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MODEL-THEORETICAL CONSTRUCTIONS IN AST I

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Abstract: In this paper two classical model constructions are adapted for the needs of the alternative set theory (AST). The first one is the construction of an isomorphism among saturated and elementary equivalent structures,: in the other, the limit of the elementary chain of structures is constructed.

Key words: Alternative set theory, structure for a language, interpretation, saturation, elementary embedding.

Classification: Primary 03E70
Secondary 03C30, 03C50

Endomorphic universes (see [V]) play an important role in AST. In the paper [Č-Vo] there is constructed an increasing sequence of endomorphic universes with standard extension of the length ω . There is a question whether it is possible to construct a similar sequence of the length Ω . The answer is positive; such a construction will be given in the second part of this paper. The construction lies on a modification of classical model constructions for needs of AST. In AST we have namely only two infinite cardinalities and therefore it is not at all evident how to adapt classical methods, using higher cardinalities, to the spirit of AST. In the paper [S2], several modifications are presented; for our needs they are not, however, general enough.

In this work it is shown how to create - in a quite general form - some model constructions in AST. These constructions will be used for interpretations of a set-theoretical language, given by a structure. The generality lies in the fact that we admit a system of class constants in our language, too. We do not presume such a system to be codable. Thus, when investigating e.g. the ultrapower, we can understand the system of all subclasses of the original structure as a system of constants; using then the above mentioned techniques we obtain a description (by normal formulas) of interpretations of these constants in the ultrapower (ultraproduct constructions will be examined in the second part of our article).

Since we do not presuppose that all readers are able to make a direct transfer from model theory to AST, we proceed in § 1 rather slowly. We prove here the theorem on an isomorphism among saturated and elementary equivalent structures (for the language $FL_{\mathcal{S}}$) and show, among others, when an arbitrary formula of the language $FL_{\mathcal{S}}$ is transmitted by means of an isomorphism.

In the second paragraph we deal with a construction of the limit structure for an elementary chain of structures. It is proved here that this structure is an elementary extension of "preceding" structures and that, if suitable obstacles are fulfilled, it is saturated. This construction will be substantially used in the second part of this paper since it is exceptionally convenient for chains of the length Ω.

For the readers, who are familiar with a classical form of the studied model constructions, it could be interesting to take notice of the strength of applied axioms, esp. those of the type of axiom of choice, and of the "largeness" of used cardinalities (e.g. when applying these results in higher-order arithmetics).

§ 1. Let $\mathcal{S}$ be a system of classes. Writing $X \in \mathcal{S}$ we mean (when taking a formal point of view) that X is of a sort of variables $\mathcal{S}$ which is subordinate to a sort of class variables. A special case is when this sort is determined by a formula $X \in \mathcal{S} \equiv \equiv \varphi(X)$. Only such a case will be used later.

The alphabet of the language $L_{\mathcal{S}}$ consists of the following systems of signs:

1) $x_1, x_2, \ldots, X_1, X_2, \ldots$ - variables for sets and classes

2) $\&, \vee, \neg, \Rightarrow, \equiv, \forall, \exists$

3) $=, \in$

4) C,D,... (event. with indexes) - special constants for classes from $\mathcal{S}$.

Formulas of the language $FL_{\mathcal{S}}$ are such formulas (of a finite length) which arise from formulas of the language FL (see [V]) by an incidental replacing of some free occurrences of variables for classes by constants for classes from $\mathcal{S}$. The language which we obtain from $FL_{\mathcal{S}}$ by a restriction to its normal formulas (i.e. formulas in which we admit only the quantification of set variables) will be called the language $NFL_{\mathcal{S}}$. If we limit ourselves in $NFL_{\mathcal{S}}$ only on such formulas which have no variables for classes (we admit, of course, constants for classes from $\mathcal{S}$), we speak about the language $SFL_{\mathcal{S}}$.

Further we shall introduce the notions of a structure for the language $FL_{\mathcal{S}}$ and of an interpretation of formulas of $FL_{\mathcal{S}}$ determined by a given structure. For the language FL, both the notions are introduced in [S1].

A structure $\mathfrak{A}$ for the language $FL_{\mathcal{S}}$ (briefly only a structure) is a triplet $\{A,E,I\}$, where $(E \cup I) \subseteq A^2$, together with such a system of classes $\mathcal{S}^{\mathfrak{a}}$ that to each $C \in \mathcal{S}$ there is just one class $D \in \mathcal{S}^{\mathcal{A}}$ such that $D \subseteq A$ and the following holds (i.e. saturation of classes w.r.t. identity):

$$(\forall x,y \in A)(x \in D \;\&\; \langle x,y\rangle \in I) \Rightarrow y \in D,$$

we denote the class D by $C^{\mathcal{A}}$. We say that A is a support of $\mathcal{U}$.

Sometimes, it will be useful to write instead of $\mathcal{U}$ more precisely $\mathcal{U} = \{A,E,I,\mathcal{S}^{\mathcal{A}}\}$ or even $\mathcal{U} = \{A,E^{\mathcal{A}},I^{\mathcal{A}},\mathcal{S}^{\mathcal{A}}\}$.

Let $\mathcal{U}$ be a structure. The symbol $\mathcal{A}$ denotes the <u>interpretation of formulas</u> of the language $FL_{\mathcal{S}}$ determined by $\mathcal{U}$. We define it as follows:

1) $Cls^{\mathcal{A}}(X) \equiv (X \subseteq A \;\&$ X is saturated w.r.t. I); writing $X^{\mathcal{A}}$ we mean that $X^{\mathcal{A}}$ is a class and $Cls^{\mathcal{A}}(X^{\mathcal{A}})$

2) $X^{\mathcal{A}} \in^{\mathcal{A}} Y^{\mathcal{A}} \equiv (\exists x \in Y^{\mathcal{A}})(X^{\mathcal{A}} = E''\{x\})$

3) $X^{\mathcal{A}} =^{\mathcal{A}} Y^{\mathcal{A}} \equiv X^{\mathcal{A}} = Y^{\mathcal{A}}$

4) For $C \in \mathcal{S}$ let $C^{\mathcal{A}}$ denote the class which corresponds to C in $\mathcal{S}^{\mathcal{A}}$, then $C^{\mathcal{A}}$ is the interpretation of C.

5) The symbol $\varphi^{\mathcal{A}}$ denotes the formula which is the interpretation of φ, i.e. $\varphi^{\mathcal{A}}$ is such a formula in which $\in$ and $=$ are substituted in the above mentioned way, $(\forall X)$ ... is replaced by $(\forall X)(Cls^{\mathcal{A}}(X) \Rightarrow \ldots)$ and $C_1, C_2, \ldots$ by $C_1^{\mathcal{A}}, C_2^{\mathcal{A}}, \ldots$. Let us note that the symbol $\mathcal{A}$ was used before, namely for the interpretation of constants from $\mathcal{S}$.

Evidently, $Cls^{\mathcal{A}}(\emptyset)$, $Cls^{\mathcal{A}}(A)$, $Cls^{\mathcal{A}}(C^{\mathcal{A}})$. Notice that $Cls^{\mathcal{A}}(X)$ is described by a normal formula with parameters A, E, I.

