal{S} \subseteq A \times B$ be a relation such that $\mathcal{S}''\{b\}$ is the elementary domain of a connectedness, specified by b. A forming intention can be considered as a subclass of B. The domain of the form $\mathcal{S}''\{b\}$ are not sharp. Thus, to satisfy a claim of a certainty of the actualization of an object in question, we cannot define such an object, formed by an intention $\mathcal{Y} \subseteq B$, as $\mathcal{S}''\mathcal{Y}$. Calling this object $\mathcal{S}$-realization of $\mathcal{Y}$, we define it in the § 2.

Note that the case A=B can be interpreted by such a way that a point $a \in A$ is identified with an attention, specifying a domain of connectedness of a. This comprehensive and important situation can be called simple. It is extensively investigated under presumption that $\mathcal{S}$ is an indiscernibility equivalence. (See [V].)

Later, we shall work with a more complicated situation. We do not deal with a structure $\langle A,B,\mathcal{S}\rangle$ only, but we suppose that A is furnished with a relation of connectedness $\mathcal{R} \subseteq A \times A$ (and subject to the condition of the reflexivity and the symmetry). This situation enables us to study a difference between some actualization of a given object in a context of the simple situation $\langle A,\mathcal{R}\rangle$ and of the actualization, realized in $\langle A,B,\mathcal{S}\rangle$ (by using a translation of the intentions of A into those of B).

Note that only the case where $\mathcal{R}$ and $\mathcal{S}$ are π-classes, is discussed in depth. Moreover, the subject is much more extensive than is presented below.

§ 1. Preliminaries. Our investigation is founded on the existence of two kinds of classes. The classes of the first one are formally short classes which can be seen as elements of a system $\mathfrak{M}$ of classes, satisfying the following conditions: (1) $V \subseteq \mathfrak{M}$, (2) $\{x;\varphi(x)\} \in \mathfrak{M}$ for every normal formula φ of the language $FL_{\mathfrak{M}}$, (3) $(\forall X \subseteq N)$ $(X \neq 0 \longrightarrow$ there exists the first element of X), (4) $(\forall\{X_n\} \subseteq \mathfrak{M})(\exists X \in \mathfrak{M})(FN \subseteq dom(X)\ \&$ $\&\ (\forall n)(X_n = X''\{n\}))$. Note that $\mathfrak{M} \models GB_{fin}$, there is no proper semiset in $\mathfrak{M}$ and every class from $\mathfrak{M}$ is fully revealed. (See [M], [S-V].)

Convention, Throughout this paper, let capital block-letters be ranging over elements of $\mathfrak{M}$. The usual notation of sets, natural numbers and finite natural numbers is accepted.

The classes of second kind are the remaining classes of the universe of classes of the AST.

Convention. The script capital letters denote classes and the letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{X}', \mathcal{X}_1, \dots$ ranging over (coded) systems of classes. We denote by $\mathcal{T}[\mathcal{X}]$ the class $\{\mathcal{T}''\mathcal{X}, \mathcal{X} \in \mathcal{X}\}$.

By a class of the type π we mean a class $\mathcal{X}$, which is an intersection of a sequence $\{X_n\} \in \mathfrak{M}$. A string of the type π over $\mathcal{X}$ is a class $\{X_\alpha\}_{\alpha \leq \eta} \in \mathfrak{M}$ such that $X_{\alpha+1} \subseteq X_\alpha$ holds for each $\alpha < \eta$ and $\cap X_n = \mathcal{X}$. A formula $\Psi(\mathcal{X})$ has $\nearrow$-property iff $(\forall \mathcal{X},\mathcal{Y})(\Psi(\mathcal{X})\ \&\ \mathcal{Y} \supseteq \mathcal{X} \rightarrow \Psi(\mathcal{Y}))$. It is easy to prove the

Proposition. Let $\mathcal{X}$ be of the type π, and assume that Ψ has $\nearrow$-property. Then

$(\forall X \supseteq \mathcal{X})\,\Psi(X) \longleftrightarrow (\exists$ string $\{X_\alpha\}$ of the type π over $\mathcal{X})$
$(\forall n)\,\Psi(X_n) \longleftrightarrow (\forall$ string $\{X_\alpha\}$ of the type π over $\mathcal{X})$
$(\forall n)\,\Psi(X_n)$.

