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ON THE TIGHTNESS OF CHAIN-NET SPACES

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Abstract: We give a general construction that yields (in ZFC)
1) a 0-dimensional T_2 chain net space of countable tightness that is not sequential;
2) a 0-dimensional T_2 chain net space X for which $t(X) \neq t_s(X)$.
1) answers a problem from [1] and 2) from [2].

Key words: Chain-net space, sequential space, tightness.

Classification: 54A20, 54A25

The aim of this paper is to solve two problems raised in [1] and [2], respectively, both connected with the tightness of chain-net spaces. The first problem asks whether a chain-net space of countable tightness is sequential. This problem was partially solved in [3], where consistent T_2 counter examples were given. The example given below has the advantage over these these that it is both constructed in ZFC and T_3 (in fact, 0-dimensional T_2). The second problem asks whether $t(X)=t_s(X)$ holds for a chain-net space X. (We recall that $t_s(X)$ is the smallest cardinal τ such that whenever $p \in \bar{A}$ in X then there is a family $\mathcal{A}$ of subsets of A such that $p \in \overline{\cup\mathcal{A}}$ but $p \notin \cup\{\bar{B} : B \in \mathcal{A}\}$.) The example will again be 0-dimensional T_2 and obtained in ZFC. It is known (cf. [4]) that the size of a counter example must be bigger than $\aleph_\omega$, and our example has cardinality $\beth_{\omega+1}$. The question whether an example of size $\aleph_{\omega+1}$ may be obtained in ZFC thus remains open.

Both examples will be obtained from a general construction that will now be given.

Theorem. Let $\varkappa$, λ be cardinals such that $cf(\varkappa)=\omega$, if $\varkappa > \omega$ then $\mu^\omega < \varkappa$ for each $\mu < \varkappa$, furthermore $\lambda^\varkappa = \lambda$. Let

$\langle X, \tau \rangle$ be a topological space with $|X| = \lambda$ and satisfying properties 1)-4) below:

1) for every closed set $F \subset X$ we have either $|F| \le \varkappa$ or $|F| = \lambda$;

2) $hd(X) \le \varkappa$;

3) X is T_2 and first countable (in fact, for each $p \in X$ we fix a countable τ-neighbourhood base $\{U_n(p) : n \in \omega\}$);

4) if $F \subset X$ is closed with $|F| \le \varkappa$ then F is the intersection of countably many τ-clopen sets.

Then there is a locally countable and locally compact topology $\rho \supset \tau$ on X such that

(i) if $S \subset X$ and $|\overline{S}^{\tau}| = \lambda$ then $|\overline{S}^{\rho}| = \lambda$ as well, and

(ii) if $F \subset X$ is ρ-closed and $|F| \le \varkappa$ then there is a ρ-clopen set Z with $S \subset Z$ and $|Z| \le \varkappa$.

<u>Proof</u>. Let S be the family of all sets $S \in [X]^{\varkappa}$ with $|\overline{S}^{\tau}| = \lambda$, then by $\lambda^{\varkappa} = \lambda$ we can write $\mathcal{S} = \{S_\alpha : \alpha \in \lambda\}$ where for each $S \in \mathcal{S}$ we have $|\{\alpha \in \lambda : S = S_\alpha\}| = \lambda$. We also fix a well-ordering $\prec$ of X in type λ .

Our aim is now to define, by induction on $\alpha \in \lambda$ points $p_\alpha \in X$ and topologies ρ_α on $X_\alpha = \{p_\beta : \beta \in \alpha\}$ that satisfy the following inductive hypotheses:

$I(\alpha)$: ρ_α is a locally countable and locally compact refinement of $\tau_\alpha = \tau \restriction X$ (i.e. $\rho_\alpha \supset \tau_\alpha$):

$J(\alpha)$: for all $\beta \in \alpha$ we have $\rho_\beta = P(X_\beta) \cap \rho_\alpha$.

If $\alpha \in \lambda$ is limit and p_β, ρ_β have been suitably defined for every $\beta \in \alpha$ then ρ_α is the topology generated by $\cup\{\rho_\beta : \beta \in \alpha\}$ on X_α. Clearly, $I(\alpha)$ and $J(\alpha)$ will hold then.