Let $\mathcal{U}, \mathcal{L}$ be structures for the language $FL_{\mathcal{S}}$ ($\mathcal{A}, \mathcal{B}$ corresponding interpretations). $\mathcal{U}$, $\mathcal{L}$ are called <u>$\mathcal{S}$-elementary equivalent structures</u> iff

$$(\forall \varphi \in SFL_{\mathcal{S}}) \quad \varphi^{\mathcal{A}} \equiv \varphi^{\mathcal{B}}$$

Let $\mathcal{U} = \{A,E^{\mathcal{A}},I^{\mathcal{A}},\mathcal{S}^{\mathcal{A}}\}$ be a structure for the language $FL_{\mathcal{S}}$. We say that a structure $\mathcal{L} = \{B,E^{\mathcal{B}},I^{\mathcal{B}},\mathcal{S}^{\mathcal{B}}\}$ (for the same language) is a <u>substructure</u> of $\mathfrak{A}$ iff $B \subseteq A$, $E^{\mathfrak{B}} = E^{\mathfrak{A}} \cap B^2$, $I^{\mathfrak{B}} = I^{\mathfrak{A}} \cap B^2$ and $\mathcal{S}^{\mathfrak{B}} = \{C^{\mathfrak{A}} \cap B;\ C \in \mathcal{S}\}$

A structure $\mathfrak{B}$ is an <u>elementary substructure</u> of $\mathfrak{A}$ iff $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ and $\mathfrak{A}$, $\mathfrak{B}$ are $\mathcal{S}$-elementary equivalent structures.

Let $\mathfrak{A}$ be a structure. Remind that a class x is a set in the sense of $\mathfrak{A}$ iff x is a class in the sense of $\mathfrak{A}$ and if there exists $t \in A$ such that $x = E''\{t\}$.

Denote

$A_m = \{t; t \in A \ \&\ Cls^{\mathfrak{A}}(E''\{t\})\}$

The class A_m is described by a normal formula with parameters A, E, I. Realize that A_m is in fact the system of all codes for sets (A_m has not to be a class in the sense of $\mathfrak{A}$).

For the sake of typing we take the following agreement (which will not be, however, kept in principle):

Let $Set^{\mathfrak{A}}(x)$, then the code of x (more precisely, some from the codes of x) will be denoted t_x. If x belongs to an indexed system, e.g. $x = x_{\alpha}$, then instead of $t_{x_{\alpha}}$ we shall write only t_{α}. On the contrary, if we use for a code the notation t, then the set to which this code corresponds will be denoted x_t. When working with an indexed system of codes t_{α}, we write briefly x_{α} instead of $x_{t_{\alpha}}$.

Further we show that if in a structure $\mathfrak{A}$ the axiom of extensionality holds, we can limit ourselves, when working with the support A, to A_m.

<u>Lemma 1</u>. Let $(Ext)^{\mathfrak{A}}$ hold in $\mathfrak{A}$. Then

$(\forall X^{\mathfrak{A}}, Y^{\mathfrak{A}})\ (X^{\mathfrak{A}} = Y^{\mathfrak{A}}) \equiv X^{\mathfrak{A}} \cap A_m = Y^{\mathfrak{A}} \cap A_m$

<u>Proof</u>. The implication $\Rightarrow$ is obvious. For proving $\Leftarrow$ use $(Ext)^{\mathfrak{A}}$.

Lemma 2. Let $\varphi \in FL_{\mathcal{S}}$ and $Cls^{\mathcal{A}}(X_i)$, $i = 1,\dots,m$. Then

(1) $((\exists x)\,\varphi(x,C_1,\dots,C_n,X_1,\dots,X_m))^{\mathcal{A}} \equiv$
$\equiv (\exists t \in A_m)(\varphi^{\mathcal{A}}(E''\{t\},C_1,\dots,C_n,X_1,\dots,X_m))$.

Proof. Obviously

$((\exists x)\,\varphi(x,C_1,\dots,C_n,X_1,\dots,X_m))^{\mathcal{A}} \equiv (\exists X)\,[(\exists t \in A)(X = E''\{t\}\ \&$ $\&\ Cls^{\mathcal{A}}(X)\ \&\ \varphi^{\mathcal{A}}(x,C_1,\dots,C_n,X_1,\dots,X_m)] \equiv (\exists t \in A_m)(\varphi^{\mathcal{A}}(E''\{t\},$ $C_1,\dots,C_n,X_1,\dots,X_m))$.

Theorem 1. Let $\varphi \in NFL_{\mathcal{S}}$. Then it is possible to express $\varphi^{\mathcal{A}}(X_1^{\mathcal{A}},\dots,X_n^{\mathcal{A}})$ by a normal formula with parameters A, E, I, interpretations of constants from $\mathcal{S}$, which occur in φ, and classes $X_i^{\mathcal{A}}$ $(i = 1,\dots,n)$.

Proof can be done by induction. Its individual steps - except $(\exists x)\varphi$ which was investigated in Lemma 2 - follow directly from the definition of the interpretation.

The fact that model constructions can be expressed by normal formulas (with given parameters) is important. It is used e.g. when making iterations of those constructions. The mere existence (i.e. a description by not-normal formulas) could namely claim a strong form of an axiom of the type of choice.

Now we shall examine some properties of structures (for the language $FL_{\mathcal{S}}$).

Let $\mathcal{S} \subseteq \mathcal{S}_1$. We say that a structure $\mathfrak{A}_1 = \{A,E,I,\mathcal{S}_1^{\mathcal{A}_1}\}$ is an expansion of the structure $\mathfrak{A} = \{A,E,I,\mathcal{S}^{\mathcal{A}}\}$ for the language $FL_{\mathcal{S}_1}$, which we call an expansion of the language $FL_{\mathcal{S}}$ iff for each $C \in \mathcal{S}$ the condition $C^{\mathcal{A}} = C^{\mathcal{A}_1}$ holds ($\mathcal{A}$, $\mathcal{A}_1$ are the interpretations determined by $\mathfrak{A}$, $\mathfrak{A}_1$, respectively).

When we interpret sets in an expanded language, it is convenient to use their codes instead of corresponding constants. So if we have e.g. $Z \subseteq A_m$ and $t \in Z$ then, in the language $FL_{\mathcal{S} \cup Z}$, we shall interpret t as $E''\{t\}$.

Let $\{\varphi_n; n \in FN\}$ be a sequence of formulas (of one free variable) of the language $SFL_{\mathcal{S}}$. We say that this <u>sequence</u> is <u>consistent</u> in $\mathfrak{A}$ iff $(\forall n \in FN)[(\exists x)(\varphi_0(x) \& \varphi_1(x) \& \ldots \& \varphi_n(x))]^{\mathfrak{a}}$.

A structure $\mathfrak{A}$ is called an <u>$\mathcal{S}$-saturated structure</u> iff for each consistent sequence $\{\varphi_n; n \in FN\}$ of the language $SFL_{\mathcal{S} \cup A_m}$ it is true:

$$(\exists t \in A_m)(\forall n \in FN)\, \varphi_n^{\mathfrak{a}}(x_t)$$

($\mathfrak{a}$ denotes here the interpretation of formulas of the language $SFL_{\mathcal{S} \cup A_m}$).

The reader may be here in doubt about the correctness of our considerations. It looks like we need the relation of satisfaction. But as the formulas in question are normal and contain only at most countably many class parameters (they can be coded in one class), we are able to construct to them (due to the Morse´s scheme) the required relation of satisfaction. We can also understand the above definition as follows: If we fix (in AST) constants for $\mathfrak{A}$, then assuming formulas describing the consistency of a sequence of formulas, we must be able to deduce the existence of $t \in A_m$ (see the definition).