By a <u>mapping</u> Θ <u>of classes</u> from $\mathcal{A}$ to $\mathcal{B}$ we mean the existence of a formula $\Phi(\mathcal{X},\mathcal{Y})$ such that $(\forall \mathcal{X} \subseteq \mathcal{A})(\exists!\, \mathcal{Y} \subseteq \mathcal{B})$. $\Theta(\mathcal{X})$ is, for every $\mathcal{X} \subseteq \mathcal{A}$, a subclass of $\mathcal{B}$ such that $\Phi(\mathcal{X},\Theta(\mathcal{X}))$ holds. We shall write $\Theta \mathbin{\S} \mathcal{A} \longrightarrow \mathcal{B}$.

<u>Convention</u>. Throughout this paper, let A, B be two fixed classes from $\mathfrak{M}$ and let $\mathcal{T} \subseteq A \times B$ be a relation such that $\mathrm{rng}(\mathcal{T})=A$, $\mathrm{dom}(\mathcal{T})=B$.

Let a (b resp.) be ranging over elements of A (B resp.) and $\mathcal{X}$ ($\mathcal{Y}$ resp.) let be designated subclasses of A (B resp.). Let us use, similarly, $\mathfrak{X}$ ($\mathfrak{Y}$ resp.) as a designation of a system of subclasses of A (B resp.).

Let us, finally, $\{T_\alpha\}_\eta$ denote a string of the type π of relations $T_\alpha \subseteq A \times B$, $\alpha \leq \eta$.

§ 2. $\mathcal{T}$-<u>realization</u>

<u>Definition</u>. (1) a is said to be $\mathcal{T}$-<u>separated from</u> $\mathcal{Y}$, $\mathrm{Sep}(\mathcal{T},a,\mathcal{Y})$, iff $(\exists U \supseteq \mathcal{T}^{-1}{}''\{a\})(U \cap \mathcal{Y} = 0)$ holds,

(2) We define $\widetilde{\mathcal{Y}}^{\mathcal{T}} = \{a \in A; \neg \mathrm{Sep}(\mathcal{T},a,\mathcal{Y})\}$ and call this class $\mathcal{T}$-<u>realization of</u> $\mathcal{Y}$.

We have, evidently, $\sim^{\mathcal{T}^{-1}} \mathbin{\S} A \longrightarrow B$ and $\sim^{\mathcal{T}} \mathbin{\S} B \longrightarrow A$.

<u>Proposition</u>. (1) $\neg \mathrm{Sep}(\mathcal{T},a,\mathcal{Y})$ has $\nearrow$-property in $\mathcal{T}$.
(2) Let $\{T_\alpha\}$ be over $\mathcal{T}$. Then $\widetilde{\mathcal{Y}}^{\mathcal{T}} = \bigcap_n T_n{}''\mathcal{Y}$.
(3) $\mathcal{T}''\mathcal{Y} \subseteq \widetilde{\mathcal{Y}}^{\mathcal{T}}$.
The proof is easy and we omit it.

<u>Definition</u>. A clas $\mathcal{X}$ is said to be $\mathcal{T}$-<u>overt</u> iff $(\forall b \in \mathcal{T}^{-1}{}''\mathcal{X})(\exists U \supseteq \mathcal{T}''\{b\})(U \subseteq \mathcal{X})$.

<u>Proposition</u>. (1) $\mathcal{X}$ is $\mathcal{T}$-overt $\longrightarrow (\mathcal{T} \circ \mathcal{T}^{-1})''\mathcal{X} \subseteq \mathcal{X}$.
(2) Let $\{T_\alpha\}$ be over $\mathcal{T}$. Then $\mathcal{X}$ is $\mathcal{T}$-overt $\longleftrightarrow$
$\longleftrightarrow (\forall b \in \mathcal{T}^{-1}{}''\mathcal{X})(\exists n)(T_n{}''\{b\} \subseteq \mathcal{X})$.
The proof is routine.

Definition. (1) $\mathcal{X}$ is said to be $\mathcal{T},\Theta$-shut iff we have $(\forall b \in B-\Theta(\mathcal{X}))$ Sep$(\mathcal{T}^{-1},b,\mathcal{X})$.

(2) We call $\mathcal{X}$ fully $\mathcal{T}$-shut iff $\mathcal{X}$ is $\mathcal{T}$, $\Theta_{\mathcal{T}}$-shut, where we have $\Theta_{\mathcal{T}}: A \to B$ defined by $\Theta_{\mathcal{T}}(\mathcal{X}) = B - \mathcal{T}^{-1}{}''(A-\mathcal{X})$.