Now, if $\alpha = \beta + 1$, we distinguish two cases. If $S_\beta \subset X_\beta$ then we choose p_β as the $\prec$-first element of $\overline{S}_\beta^{\tau} \setminus X_\beta$ and then choose a sequence $q_n \in S_\beta$ such that $q_n \in U_n(p_\beta)$. Using $I(\beta)$ we can choose for each $n \in \omega$ a compact open (hence countable) ρ_β neighbourhood K_n of q_n with $K_n \subset U_n(p_\beta)$. A ρ_α-neighbourhood base of p_β in X_α is then formed by the sets $\cup\{K_i : i \in \omega \setminus n\} \cup \{p_\beta\}$ for all $n \in \omega$. If $S_\beta \not\subset X_\beta$ then we take as p_β the $\prec$-first element of $X \setminus X_\beta$ and define ρ_α by declaring p_β isolated in X_α. It is easy to check that $I(\alpha)$ and $J(\alpha)$ will be valid in either case

Having finished the induction we then define ρ as the topology generated by $\cup\{\rho_\alpha : \alpha \in \lambda\}$ on X. Since we made sure that every point of X occurs as some p_α we clearly have that ρ is a locally countable and locally compact refinement of τ . (i) follows since for every $S \in \mathcal{S}$ we have $S=S_\beta \subset X_\beta$ for λ many $\beta \in \lambda$ (this makes use of the fact that $cf(\lambda) > \varkappa$ since $\lambda = \lambda^\varkappa$), moreover, by 2), for every $A \subset X$ there is an $S \subset A$ with $|S| \le \varkappa$ such that $\overline{A}^\tau = \overline{S}^\tau$.

To prove (ii), let us first observe that for every ρ-closed F with $|F| \le \varkappa$ we have by (i) that $|\overline{F}^\tau| \le \varkappa$ as well, hence it suffices to prove that every τ-closed set F with $|F| \le \varkappa$ can be covered by a ρ-clopen set Z with $|Z| \le \varkappa$.

It is straightforward from the local countability and first countability of ρ that any set $H \subset X$ with $|H| < \varkappa$ can be covered by a ρ-clopen set of size $\le \varkappa$. Indeed, if $\varkappa = \omega$ this simply follows from the 0-dimensionality and local countability of ρ , and if $\varkappa > \omega$ then we simply may iterate ω_1 times taking closures and covering by countable open sets, and then taking the union, which will be clopen and of size $\le \varkappa$ by $\mu^\omega < \varkappa$ whenever $\mu = |H| < \varkappa$.

Next, assume that F is τ-closed and $|F| = \varkappa$. By 4) there is a decreasing sequence $\{U_n : n \in \omega\}$ of τ-clopen sets with $F = \bigcap_{n \in \omega} U_n$. Also, we may write $F = \bigcup_{n \in \omega} F_n$ where $|F_n| < \varkappa$ for each $n \in \omega$. By the above we may then find for every n a ρ-clopen set $Z_n \supset F_n$ with $|Z_n| \le \varkappa$. We claim that

$$Z = \cup\{Z_n \cap U_n : n \in \omega\}$$

is as required. Indeed, we clearly have $F \subset Z$ and $|Z| \le \varkappa$, and that Z is ρ-open. To show that Z is also ρ-closed, pick any point $x \in X \setminus Z$ and choose $n \in \omega$ such that $x \notin U_n$. But then

$$V = X \setminus [\bigcup_{i<n} (Z_i \cap U_i) \cup U_n]$$

is clearly a ρ-clopen set containing x with $V \cap Z = \emptyset$, hence Z is also ρ-closed.

<u>Corollary 1</u>. There exists a 0-dimensional T_2^* chain-net space X of countable tightness that is not sequential.