Now we show that if $\mathfrak{A}$ is an $\mathcal{S}$-saturated structure then its support can be ordered by the type Ω.

<u>Lemma 3</u>. Let $\mathfrak{A}$ be an $\mathcal{S}$-saturated structure. Let K be such a class that

$$(\forall \bar{t} \in A_m)(\exists t \in K)(x_t = x_{\bar{t}}) \& (\forall t_1, t_2 \in K) x_{t_1} \neq x_{t_2}.$$

Then K is either finite or uncountable.

<u>Proof</u>. Suppose K is countable. Let us enumerate all its elements - $t_1, t_2, \ldots, t_n, \ldots$ .. Denote $\varphi_n(x)$ ν $x \neq t_n$. Then $\{\varphi_n; n \in FN\}$

is a countable system of formulas of the language $SFL_{\mathcal{S}\cup A_m}$. It follows, from $\mathcal{S}$-saturation of $\mathfrak{A}$, that there is $t \in A_m$ such that for each $n \in FN$ the formula $\varphi_n^{\mathcal{A}}(x_t)$ is valid - which is in contradiction to the definition of K.

In further considerations, the notion of isomorphism among structures is important. Let us remind firstly its definition (of course, formulated in our terminology).

Let $\mathfrak{A} = \{A, E^{\mathcal{A}}, I^{\mathcal{A}}, \mathcal{S}^{\mathcal{A}}\}$, $\mathfrak{B} = \{B, E^{\mathcal{B}}, I^{\mathcal{B}}, \mathcal{S}^{\mathcal{B}}\}$ be structures for the language $SFL_{\mathcal{S}}$ ($\mathcal{A}, \mathcal{B}$ corresponding interpretations). A one-one mapping F is called a partial isomorphism iff $dom(F) \subseteq A_m$, $rng(F) \subseteq B_m$ and the following holds:

(2) $(\forall \varphi \in SFL_{\mathcal{S}})(\varphi^{\mathcal{A}}(x_{t_1}, \dots, x_{t_n}) \equiv \varphi^{\mathcal{B}}(y_{F(t_1)}, \dots, y_{F(t_n)}))$,

where $y_{F(t_i)} = E^{\mathcal{B}}{}'' \{F(t_i)\}$, $i = 1, \dots, n$.

A partial isomorphism F will be called a total isomorphism iff

(3) $[(\forall t \in A_m)(\exists \bar{t} \in dom(F))\ x_t = t_{\bar{t}}]\ \&$

$\&[(\forall u \in B_m)(\exists \bar{u} \in rng(F))\ y_u = y_{\bar{u}}]$

Structures $\mathfrak{A}, \mathfrak{B}$ are isomorphic (denotation $\mathfrak{A} \approx \mathfrak{B}$) iff there is a total isomorphism F between them (we denote $F: \mathfrak{A} \approx \mathfrak{B}$).

From these definitions it follows that if F is a partial (or a total) isomorphism then F^{-1} is a partial (or a total) isomorphism.

Lemma 4. Let $F: \mathfrak{A} \approx \mathfrak{B}$. Then

$$(\forall t)(t \in C^{\mathcal{A}} \equiv F(t) \in C^{\mathcal{B}})$$

Proof. Realize that from the definition of isomorphism it follows:

$$(x_t \in C)^{\mathcal{A}} \equiv (y_{F(t)} \in C)^{\mathcal{B}}$$

When investigating substructures of given structures we need the following notion.

Let $\mathfrak{A}$, $\mathfrak{B}$ be structures of the language $SFL_{\mathcal{S}}$. Let $\mathfrak{A}_1$ be such an elementary substructure of $\mathfrak{A}$ that $F: \mathfrak{B} \approx \mathfrak{A}_1$: Then we call F an <u>elementary embedding</u> $\mathfrak{B}$ into $\mathfrak{A}$ (in symbols $F: \mathfrak{B} \precsim \mathfrak{A}$). We say then that $\mathfrak{B}$ is <u>elementarily embedded</u> into $\mathfrak{A}$ (denotation $\mathfrak{B} \precsim \mathfrak{A}$).

Lemma 4 asserts that "isomorphism transmits constants". Now we prove that the same is true also for the relation E. As to identity, the situation is more complicated and will be investigated later.

<u>Lemma 5</u>. Let $\mathfrak{A} = \{A, E^{\mathfrak{A}}, I^{\mathfrak{A}}, \mathcal{S}^{\mathfrak{A}}\}$, $\mathfrak{B} = \{B, E^{\mathfrak{B}}, I^{\mathfrak{B}}, \mathcal{S}^{\mathfrak{B}}\}$ be such structures that $F: \mathfrak{A} \precsim \mathfrak{B}$. Then

$$(\forall t,u \in A_m)\ t\ E^{\mathfrak{A}} u \equiv F(t)\ E^{\mathfrak{B}} F(u).$$

<u>Proof</u>. Let $t,u \in A_m$ and $t\ E^{\mathfrak{A}} u$. Then $(x_t \in x_u)^{\mathfrak{A}}$ and therefore (see the definition of isomorphism) $(F(x_t) \in F(x_u))^{\mathfrak{B}}$; hence $F(t) E^{\mathfrak{B}} F(u)$. The converse implication follows from the fact that F^{-1} is an isomorphism, too.

<u>Theorem 2</u>. Let $\mathcal{S}$ be a countable system of classes. Let $\mathfrak{A}$, $\mathfrak{B}$ be $\mathcal{S}$-saturated and $\mathcal{S}$-elementary equivalent classes. Then $\mathfrak{A} \approx \mathfrak{B}$.

At first we remind a notion and prove two auxiliary assertions.

Let $\leq$ be such a partial ordering of X that

$(\forall x,y \in X)(\exists z \in X)(x \leq z \ \&\ y \leq z)$.

Then X is called <u>directed</u> by $\leq$.

<u>Lemma 6</u>. Let $\mathfrak{A}$, $\mathfrak{B}$ be structures of the language $SFL_{\mathcal{S}}$ and let $\mathcal{J}$ be a directed class (by $\leq$). Suppose $\{F_j\}_{j \in \mathcal{J}}$ is such a system of partial isomorphisms that

$$(\forall j_1, j_2 \in \mathcal{J})(j_1 \leq j_2 \Rightarrow F_{j_1} \subseteq F_{j_2}).$$

Denote $F = \bigcup \{F_j\}_{j \in \mathcal{J}}$. Then F is a partial isomorphism

Proof. Obviously F is a one-one mapping. Suppose that $\mathcal{J}$ has not the largest element (otherwise is the assertion trivial). Let further $\varphi \in SFL_{\mathcal{G}}$, $t_1, \ldots, t_n \in dom(F)$. Then there is a partial isomorphism $F_{j_k} \subseteq F$ such that $t_1, \ldots, t_n \in dom(F_{j_k})$. For completing the proof it suffices now to apply formula (2), from the definition of isomorphism, to F_{j_k}.

In the previous lemma it was proved that the union of a system of partial isomorphisms is a partial isomorphism. We show now how it is possible to prolong a partial isomorphism "by one step" (the properties of isomorphism will be, of course, preserved).