Proposition. $\mathcal{X}$ is fully $\mathcal{T}$-shut iff $A-\mathcal{X}$ is $\mathcal{T}$-overt.

The proof is easy,

Proposition. Let $\mathcal{X}$ be $\mathcal{T}$, Θ-shut. Then

(1) $\mathcal{T}^{-1}{}''\mathcal{X} \subseteq \tilde{\mathcal{X}}^{\mathcal{T}^{-1}} \subseteq \Theta(\mathcal{X})$

(2) $\Theta(\mathcal{X}) \subseteq \mathcal{T}^{-1}{}''\mathcal{X} \longrightarrow \Theta(\mathcal{X}) = \mathcal{T}^{-1}{}''\mathcal{X} = \tilde{\mathcal{X}}^{\mathcal{T}^{-1}}$

Proof: (1) Assume $b \in B-\Theta(\mathcal{X})$. Then Sep$(\mathcal{T}^{-1},b,\mathcal{X})$ and, consequently, $b \notin \tilde{\mathcal{X}}^{\mathcal{T}^{-1}}$. Thus, the second inclusion is proved. The first is a consequence of a previous proposition.

(2) We have $\Theta(\mathcal{X}) \subseteq \mathcal{T}^{-1}{}''\mathcal{X} \subseteq \tilde{\mathcal{X}}^{\mathcal{T}^{-1}} \subseteq \Theta(\mathcal{X})$.

Let us introduce relations between the notions presented in the case that A=B and $\mathcal{T}$ is an equivalence on A.

Proposition. Let $\mathcal{T}$ be an equivalence on A, A=B. Then

(1) $(\mathcal{T} \circ \mathcal{T}^{-1})''\mathcal{X} \subseteq \mathcal{X} \Longleftrightarrow \mathcal{T}''\mathcal{X} = \mathcal{X}$

(2) ($\mathcal{X}$ is $\mathcal{T}$, Id-shut $\vee$ $\mathcal{X}$ is fully $\mathcal{T}$-shut $\vee$ $\mathcal{X}$ is $\mathcal{T}$-overt)$\rightarrow$ $\longrightarrow \mathcal{T}''\mathcal{X} = \mathcal{X}$.

(3) $\mathcal{X}$ is $\mathcal{T}$, Id-shut $\longleftrightarrow$ $\mathcal{X}$ is fully $\mathcal{T}$-shut $\longleftrightarrow$ $A-\mathcal{X}$ is $\mathcal{T}$-overt.

Proof. (1) is trivial. (2) is an easy consequence from the following assertions, concerning the general case of the relation $\mathcal{T}$:

(α) $\mathcal{X}$ is $\mathcal{T}$-overt $\longrightarrow (\mathcal{T} \circ \mathcal{T}^{-1})''\mathcal{X} \subseteq \mathcal{X}$, ($\beta$) $\mathcal{X}$ is $\mathcal{T}$, Id-shut $\longrightarrow \mathcal{T}^{-1}{}''\mathcal{X} \subseteq \mathcal{X}(=\mathrm{Id}(\mathcal{X}))$, ($\gamma$) $\mathcal{X}$ is fully $\mathcal{T}$-shut $\longrightarrow$ $\longrightarrow \mathcal{T}^{-1}{}''\mathcal{X} \subseteq B-\mathcal{T}^{-1}{}''(A-\mathcal{X}) (= \Theta_{\mathcal{T}}(\mathcal{X}))$, and from ($\delta$) $\mathcal{T}$ is an equivalence on A, A=B $\longrightarrow (\mathcal{T}''\mathcal{X} \subseteq A-\mathcal{T}''(A-\mathcal{X}) \longrightarrow \mathcal{T}''\mathcal{X} = \mathcal{X})$.

Finally, (3) follows immediately from (1),(2) and from the definitions of the notions in question.

Definition. (1) We say that $\mathcal{X}$ $\mathcal{T}$-covers B iff $\bigcup \mathcal{T}^{-1}[\mathcal{X}] = B$ holds.

(2) $\mathcal{X}$ is said to be $\mathcal{T}$-centred iff every finite subsystem

$\mathfrak{X}' \subseteq \mathfrak{X}$ satisfies $\bigcap \mathcal{T}^{-1}[\mathfrak{X}'] \neq 0$.