Proof. Let us first apply our theorem (with $\varkappa=\omega$ and $\lambda=c$) to e.g. the Cantor set $\mathbb{C}$, we then obtain a topology ρ on $\mathbb{C}$ satisfying (i) and (ii). We then put $X=\mathbb{C}\cup\{p\}$ (with $p\notin\mathbb{C}$) with the topology that agrees with ρ on $\mathbb{C}$ and has neighbourhoods for p of the form $(X\setminus Z)\cup\{p\}$ where $Z\subset\mathbb{C}$ is ρ-clopen and $|Z|\le\omega$.

It is straightforward from (i) and (ii) that X is 0-dimensional T_2 and not sequential because no ω-sequence from $\mathbb{C}$ can converge to p.

X is chain-net because if $F\subset\mathbb{C}$ is ρ-closed and not closed then $|F|>\omega$ by (ii), and it is clear that every ω_1-sequence of distinct elements of F converges to p.

Finally, X has a countable tightness because if $A\subset\mathbb{C}$ and $p\in\overline{A}$ then $|\overline{A}^{\rho}|=|\overline{A}^{\tau}|=c$, hence there is a countable set $B\subset A$ with $|\overline{B}^{\rho}|=|\overline{B}^{\tau}|=c$, hence B cannot be covered by a countable ρ-clopen set, i.e. $p\in\overline{B}$ as well.

Corollary 2. Suppose $cf(\varkappa)=\omega<\varkappa$ and $\lambda=\varkappa^{\omega}$ are such that $\mu<\varkappa$ implies $\mu^{\omega}<\varkappa$ and $\lambda^{\varkappa}=\lambda$. Then there is a 0-dimensional T_2 chain-net space X with $|X|=\lambda$ for which $t_s(X)\neq t(X)$.

Proof. Let us apply in this case our theorem to the space $B(\varkappa)=D(\varkappa)^{\omega}$, i.e. the Baire space of weight $\varkappa$. 2) and 3) are now obvious, 1) holds because every closed set F in $B(\varkappa)$ is a complete metric space, hence by [5] we have $|F|\le\varkappa$ or $|F|=\lambda$. Finally 4) holds because $Ind(B(\varkappa))$, the large inductive dimension of $B(\varkappa)$ is equal to 0.

Now let ρ be the topology on $B(\varkappa)$ that satisfies (i) and (ii). Our space X will be of the form $X=B(\varkappa)\cup\{p\}$ with the topology that agrees with ρ on $B(\varkappa)$ and has as neighbourhoods for p sets of the form $(B(\varkappa)\setminus Z)\cup\{p\}$, where Z is a ρ-clopen and $|Z|\le\varkappa$.

It is clear that X is 0-dimensional and T_2. To show that it is chain-net take any ρ-closed set $F\subset B(\varkappa)$ that is not closed in X, i.e. $p\in\overline{F}$. Then by (ii) we must have $|F|=\lambda>\varkappa$, and it is clear that every $\varkappa^{+}$-sequence of distinct points from F converges to p.

Next we show that $t_s(p,X)=\omega$; hence $t_s(X)=\omega$ as well. Indeed, if $A\subset B(\varkappa)$ and $p\in\overline{A}$ then we must have, by (ii), $|\overline{A}^{\rho}|=\lambda=|\overline{A}^{\tau}|$. But we can find a set $S\subset A$ with $|S|=\varkappa$ and $\overline{S}^{\tau}=\overline{A}^{\tau}$, hence $|\overline{S}^{\rho}|=\lambda$ as well. Consequently we have $p\in\overline{S}$ since no ρ-clopen Z with $|Z|\leq\varkappa$ may cover S, but if we write $S=\bigcup\{S_n : n\in\omega\}$ with $|S_n|<\varkappa$ for all n then $p\notin\overline{S}_n$ for any n by (ii) and $|\overline{S}_n|<\varkappa$, hence indeed $t_s(p,X)=\omega$.

Finally we show that $t(p,X)=\varkappa$, and this will complete our proof. It suffices, of course, to show $t(p,X)\geq\varkappa$. This, however, is now trivial because, as we have seen above, for all $A\subset B(\varkappa)$ with $|A|<\varkappa$ we have $p\notin\overline{A}$.

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