Lemma 7. Let F be an at most countable partial isomorphism. Then, under the assumptions of Theorem 2, just one of the following conditions holds:

(a) F is even a total isomorphism;

(b) for each $t \in A_m$ such that for no $\bar{t} \in dom(F)$ the condition $x_t = x_{\bar{t}}$ is valid, there exists such an element $u \in B_m$ that $F \cup \langle u, t \rangle$ is a partial isomorphism.

Proof. Let $t \in A_m$, $t_1, t_n, \ldots$ be all elements of dom(F). Examine all formulas $\varphi^{\mathcal{a}}(x_t, x_1, \ldots, x_n) \in SFL_{\mathcal{G}}$ which hold for x_t. Enumerate them $\varphi_1^{\mathcal{a}}, \ldots, \varphi_n^{\mathcal{a}}, \ldots$ (there is only a countable amount of them). Let us investigate formulas

(4) $((\exists x \in A)\, \varphi_i(x, x_1, \ldots, x_n))^{\mathcal{a}}$, $i \in FN$.

Since F is an isomorphism, we obtain from the validity of (4) that

(5) $((\exists y \in B)\, \varphi_i(y, y_1, \ldots, y_n))^{\mathcal{B}}$, $i \in FN$

hold. It follows from $\mathcal{G}$-saturation of $\mathcal{B}$ that there is $y \in B$ such that all formulas (5) hold for y. Denote u the code of y. Then $F \cup \langle u,t \rangle$ (where t is the code of x) is the partial isomorphism we looked for.

Proof of Theorem 2. Suppose that we have a well-ordering of the type Ω on A_m and B_m (see Lemma 3). The searched isomorphism will be constructed by the "zig-zag" method.

Let $\{F_\alpha\}_{\alpha\in\beta}$ (β is an ordinal number) be an increasing sequence of partial isomorphisms. Suppose, firstly, that β is a limit ordinal number. Let us construct $\bigcup\{F_\alpha\}_{\alpha\in\beta}$; this mapping is, according to Lemma 6, a partial isomorphism. Suppose, further, that β is an isolated ordinal number.

If β is odd ("step zig") then either $F_{\beta-1}$ is a total isomorphism and the proof is finished or $F_{\beta-1}$ is a partial isomorphism. In the second case, let t be the smallest element of A_m such that there is no $\tilde{t} \in dom(F_{\beta-1})$ that codes the same set as t. Then we can, in accordance with Lemma 7, prolong $F_{\beta-1}$ "by one step". Put $F = F_{\beta-1} \cup \langle u,t \rangle$, where $u \in B_m$ is such an element that F is a partial isomorphism.

If β is even ("step zag") nonlimit ordinal, then again either $F_{\beta-1}$ is a total isomorphism, which ends the proof, or $F_{\beta-1}$ is a partial isomorphism. Let then u be the smallest element of B_m such that there is no $\tilde{u} \in rng(F_{\beta-1})$ that codes the same set as u. Applying Lemma 7 on $(F_{\beta-1})^{-1}$ we obtain again a prolongation "by one step".

Provided the above construction stops on an ordinal, we obtain a total isomorphism. In the opposite case (i.e. if it goes cofinally to Ω) we have an increasing chain of partial isomorphisms. Their union - denote if F - is, however (see Lemma 6), a partial isomorphism. We show that F is even a total isomorphism.

Let e.g. $(x \in A)^{\mathfrak{A}}$. Suppose that there is no odd isolated ordinal β such that, for each code t_x, $t_x \notin \mathrm{dom}(F_{\beta-1})$. Then all odd isolated ordinals less than t_x form a countable sequence which is cofinal with Ω - a contradiction. Thus for each x, such that $(x \in A)^{\mathfrak{A}}$ there is $t_x \in \mathrm{dom}(F)$; similarly we can verify formula (3), from the definition of isomorphism, for elements from B. This concludes the proof.

In the last part of this paragraph we show that such an isomorphism which "transmits" formulas of the language $SFL_{\mathcal{S}}$ "transmits" - under certain conditions - all formulas of the language $FL_{\mathcal{S}}$, too. For this we need, however, to accept a demand. Realize that up to this time we have not used the fact that the relation I represents an equality on sets (more precisely, on codes for sets). We used, e.g. in the definition of isomorphism, only an implicitly introduced equality (defined by means of the relation E). In further considerations we shall ask for so called identity of both these equalities, i.e. for the validity of the following formula

$(*)\quad (\forall t,u \in A_m)\ t\ I\ u \equiv E''\{t\} = E''\{u\}$.

This requirement is in [S1] expressed by the notation $\mathfrak{A} \models (\mathrm{Ext})$.

Now we can prove the assertion about "the transmission" of I by means of isomorphism.

Lemma 8. Let in $\mathfrak{A} = \{A, E^{\mathfrak{A}}, I^{\mathfrak{A}}, \mathcal{S}^{\mathfrak{A}}\}$ and in $\mathfrak{B} = \{B, E^{\mathfrak{B}}, I^{\mathfrak{B}}, \mathcal{S}^{\mathfrak{B}}\}$ the condition $(*)$ holds. Let $F: \mathfrak{A} \precsim \mathfrak{B}$. Then

$(\forall t,u \in A_m)\ t\ I^{\mathfrak{A}}\ u \equiv F(t) I^{\mathfrak{B}} F(u)$.

Proof. Let $t,u \in A_m$ such that $t\ I^{\mathfrak{A}}\ u$. Then (owing to $(*)$) we have $(x_t = x_u)^{\mathfrak{A}}$. From the definition of isomorphism it follows that $(F(x_t) = F(x_u))^{\mathfrak{B}}$. Thus (again by means of $(*)$), we obtain $F(t)I^{\mathcal{B}}F(u)$. The converse implication follows from the fact that F^{-1} is an isomorphism.

Notice that - under the assumption of the validity of $(*)$ in $\mathcal{O}\!\iota$, $\mathcal{L}$ - the totality of $F: \mathcal{O}\!\iota \approx \mathcal{L}$ means that from each class of equivalence $I^{\mathcal{a}}$ ($(*)$ implies that relations $I^{\mathcal{a}}$, $I^{\mathcal{B}}$ are equivalences) at least one element falls into dom(F); analogously for $I^{\mathcal{B}}$ and rng(F). From this consideration it follows directly

Lemma 9. Let $F: \mathcal{O}\!\iota \approx \mathcal{L}$. Then
$(\forall X^{a})\ X^{a} = I^{a}{}''(F^{-1}(I^{\mathcal{B}}{}''(F''X^{a})))$.

Thus, if we put $Y^{\mathcal{B}} = I^{\mathcal{B}}{}''(F''X^{a})$ we obtain a sensible "transmission" of classes. Therefore we can extend the definition of isomorphism that was formulated above as a mapping between sets, also on classes. A concrete realization gives the next theorem

Theorem 3. Let $\mathcal{O}\!\iota = \{A, E^{a}, I^{a}, \mathcal{S}^{a}\}$, $\mathcal{L} = \{B, E^{\mathcal{B}}, I^{\mathcal{B}}, \mathcal{S}^{\mathcal{B}}\}$ be structures for the language $FL_{\mathcal{S}}$ satisfying $(*)$, $(Ext)^{a}$ and $(Ext)^{\mathcal{B}}$. Let $F: \mathcal{O}\!\iota \approx \mathcal{L}$. Let $\mathcal{S}'$ be an arbitrary system of classes in the sense of a and let $\mathcal{O}\!\iota'$, $\mathcal{L}'$ arise from $\mathcal{O}\!\iota$, $\mathcal{L}$, respectively, by the following expansion:

we interpret $X \in \mathcal{S}'$ in $\mathcal{O}\!\iota$ as X, in $\mathcal{L}$ as $I^{\mathcal{B}}{}''(F''X)$ (thus we put $X^{a} = X$, $X^{\mathcal{B}} = I^{\mathcal{B}}{}''(F''X)$).
Then $F: \mathcal{O}\!\iota' \approx \mathcal{L}'$.