Lemma. Assume $(\mathcal{T} \circ \mathcal{T}^{-1})'' \mathcal{X} \subseteq \mathcal{X}$. Then $\mathcal{T}^{-1''}\mathcal{X} \cap \mathcal{T}^{-1''}(A-\mathcal{X}) = 0$ and $\mathcal{T}^{-1''}(A-\mathcal{X}) = B - \mathcal{T}^{-1''}\mathcal{X}$ hold.

We omit the routine proof.

Proposision. Let $\mathfrak{X}$ be a system such that (i) $\mathcal{X} \in \mathfrak{X} \longrightarrow (\mathcal{T} \circ \mathcal{T}^{-1})'' \mathcal{X} \subseteq \mathcal{X}$, (ii) $\mathcal{X} \in \mathfrak{X} \longrightarrow \mathcal{X}$ is fully $\mathcal{T}$-shut. Then $\bigcap \mathcal{T}^{-1}[\mathfrak{X}] = 0 \longleftrightarrow \{A-\mathcal{X}; \mathcal{X} \in \mathfrak{X}\}$ $\mathcal{T}$-covers B.

The following sequence of equivalences is a proof of our proposition:
$\bigcap \mathcal{T}^{-1}[\mathfrak{X}] = 0 \longleftrightarrow B - \bigcap \mathcal{T}^{-1}[\mathfrak{X}] = B \longleftrightarrow \bigcup\{B - \mathcal{T}^{-1''}\mathcal{X}; \mathcal{X} \in \mathfrak{X}\} = B \longleftrightarrow \bigcup\{\mathcal{T}^{-1''}(A-\mathcal{X}); \mathcal{X} \in \mathfrak{X}\} = B \longleftrightarrow \bigcup \mathcal{T}^{-1}[\{A-\mathcal{X}; \mathcal{X} \in \mathfrak{X}\}] = B$.

§ 3. β-structure

Definition. β-structure in $A \times B$ consists of two relations $\mathcal{R}$ and $\mathcal{S}$ and a class $\mathcal{W} \subseteq B$ such that

(1) $\mathcal{R} \subseteq A \times A$ is reflexive and symmetric,

(2) $\mathcal{S} \subseteq A \times B$ satisfies $\mathrm{dom}(\mathcal{S}) = B$ and $\mathrm{rng}(\mathcal{S}) = A$,

(3) $(\forall y \in \mathcal{W})(\mathcal{R}'' \mathcal{S}'' \{y\} \subseteq \mathcal{S}'' \{y\})$,

(4) $\{x; x \in A \,\&\, \{x\} \neq \mathcal{R}''\{x\}\} \subseteq \mathcal{S}'' \mathcal{W}$.

We denote briefly such a β-structure $\mathcal{L} = \mathcal{L}(A, B, \mathcal{R}, \mathcal{S}, \mathcal{W})$; throughout this paper let $A, B, \mathcal{R}, \mathcal{S}, \mathcal{W}$ be fixed and satisfying (1)-(4). $\mathcal{L}$ is of the type π iff $\mathcal{R}, \mathcal{S}$ and $\mathcal{W}$ are.

Definition. β-string in $A \times B$ of the type π consists of three strings $\{R_\alpha\}_\eta$, $\{S_\alpha\}_\eta$, $\{W_\alpha\}_\eta$ of the type π such that the following holds for every $\alpha < \eta$: (0) $W_\alpha \subseteq B$,

(1) $R_\alpha \subseteq A \times A$ is reflexive and symmetric,

(2) $S_\alpha \subseteq A \times B$ satisfies $\mathrm{dom}(S_\alpha) = B$, $\mathrm{rng}(S_\alpha) = A$,

(3) $R_{\alpha+1} \circ (S_{\alpha+1} \upharpoonright W_{\alpha+1}) \subseteq S_\alpha \upharpoonright W_\alpha$,

(4) $\{x \in A; \{x\} \neq R_{\alpha+1}''\{x\}\} \subseteq S_\alpha'' W_\alpha$.

Throughout this paper, let $\{R_\alpha\}_\eta$, $\{S_\alpha\}_\eta$, $\{W_\alpha\}_\eta$ be a β-string in $A \times B$ of the type π. We say that this string is over $\mathcal{L}$, iff $\bigcap R_n = \mathcal{R}$, $\bigcap S_n = \mathcal{S}$ and $\bigcap W_n = \mathcal{W}$.

Proposition. Let $\mathcal{L}$ be of the type π. Then there exists a β-string in $A \times B$ of the type π which is over $\mathcal{L}$.

Proof. Let $\{\hat{R}_\alpha\}_{\vartheta}$, $\{\hat{S}_\alpha\}_{\vartheta}$, $\{\hat{W}_\alpha\}_{\vartheta}$ be strings of the type π over $\mathcal{R}$, $\mathcal{S}$, $\mathcal{W}$ resp., such that $R_\alpha \subseteq A\times A$ is reflexive and symmetric, $S_\alpha \subseteq A\times B$ satisfies $dom(S_\alpha)=B$ and $rng(S_\alpha)=A$, $W_\alpha \subseteq B$ holds for every $\alpha \in \vartheta$. Put $\tilde{R}_0=\hat{R}_0$, $\tilde{S}_0=\hat{S}_0$ and $\tilde{W}_0=\hat{W}_0$. We have $(\forall\alpha \notin FN)(\hat{R}_\alpha \circ (\hat{S}_\alpha \wedge \hat{W}_\alpha) \subseteq \tilde{S}_0 \wedge \tilde{W}_0 \;\&\; \{x\in A;\{x\}\neq \hat{R}_\alpha''\{x\}\subseteq \tilde{S}_0''\tilde{W}_0\}$ and, consequently, there exists an n such that $\hat{R}_n \circ (\hat{S}_n \wedge \hat{W}_n) \subseteq \subseteq \tilde{S}_0 \wedge \tilde{W}_0 \;\&\; \{x;\hat{R}_n''\{x\}\neq\{x\}\} \subseteq \tilde{S}_0''\tilde{W}_0$. Put $\tilde{R}_1=\hat{R}_n$, $\tilde{S}_1=\hat{S}_n$, $\tilde{W}_1=\hat{W}_n$. By this way we can construct three sequences $\{\tilde{R}_m\}_{FN}$, $\{\tilde{S}_m\}_{FN}$, $\{\tilde{W}_m\}_{FN}$ cofinal in $\{\hat{R}_\alpha\}_{FN}$, $\{\hat{S}_\alpha\}_{FN}$, $\{\hat{W}_\alpha\}_{FN}$ resp. and satisfying the conditions: $\tilde{R}_{m+1} \circ (\tilde{S}_{m+1} \wedge \tilde{W}_{m+1}) \subseteq \tilde{S}_m \wedge \tilde{W}_m$, $\{x\in A;\{x\}\neq \tilde{R}_{m+1}''\{x\}\} \subseteq \subseteq \tilde{S}_m''\tilde{W}_m$.

Some prolongations $\{\tilde{R}_\alpha\}_{\tilde{\vartheta}}$, $\{\tilde{S}_\alpha\}_{\tilde{\vartheta}}$, $\{\tilde{W}_\alpha\}_{\tilde{\vartheta}}$ of $\{\tilde{R}_m\}_{FN}$, $\{\tilde{S}_m\}_{FN}$, $\{\tilde{W}_m\}_{FN}$ resp., have the required properties.

Further we accept that at least one of the relations $\mathcal{R}$, $\mathcal{S}$ is combinatorically simple in the sense of the following definition:

<u>Definition</u>. $\mathcal{T}$ is said to be <u>uniformly reducing on</u> $\mathfrak{X}$ iff

$(\forall T \supseteq \mathcal{T})(\exists v \in P_f(B))(\mathfrak{X} \subseteq T''v)$.

Note that this condition has $\nearrow$-property in $\mathcal{T}$.

We mean, in the previous definition, the uniformity of the approximation $T \supseteq \mathcal{T}$ of $\mathcal{T}$ which is, roughly speaking, globally distinct, compared with this one $\mathcal{K} \supseteq \mathcal{T}$, which can be generally only locally distinct.

We accept the

<u>Definition</u>. $\mathcal{K} \supseteq \mathcal{T}$ is said to be <u>locally compatible</u> with $\mathcal{T}$ iff we have

$(\forall y \in B)(\exists U)(\mathcal{T}''\{y\} \subseteq U \subseteq \mathcal{K}''\{y\})$.

<u>Definition</u>. $\mathcal{T}$ is <u>reducing on</u> A iff $(\forall \mathcal{K} \supseteq \mathcal{T})$ ($\mathcal{K}$ is locally compatible with $\mathcal{T} \longrightarrow (\exists v \in P_f(B)(A \subseteq \mathcal{K}''v))$.