At first we shall prove the assertion on "the transmission" of classes for sets and constants.

Lemma 10. Under the assumptions of Theorem 3 we have

(6) (a) $(\forall t \in A_m) I^{\mathcal{B}}{}''(F''(E^{a}{}''\{t\})) = E^{\mathcal{B}}{}''\{F(t)\}$.

(b) Let C be a constant for a class from $\mathcal{S}$. Then

(7) $$I^{\mathcal{B}}{}''(F''C^{a}) = C^{\mathcal{B}}.$$

Proof. (a) We have (owing to $(*)$)

$$u \in I^{\mathcal{B}''}(F''(E^{\mathcal{A}''}\{t\})) \equiv (\exists u_1 \in B)\,[(y_u = y_{u_1})^{\mathcal{B}} \,\&\, (x_{F^{-1}(u_1)} \in x_t)^{\mathcal{A}}].$$

From the definition of interpretation we obtain

$$u \in E^{\mathcal{B}''}\{F(t)\} \equiv (y_u \in y_{F(t)})^{\mathcal{B}}.$$

Now it is sufficient to realize that F is an isomorphism.

(b) The proof is easy and can be left to the reader.

Proof of Theorem 3 will be done by induction. At first we shall examine atomic formulas. Owing to Lemma 10 it suffices to restrict ourselves to cases $(X^{\mathcal{A}} = Y^{\mathcal{A}})^{\mathcal{A}}$ and $(X^{\mathcal{A}} \in Y^{\mathcal{A}})^{\mathcal{A}}$.

For proving

$$(X^{\mathcal{A}} = Y^{\mathcal{A}})^{\mathcal{A}} \equiv [\,I^{\mathcal{B}''}(F''\,X^{\mathcal{A}}) = I^{\mathcal{B}''}(F''\,Y^{\mathcal{A}})]^{\mathcal{B}}$$

use the remark behind Lemma 9.

Further we have to verify

$$(X^{\mathcal{A}} \in Y^{\mathcal{A}})^{\mathcal{A}} \equiv [I^{\mathcal{B}''}(F''\,X^{\mathcal{A}}) \in I^{\mathcal{B}''}(F''\,Y^{\mathcal{A}})]^{\mathcal{B}}.$$

We know that

$$(X^{\mathcal{A}} \in Y^{\mathcal{A}})^{\mathcal{A}} \equiv (\exists t \in Y^{\mathcal{A}})(X^{\mathcal{A}} = E^{\mathcal{A}''}\{t\}).$$

From totality of F and from the fact that $Y^{\mathcal{A}}$ is a class in the sense of $\mathcal{A}$ it follows that we can suppose $t \in \mathrm{dom}(F)$. Then (see Lemma 10) we obtain $I^{\mathcal{B}''}(F''\,X^{\mathcal{A}}) = E^{\mathcal{B}''}\{F(t)\}$. Moreover, $F(t) \in I^{\mathcal{B}''}(F''\,Y^{\mathcal{A}})$ since even $F(t) \in F''\,Y^{\mathcal{A}}$. The proof of the converse implication is analogous.

For connectives $\neg$, & is the proof obvious. Thus it remains to verify the theorem for the quantifier $\exists$ (binding sets - cf. the definition of isomorphism).

Let us consider the formula

$(\exists x)\,\varphi(x, x_1, \ldots, x_n, C_1^{\mathcal{A}}, \ldots, C_m^{\mathcal{A}}, X_1^{\mathcal{A}}, \ldots, X_k^{\mathcal{A}})$, where $X_i \in \mathcal{S}'$.

Denote $F(x_i) = y_{F(t_{x_i})}$, where t_{x_i} are codes for x_i ($i = 1, \ldots, n$). We have to prove

(8) $\quad (\exists x)\,\varphi(x, x_1, \ldots, x_n, C_1^{\mathcal{A}}, \ldots, C_m^{\mathcal{A}}, X_1^{\mathcal{A}}, \ldots, X_k^{\mathcal{A}})]^{\mathcal{A}} \equiv$

$\equiv [(\exists y)\varphi(y, F(x_1), \dots, F(x_n), C_1^{\mathcal{B}}, \dots, C_m^{\mathcal{B}}, X_1^{\mathcal{B}}, \dots, X_k^{\mathcal{B}})]^{\mathcal{B}}$.

For each x we get (by the induction hypothesis)

$$\varphi^{\mathcal{A}}(x, x_1, \dots, x_n, C_1^{\mathcal{A}}, \dots, C_m^{\mathcal{A}}, X_1^{\mathcal{A}}, \dots, X_k^{\mathcal{A}}) \equiv$$
$$\equiv \varphi^{\mathcal{B}}(F(x), F(x_1), \dots, F(x_n), C_1^{\mathcal{B}}, \dots, C_m^{\mathcal{B}}, X_1^{\mathcal{B}}, \dots, X_k^{\mathcal{B}}),$$

where $F(x) = y_{F(t_x)}$.

We shall prove $\Longrightarrow$ of (8). Suppose that the left-hand side of it holds. Then there is such x that $\varphi^{\mathcal{A}}(x, x_1, \dots, x_n, C_1, \dots$ $\dots, C_m, X_1, \dots, X_k)$ is fulfilled. Let t_x be the code of x. Since F is a total isomorphism and ($*$) holds in $\mathcal{A}$, we can assume that $t_x \in dom(F)$. Put $F(x) = y_{F(t_x)}$. In this way we have obtained the required element (lying in $\mathcal{B}$ and satisfying the right-hand side of (8)). For proving $\Longleftarrow$ realize that F^{-1} is a total isomorphism.

In the last part of this section we shall deal with the theorem on "a transmission" of formulas of the language $FL_{\mathcal{G}}$.

<u>Theorem 4</u>. Let $\mathcal{A} = \{A, E^{\mathcal{A}}, I^{\mathcal{A}}, \mathcal{G}^{\mathcal{A}}\}$, $\mathcal{B} = \{B, E^{\mathcal{B}}, I^{\mathcal{B}}, \mathcal{G}^{\mathcal{B}}\}$ be structures for the language $FL_{\mathcal{G}}$ satisfying ($*$), $(Ext)^{\mathcal{A}}$ and $(Ext)^{\mathcal{B}}$. Let $F: \mathcal{A} \approx \mathcal{B}$. Then for each formula $\varphi \in FL_{\mathcal{G}}$ it is true

$$\varphi^{\mathcal{A}}(x_1, \dots, x_n, X_1^{\mathcal{A}}, \dots, X_m^{\mathcal{A}}) \equiv \varphi^{\mathcal{B}}(F(x_1), \dots, F(x_n), (F''\ X_1)^{\mathcal{B}}, \dots, (F''\ X_m)^{\mathcal{B}}), \tag{9}$$

where we put $(F''\ X_i)^{\mathcal{B}} = I^{\mathcal{B}''}(F''\ X_i^{\mathcal{A}})$, $i = 1, \dots, m$.