<u>Proposition</u>. Let $\mathcal{T}$ be reducing on A.

(1) Let $\mathfrak{X}$ be a $\mathcal{T}$-covering of B such that every $\mathcal{X} \in \mathfrak{X}$ is $\mathcal{T}$-overt. Then there exists a finite $\mathfrak{X}' \subseteq \mathfrak{X}$ and $\mathfrak{X}'$ $\mathcal{T}$-covers B.

(2) Let $\mathfrak{X}$ be $\mathcal{T}$-centred such that $\mathcal{X} \in \mathfrak{X}$ is fully $\mathcal{T}$-shut and we have $(\mathcal{T} \circ \mathcal{T}^{-1})''\,\mathcal{X} \subseteq \mathcal{X}$. Then $\bigcap \mathcal{T}^{-1}[\mathfrak{X}] \neq 0$.

Proof. (1) Let, for every $b \in B$, $\mathcal{X}_b \in \mathfrak{X}$ be such that

$b \in \mathcal{T}^{-1}{}'' \mathcal{X}_b$ Then $\mathcal{K} = \bigcup\{\mathcal{X}_b \times \{b\}; b \in B\}$ is locally compatible with $\mathcal{T}$. Let $v \in P_f(B)$ be such that $A = \mathcal{K}''v$. Then $\mathfrak{X}' = \{\mathcal{X}_b : b \in v\}$ has the required properties. (We use the presumption that $\mathcal{T}^{-1}{}''A = B$.) (2) Let, for $\mathfrak{X}' \subseteq \mathfrak{X}$, $\widetilde{\mathfrak{X}}'$ be the system $\{A - \mathcal{X}, \mathcal{X} \in \mathfrak{X}'\}$. Then $\widetilde{\mathfrak{X}}'$ is a system of $\mathcal{T}$-overt classes. We have, moreover, $\bigcap \mathcal{T}^{-1}[\mathfrak{X}'] = 0 \longleftrightarrow \widetilde{\mathfrak{X}}'$ $\mathcal{T}$-covers B. The relation $\bigcap \mathcal{T}^{-1}[\mathfrak{X}] = 0$ implies that $\widetilde{\mathfrak{X}}$ $\mathcal{T}$-covers B. We deduce from this and by using (1) that there exists a finite $\mathfrak{X}' \subseteq \mathfrak{X}$ with the property $\bigcap \mathcal{T}^{-1}[\mathfrak{X}'] = 0$, which is a contradiction.

<u>Theorem</u>. Let us assume that $\mathcal{L}(A, B, \mathcal{R}, \mathcal{S}, \mathcal{W})$ is of the type π and $\mathcal{R}$ is uniformly reducing on A. Then $\mathcal{S}$ is reducing on A.

Proof. Assume that $\mathcal{K}$ is locally compatible with $\mathcal{S}$. We use further the following abbreviation. We define, for a given relation $\mathcal{Z}$, $\partial\mathcal{Z} = \{x \in \mathrm{dom}(\mathcal{Z}); \{x\} \neq \mathcal{Z}''\{x\}\}$. Let, for every n, $v_n \in P_f(A)$ be such that $A = R_n'' v_n$. We have $A - \bigcup u_n \subseteq \partial\mathcal{R}$. Choose, for every $a \in A$, an element $\mathcal{F}(a) \in B$ such that $a \in \mathcal{S}(\mathcal{F}(a))$ and $\mathcal{F}''\partial\mathcal{R} \subseteq \mathcal{W}$, and let $m(a) \in FN$ be such that

$$(1) \qquad S''_{m(a)}\{\mathcal{F}(a)\} \subseteq \mathcal{K}\{\mathcal{F}(a)\}.$$

We have $A = \mathcal{D} \cup \partial\mathcal{R}$, where $\mathcal{D} = \bigcup u_n - \partial\mathcal{R}$ is an at most countable class. We define

$$\mathcal{Z}_1 = \{S^*_{m(a)}\{\mathcal{F}(a)\}; a \in \mathcal{D}\}.$$

It is easy that $\bigcup \mathcal{Z}_1 \supseteq \mathcal{D}$. Let, for every $a \in \partial\mathcal{R}$, $a^* \in u_{m(a)+2}$ be such that $a^* \in R_{m(a)+2}\{a\}$. Then $\mathcal{F}(a) \in \mathcal{W} \subseteq W_{m(a)+1}$ and, consequently, $a^* \in S_{m(a)+1}\{\mathcal{F}(a)\}$ holds. We deduce quite similarly that

$$(2) \qquad R''_{m(a)+1}\{a^*\} \subseteq S''_{m(a)}\{\mathcal{F}(a)\}.$$

We define $\mathcal{Z}_2 = \{R_{m(a)+1}\{a^*\}; a \in \partial\mathcal{R}\}$. We can see by using the symmetry of R_i that $\bigcup\mathcal{Z}_2 \supseteq \partial\mathcal{R}$. Consequently, $\mathcal{Z}_1 \cup \mathcal{Z}_2 \subseteq \mathfrak{M}$ is at most countable and $\bigcup(\mathcal{Z}_1 \cup \mathcal{Z}_2) = A$. Thus, there exists $v_1 \in P_f(B)$ and $v_2 \in P_f(\partial\mathcal{R})$ such that $\bigcup\{S''_{m(a)}\{\mathcal{F}(a)\}; a \in v_1\} \cup \bigcup\{R''_{m(a)+1}\{a^*\}; a \in v_2\} = A$. We deduce from this and by using (1),(2) that $\bigcup\{\mathcal{K}''\{\mathcal{F}(a)\}; a \in v_1 \cup v_2\} = A$.

Proposition. Let $\mathcal{Y} \subseteq \mathcal{W}$ and let $\mathcal{L}$ be of the type π.
(1) The class $\widetilde{\mathcal{Y}}^{\mathcal{S}}$ is $\mathcal{R}$-shut.

(2) Assume $\mathcal{X} \subseteq \tilde{\mathcal{Y}}^{\mathcal{S}}$. Then $\tilde{\mathcal{X}}^{\mathcal{R}} \subseteq \tilde{\mathcal{Y}}^{\mathcal{S}}$ holds, too.

Proof. (1) Choose $a \notin \tilde{\mathcal{Y}}^{\mathcal{S}}$. There exists n such that $a \notin S''_n \mathcal{Y}$. The proof will be finished if the relation $R''_{n+1}\{a\} \cap \tilde{\mathcal{Y}}^{\mathcal{S}} = 0$ is proved.

Suppose that $x \in R''_{n+1}\{a\} \cap \tilde{\mathcal{Y}}^{\mathcal{S}}$. Then there is a $y \in \mathcal{Y}$ such that $x \in S''_{n+1}\{y\}$. We have $y \in \mathcal{Y}$ and, consequently, $R''_{n+1}S''_{n+1}\{y\} \subseteq S''_n\{y\}$. But $a \in R''_{n+1}S''_{n+1}\{y\}$ holds. Thus $a \in S''_n\{y\}$, which is a contradiction. (2) We deduce from the fact that $\tilde{\mathcal{Y}}^{\mathcal{S}}$ is $\mathcal{R}$-shut, the following: $\tilde{\mathcal{X}}^{\mathcal{R}} \subseteq \tilde{\tilde{\mathcal{Y}}}^{\mathcal{S}\mathcal{R}} = \tilde{\mathcal{Y}}^{\mathcal{S}}$.

<u>Proposition</u>. Let $\mathcal{L}$ be of the type π and let $\mathcal{X} \subseteq A$ be revealed, $\mathcal{R}''\mathcal{X} = \mathcal{X}$. Then $\mathcal{X}$ is $\mathcal{R}$-shut.

Proof. Assume $a \in A - \mathcal{X}$ and suppose that

$$(*) \qquad \neg\, \mathrm{Sep}(\mathcal{R}, a, \mathcal{X}).$$

Then $(\forall n)(R''_n\{a\} \cap \mathcal{X} \neq 0)$ holds; put, for ev ery n, $x_n \in \mathcal{X}$ such that $x_n \in R_n\{a\}$. Let $d \subseteq \mathcal{X}$ be a set such that $\{x_n\} \subseteq d$. We have $(\forall n)(R''_n\{a\} \cap d \neq 0)$ and, consequently, there exists an $\alpha \notin FN$ with $R''_\alpha\{a\} \cap d \neq 0$. Thus $a \in R''_\alpha \mathcal{X}$ and $R''_\alpha \mathcal{X} \subseteq \mathcal{X}$, which is a contradiction.