<u>Proof</u>. Firstly we shall make an expansion of the language $FL_{\mathcal{G}}$ - we add constants for all classes $X^{\mathcal{A}}$. We shall interpret these constants in $\mathcal{A}$ as $X^{\mathcal{A}}$, in $\mathcal{B}$ as $I^{\mathcal{B}''}(F''\ X^{\mathcal{A}})$. Then each class $Y^{\mathcal{B}}$ is an interpretation of a constant (since F^{-1} is a total isomorphism).

The proof will be done by induction. Since for formulas of the language $NFL_{\mathcal{G}}$ is the validity of (9) an immediate consequence of Theorem 3, it is sufficient to verify the case $(\exists X)\varphi$, i.e.:

$$(10)\quad [(\exists X)\varphi(x_1,\dots,x_n,X,X_1^{\mathcal{A}},\dots,X_m^{\mathcal{A}})]^{\mathcal{A}} \equiv$$
$$\equiv [(\exists Y)\varphi(F(x_1),\dots,F(x_n),Y,(F''\,X_1)^{\mathcal{B}},\dots,(F''\,X_m)^{\mathcal{B}}]^{\mathcal{B}}.$$

By the induction hypothesis we know that for each X such that $Cls^{\mathcal{A}}(X)$ the formula

$$\varphi^{\mathcal{A}}(x_1,\dots,x_n,X^{\mathcal{A}},X_1^{\mathcal{A}},\dots,X_m^{\mathcal{A}}) \equiv \varphi^{\mathcal{B}}(F(x_1),\dots,F(x_n),(F''\,X)^{\mathcal{B}},(F''\,X_1)^{\mathcal{B}},\dots,(F''\,X_m)^{\mathcal{B}})$$

holds.

For proving $\Rightarrow$ in (10) it suffices to realize that if X is a class in the sense of $\mathcal{A}$, then the class which corresponds to X, i.e. $(F''\,X)^{\mathcal{B}}$, is a class in the sense of $\mathcal{B}$ (as F is a total isomorphism). The implication $\Leftarrow$ can be proved similarly by means of the fact that F^{-1} is a total isomorphism. This completes the proof.

§ 2. In this section we shall construct, from a given system of structures, a new structure - namely such a one that each starting structure will be immersed inside by means of a suitable embedding. At the same time we shall suppose that in each initial structure the axiom of extensionality and ($*$) are valid (then relations of identity will be equivalences).

For a more simple notation we shall write further instead of $\mathfrak{A}_\alpha = \{A_\alpha, E^{\mathcal{A}_\alpha}, I^{\mathcal{A}_\alpha}, \mathcal{S}^{\mathcal{A}_\alpha}\}$ only $\mathfrak{A}_\alpha = \{A_\alpha, E_\alpha, I_\alpha, \mathcal{S}_\alpha\}$.

<u>A coded system of structures</u> $\mathfrak{A}_\alpha = \{A_\alpha, E_\alpha, I_\alpha, \mathcal{S}_\alpha\}$ of the language $FL_{\mathcal{S}}$ (with interpretations $\mathcal{A}_\alpha$) - let us denote it $\{\mathfrak{A}_\alpha ; \alpha \in K\}$, where K is a coding class - consists of coded systems of supports $\{A_\alpha ; \alpha \in K\}$, extensions $\{E_\alpha ; \alpha \in K\}$, identities $\{I_\alpha ; \alpha \in K\}$ and for each $C \in \mathcal{S}$ of a coded system (for the same coding class K) of interpretations $C^{\mathcal{A}_\alpha}$ of C.

A coded <u>system of structures</u> $\{\mathfrak{A}_\alpha ; \alpha \in K\}$ will be called <u>elementary</u> iff

1) K is a directed class

2) $(\forall \alpha, \beta \in K)(\alpha \leq \beta \Rightarrow (\exists ! F)(F: \mathfrak{A}_\alpha \precsim \mathfrak{A}_\beta))$;

this F we shall denote $F_{\alpha,\beta}$

3) $(\forall \alpha, \beta, \gamma \in K)(\alpha \leq \beta \leq \gamma \Rightarrow F_{\alpha,\beta} \circ F_{\beta,\gamma} = F_{\alpha,\gamma})$.

Let us note that 2) ensures the existence of a coded system of embeddings. Note, moreover, that 3) can be formulated (apart from identity) more generally in this way:

$$(\forall t \in dom(F_{\alpha,\beta}))(\exists t_1 \in A_\alpha)(\exists u \in A_\beta)(t \; I_\alpha \; t_1 \,\&\, u \; I_\beta \; F_{\alpha,\beta}(t) \,\&\, \& \; F_{\alpha,\gamma}(t_1) I_\gamma F_{\beta,\gamma}(u)).$$

Let $\{\mathfrak{A}_\alpha ; \alpha \in K\}$ be an elementary system of structures. The symbol $\lim \{\mathfrak{A}_\alpha ; \alpha \in K\}$ (briefly <u>$\lim \mathfrak{A}_\alpha$</u>) will denote a structure $\{A_{lim}, E_{lim}, I_{lim}, \mathcal{S}_{lim}\}$, where

1) A_{lim} is such a relation that $dom(A_{lim}) = K$ and for each $\alpha \in K$ the condition $A_{lim}'' \{\alpha\} = A_\alpha$ holds.

2) $\langle x,\alpha\rangle \; I_{lim} \; \langle y,\beta\rangle \equiv [(\exists \gamma)(\gamma \geq \alpha,\beta) \,\&$
$\&(\exists \langle x_1,\alpha\rangle, \langle y_1,\beta\rangle)(x_1 I_\alpha x \,\&\, y_1 I_\beta y \,\&\, F_{\alpha,\gamma}(x_1) I_\gamma F_{\beta,\gamma}(y_1)]$

3) $\langle x,\alpha\rangle \; E_{lim} \; \langle y,\beta\rangle \equiv [(\exists \gamma)(\gamma \geq \alpha,\beta) \,\&$
$\&(\exists \langle x_1,\alpha\rangle, \langle y_1,\beta\rangle)(x_1 I_\alpha x \,\&\, y_1 I_\beta y \,\&\, F_{\alpha,\gamma}(x_1) E_\gamma F_{\beta,\gamma}(y_1)]$.

4) For $C \in \mathcal{S}$ we define $C^{a_{lim}}$ as such a relation that $dom(C^{a_{lim}}) = K$ and $(\forall \alpha \in K)(C^{a_{lim}''}\{\alpha\} = C^{a_\alpha})$ holds.

The interpretation determined by $\lim \mathfrak{A}_\alpha$ will be denoted a_{lim}.

Realize that $A_{lim}, E_{lim}, I_{lim}$ and $C^{a_{lim}}$ are relations and that they are described from $\{\mathfrak{A}_\alpha ; \alpha \in K\}$ by normal formulas.

Before investigating the structure $\lim \mathfrak{A}_\alpha$, we shall prove several auxiliary assertions.

<u>Lemma 1</u>. Denote $\varphi \sim (\bar{x} \in dom(F_{\alpha,\gamma}) \,\&\, \bar{y} \in dom(F_{\beta,\gamma}) \,\&\, \bar{x} \; I_\alpha \; x \,\&$

$\&\bar{y}\ I_\beta\ y)$. Then

a) the following are equivalent:

(1) $\langle x,\alpha\rangle\ I_{lim}\ \langle y,\beta\rangle$

(2) $(\forall\gamma\geq\alpha,\beta)(\exists\bar{x},\bar{y})(\varphi\ \&\ F_{\alpha,\gamma}(\bar{x})\ I_\gamma\ F_{\beta,\gamma}(\bar{y}))$

(3) $(\forall\gamma\geq\alpha,\beta)(\forall\bar{x},\bar{y})(\varphi\Rightarrow F_{\alpha,\gamma}(\bar{x})\ I_\gamma\ F_{\beta,\gamma}(\bar{y}))$

b) the following are equivalent:

(4) $\langle x,\alpha\rangle\ E_{lim}\ \langle y,\beta\rangle$

(5) $(\forall\gamma\geq\alpha,\beta)(\exists\bar{x},\bar{y})(\varphi\ \&\ F_{\alpha,\gamma}(\bar{x})E_\gamma\ F_{\beta,\gamma}(\bar{y}))$

(6) $(\forall\gamma\geq\alpha,\beta)(\forall\bar{x},\bar{y})(\varphi\Rightarrow F_{\alpha,\gamma}(\bar{x})E_\gamma\ F_{\beta,\gamma}(\bar{y}))$.

Proof. We shall prove the statement a). The implication (3) $\Rightarrow$ (1) is obvious. For proving (1) $\Rightarrow$ (2) assume (1). Then there are $\delta\geq\alpha,\beta$, $x_1\in\mathrm{dom}(F_{\alpha,\delta})$ and $y_1\in\mathrm{dom}(F_{\beta,\delta})$ such that $x_1 I_\alpha\ x, y_1 I_\beta y$ and $F_{\alpha,\delta}(x_1)I_\delta\ F_{\beta,\delta}(y_1)$. Put $\gamma\geq\alpha,\beta$ (an arbitrary fixed element). Since $F_{\alpha,\gamma}$ is an embedding, it follows from totality of $F_{\alpha,\gamma}$ that to x there is $\bar{x}$ such that $x\ I_\alpha\ \bar{x}$ and analogously to y there is $\bar{y}$ for which $y\ I_\beta\ \bar{y}$. We show that $\bar{x}$, $\bar{y}$ are the required elements. Denote

$F_{\alpha,\delta}(x_1) = x_1^\delta, F_{\beta,\delta}(y_1) = y_1^\delta, F_{\alpha,\gamma}(\bar{x}) = \bar{x}^\gamma,\ F_{\beta,\gamma}(\bar{y}) = \bar{y}^\gamma$.

Let $\varepsilon\geq\delta,\gamma$. Then (owing to the embedding of structures) we have

(7) $F_{\delta,\varepsilon}(x_1^\delta)I_\varepsilon\ F_{\delta,\varepsilon}(y_1^\delta)$.

We would like to prove

(8) $F_{\gamma,\varepsilon}(\bar{x}^\gamma)I_\varepsilon\ F_{\gamma,\varepsilon}(\bar{y}^\gamma)$.

It follows from symmetry of I_α, I_β that $\bar{x}\ I_\alpha\ x$ and $\bar{y}\ I_\beta\ y$ and from commutativity of embeddings, we receive

(9) $F_{\gamma,\varepsilon}(\bar{x}^\gamma)I_\varepsilon\ F_{\delta,\varepsilon}(x_1^\delta)\ \&\ F_{\gamma,\varepsilon}(\bar{y}^\gamma)I_\varepsilon\ F_{\delta,\varepsilon}(y_1^\delta)$.

From (7),(9) and transitivity of I_ε we obtain then (8).

Substituting now in (8) $F_{\alpha,\gamma}(\bar{x})$, $F_{\beta,\gamma}(\bar{y})$ for $\bar{x}^\gamma$, $\bar{y}^\gamma$ we get $F_{\gamma,\varepsilon}(F_{\alpha,\gamma}(\bar{x}))I_\varepsilon\ F_{\gamma,\varepsilon}(F_{\beta,\gamma}(\bar{y}))$ which implies $F_{\alpha,\gamma}(\bar{x})I_\gamma\ F_{\beta,\gamma}(\bar{y})$.

For proving (2) $\Rightarrow$ (3) let us suppose (2). Then to x, y there exist x_1, y_1, respectively, such that $x_1 I_\alpha x$, $y_1 I_\beta y$, $x_1 \in \mathrm{dom}(F_{\alpha,\gamma})$, $y_1 \in \mathrm{dom}(F_{\beta,\gamma})$ and $F_{\alpha,\gamma}(x_1) I_\gamma F_{\beta,\gamma}(y_1)$. Let further $\bar{x}$, $\bar{y}$ be arbitrary chosen elements for which the assumption of (3) holds. Since I_α, I_β are equivalences, we have $x_1 I_\alpha \bar{x}$ and $y_1 I_\beta \bar{y}$. But $F_{\alpha,\gamma}$ and $F_{\beta,\gamma}$ are embeddings; therefore $F_{\alpha,\gamma}(\bar{x}) I_\gamma F_{\alpha,\gamma}(x_1) I_\gamma F_{\alpha,\gamma}(y_1) I_\gamma F_{\beta,\gamma}(\bar{y})$.

This implies (when using transitivity of I_γ) that $F_{\alpha,\gamma}(\bar{x}) I_\gamma F_{\beta,\gamma}(\bar{y})$. This concludes the proof of a).

The statement b) can be proved similarly (only instead of transitivity of identities, we have to use (*)).

The following statement is an immediate corollary of Lemma 1.

(10) $x \in \mathrm{dom}(F_{\alpha,\gamma}) \Rightarrow \langle x,\alpha\rangle\ I_{lim} \langle F_{\alpha,\gamma}(x),\gamma\rangle$.

Lemma 2.

a) $\langle t,\alpha\rangle\ I_{lim} \langle u,\alpha\rangle \equiv t\ I_\alpha\ u$

b) $\langle t,\alpha\rangle\ E_{lim} \langle u,\alpha\rangle \equiv t\ E_\alpha\ u$

c) $\langle t,\alpha\rangle\ C^{a_{lim}} \equiv t \in C^{a_\alpha}$.

Proof. a) The implication $\Leftarrow$ is obvious (see the definition of I_{lim}). Suppose the left-hand side of a) is valid. Then $(\exists \gamma > \alpha)(\exists t_1,u_1)(t_1 I_\alpha t\ \&\ u_1 I_\alpha u\ \&\ F_{\alpha,\gamma}(t_1) I_\gamma F_{\alpha,\gamma}(u_1))$. Hence (owing to (*) and Lemma 8, § 1) we have $\langle t_1,u_1\rangle \in I_\alpha$ and since I_α is an equivalence, we receive $t\ I_\alpha\ u$.

The statement b) can be proved - using (*) and Lemma 5, § 1 - similarly. For proving c) remind the definition of $C^{a_{lim}}$.

Lemma 3. I_{lim} is an equivalence

Proof. The assertion follows directly from Lemma 1.

Lemma 4.

a) $[\langle x,\alpha\rangle\ I_{lim} \langle y,\beta\rangle\ \&\ \langle x,\alpha\rangle\ E_{lim} \langle z,\gamma\rangle] \rightarrow \langle y,\beta\rangle\ E_{lim} \langle z,\gamma\rangle$

b) $[\langle z,\gamma\rangle I_{lim} \langle y,\beta\rangle \,\&\, \langle x,\alpha\rangle E_{lim} \langle z,\gamma\rangle] \Rightarrow \langle x,\alpha\rangle E_{lim} \langle y,\beta\rangle$.