Let us suppose that A and B are tied by a mapping $\Theta : A \to B$. We modify the definition of the reduction with respect to Θ as follows:

<u>Definition</u>. Let $\Theta : A \to B$. $\mathcal{T}$ is said to be <u>uniformly reducing on</u> $\mathcal{X}$ w.r.t. Θ iff $(\forall T \geq \mathcal{T})(\exists v \in P_f(\Theta(\mathcal{X})))(\mathcal{X} \subseteq T''v)$ holds.

<u>Theorem</u>. Let $\mathcal{L}$ be of the type π, $\Theta : A \to B$, $\mathcal{X} \subseteq A$. Let us suppose that $\mathcal{X}$ is $\mathcal{R}$-shut, $\Theta(\mathcal{X}) \subseteq \mathcal{W}$, $\tilde{\mathcal{X}}^{\mathcal{R}} = \tilde{\mathcal{Y}}^{\mathcal{S}}$, and $\mathcal{S}$ is uniformly reducing on $\mathcal{X}$ w.r.t. Θ. Then

(1) $\mathcal{X}$ is of the type π,

(2) suppose, moreover, that $\Theta(\mathcal{X})$ is revealed. Then there exists a set $w \subseteq \Theta(\mathcal{X})$, $\mathcal{X} = \mathcal{S}''w$.

The assertion (1) is an immediate consequence of the

<u>Proposition</u>. Let $\mathcal{L}$ be of the type π, $\Theta : A \to B$, $\mathcal{X} \subseteq A$. Let us suppose that $\Theta(\mathcal{X}) \subseteq \mathcal{W}$, $\tilde{\mathcal{X}}^{\mathcal{R}} \subseteq \tilde{\mathcal{Y}}^{\mathcal{S}}$, and $\mathcal{S}$ is uniformly reducing on $\mathcal{X}$ w.r.t. Θ.

Then there exists a class $\mathcal{P}$ of the type π such that $\widetilde{\mathcal{X}}^{\mathcal{R}} \subseteq \mathcal{P} \subseteq \widetilde{\mathcal{Y}}^{\mathcal{S}}$ holds.

Proof. Choose, for every n, $v_n \in P_f(\Theta(\mathcal{X}))$ such that $\mathcal{X} \subseteq S"_n v_n$. We have $\{v_n\} \subseteq \mathcal{W}$ and the following relations hold:

$$R_{n+1}\mathcal{X} \subseteq R"_{n+1} S"_{n+1} v_{n+1} \subseteq S"_{n+1} v_{n+1} \subseteq S"_n \Theta(\mathcal{X}),$$

$$\widetilde{\mathcal{X}}^{\mathcal{R}} = \bigcap R"_{n+1} \mathcal{X} \subseteq \bigcap S"_n v_{n+1} \subseteq \bigcap S"_n \Theta(\mathcal{X}) = \widetilde{\Theta(\mathcal{X})}^{\mathcal{S}}.$$

Thus, the class $\mathcal{P} = \bigcap S"_n v_{n+1}$ has the required properties.

Let us prove the assertion (2) of the previous theorem. We use the notation of the previous proof. Let w be such that $\{v_n, n \in FN\} \subseteq w \subseteq \Theta(\mathcal{X})$. Then the relation $\mathcal{S}"w \subseteq \widetilde{\Theta(\mathcal{X})}^{\mathcal{S}}$ holds. Thus, the proof will be finished, if the formula

$$(*) \qquad \mathcal{X} \subseteq \mathcal{S}"w$$

is proved.

Assume that $a \in \mathcal{X}$. We have $(\forall n)(a \in S"_n v_n)$ and we deduce that $(\forall n)(\exists y \in w)(a \in S"_n\{y\})$ holds, too. T hus there exists $\alpha \notin FN$ and $b \in w$ such that $a \in S"_\alpha \{b\}$. We deduce from this and from the relation $S_\alpha \subseteq \mathcal{S}$ that $a \in \mathcal{S}"\{b\} \subseteq \mathcal{S}"w$ and $(*)$ is proved.

References

[V] P. VOPĚNKA: Mathematics in the alternative set theory, Teubner-Texte, Leipzig, 1979.

[M] J. MLČEK: Approximations of σ-classes and π-classes, Comment.Math.Univ.Carol inae 20(1979), 669-679.

[S-V] A. SOCHOR and P. VOPĚNKA: Revealments, Comment.Math.Univ. Carolinae 21(1980), 97-118.

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