Proof. a) Let the assumptions of a) hold. Let $\delta \geq \alpha, \beta, \gamma$. Choose $\bar{x} \in \mathrm{dom}(F_{\alpha,\delta})$, $\bar{y} \in \mathrm{dom}(F_{\beta,\delta})$, $\bar{z} \in \mathrm{dom}(F_{\gamma,\delta})$ in such a way that $\bar{x}\, I_\alpha\, x$, $\bar{y}\, I_\beta\, y$ and $\bar{z}\, I_\gamma\, z$. Then (see Lemma 1 a),b)) $F_{\alpha,\delta}(\bar{x}) I_\delta F_{\beta,\delta}(\bar{y})$ and $F_{\alpha,\delta}(\bar{x}) E_\delta F_{\gamma,\delta}(\bar{z})$. Since $\mathcal{O}_\delta$ satisfies ($*$), we obtain $F_{\beta,\delta}(\bar{y}) E_\delta F_{\gamma,\delta}(\bar{z})$. Now apply the definition of E_{lim}. The proof of b) is analogous.

Theorem 1.

1) $\lim \mathcal{O}_\alpha$ is a structure of the language FL_φ

2) $\lim \mathcal{O}_\alpha$ satisfies ($*$)

3) $\lim \mathcal{O}_\alpha$ satisfies $(\mathrm{Ext})^{a_{lim}}$.

Proof. For proving 1) it suffices to verify

(11) $[\langle x,\alpha\rangle I_{lim} \langle y,\beta\rangle \,\&\, \langle x,\alpha\rangle \in C^{a_{lim}}] \Rightarrow \langle y,\beta\rangle \in C^{a_{lim}}$

(i.e. saturation of interpretations of constants w.r.t. I_{lim}). Suppose the assumptions of (11) hold. Then there is $\gamma \geq \alpha, \beta$. Without loss of generality we can call for $x \in \mathrm{dom}(F_{\alpha,\gamma})$, $y \in \mathrm{dom}(F_{\beta,\gamma})$. Then $F_{\alpha,\gamma}(x) \in C^{a_\gamma}$. It follows from the definition of $\lim \mathcal{O}_\alpha$ that $\langle F_{\alpha,\gamma}(x), F_{\beta,\gamma}(y)\rangle \in I_\gamma$. Since $\mathrm{Cls}^{a_\gamma}(C^{a_\gamma})$ we have also $F_{\beta,\gamma}(y) \in C^{a_\gamma}$. But $F_{\beta,\gamma}$ is an isomorphism, hence (see Lemma 4, § 1) $y \in C^{a_\beta}$. Now it is sufficient to realize the definition of $C^{a_{lim}}$.

As to the statement 2), the implication

$\langle x,\alpha\rangle I_{lim} \langle y,\beta\rangle \Rightarrow E''_{lim}\{\langle x,\alpha\rangle\} = E''_{lim}\{\langle y,\beta\rangle\}$

is exactly the assertion b) from Lemma 4. Let us prove now the converse implication. Suppose $E''_{lim}\{\langle x,\alpha\rangle\} = E''_{lim}\{\langle y,\beta\rangle\}$. Let $\gamma \geq \alpha, \beta$. Due to Lemma 4, ($*$) and in accordance with the definition of isomorphism we may suppose $x \in \mathrm{dom}(F_{\alpha,\gamma})$ and $y \in \mathrm{dom}(F_{\beta,\gamma})$. As $\langle F_{\alpha,\gamma}(x), \gamma\rangle I_{lim} \langle x,\alpha\rangle$ and

$\langle F_{\beta,\gamma}(y),\gamma\rangle I_{lim}\langle y,\beta\rangle$ we obtain (see Lemma 4)
$E''_{lim}\{\langle F_{\beta,\gamma}(y),\gamma\rangle\} = E''_{lim}\{\langle F_{\alpha,\gamma}(x),\gamma\rangle\}$ and hence, according to Lemma 2, $E''_{\gamma}\{F_{\alpha,\gamma}(x)\} = E''_{\gamma}\{F_{\beta,\gamma}(y)\}$. In $\mathcal{O}\!\iota_{\gamma}$ is, however, $(*)$ valid; therefore $F_{\alpha,\gamma}(x) I_{\gamma} F_{\beta,\gamma}(y)$. Lemma 2 and transitivity of I_{lim} imply then $\langle x,\alpha\rangle I_{lim}\langle y,\beta\rangle$.

To prove 3) we have to verify

$$(12)\quad [(\forall X,Y)(X = Y \equiv (\forall z)(z\in X \equiv z\in Y)]^{\mathfrak{a}_{lim}}.$$

Let X, Y be chosen in such a way that they are classes in the sense of $\mathfrak{a}_{lim}$ - denote them $X^{\mathfrak{a}_{lim}}$, $Y^{\mathfrak{a}_{lim}}$. Then (see the definition of interpretation) we can reformulate (12):

$$(13)\quad (X^{\mathfrak{a}_{lim}} = Y^{\mathfrak{a}_{lim}})^{\mathfrak{a}_{lim}} \equiv [(\forall z)(z\in X^{\mathfrak{a}_{lim}} \equiv z\in Y^{\mathfrak{a}_{lim}})]^{\mathfrak{a}_{lim}}.$$

As = is absolute, it suffices to prove only

$$(\forall u \in A_{lim})\mathrm{Cls}^{\mathfrak{a}_{lim}}(E''_{lim}\{u\}),$$

but this is the consequence of Lemma 4. This completes the proof.

We shall show now that lim $\mathcal{O}\!\iota_{\alpha}$ is an elementary extension of all "preceding" structures .

<u>Theorem 2</u>. $(\forall\alpha \in K)\ \mathcal{O}\!\iota_{\alpha} \precsim \lim \mathcal{O}\!\iota_{\alpha}$.

<u>Proof</u>. Let α be a fixed, arbitrarily chosen, element of K. Define a mapping $F_{\alpha}: A_{\alpha} \longrightarrow A_{lim}$ as follows:

$$(\forall x \in A_{\alpha})F_{\alpha}(x) = \langle x,\alpha\rangle .$$

At first we shall prove that F_{α} is a total isomorphism $\mathcal{O}\!\iota_{\alpha}$ onto $\overline{\mathcal{O}\!\iota}_{\alpha}$, where $\overline{\mathcal{O}\!\iota}_{\alpha} = \lim \mathcal{O}\!\iota_{\alpha} \wedge \{\langle x,\alpha\rangle ; x\in A_{\alpha}\}$ (i.e. $\overline{\mathcal{O}\!\iota}_{\alpha}$ is a structure which arises from lim $\mathcal{O}\!\iota_{\alpha}$ by the restriction on elements with index α).

The validity of formula (2), from the definition of isomorphism, for F_{α} is, when we bear in mind atomic formulas with the predicate $\in$, an obvious **consequence of Lemma** 2. This lemma also implies that $\overline{\mathfrak{A}}_\alpha$ satisfies ($*$) and axiom of extensionality. Therefore (2) holds also for atomic formulas with the predicate = . When proving the fact that F_α is a one-one mapping and total, remind that $\mathfrak{A}_\alpha$ and $\overline{\mathfrak{A}}_\alpha$ satisfy ($*$) and use again Lemma 2. Thus $F_\alpha : \mathfrak{A}_\alpha \approx \overline{\mathfrak{A}}_\alpha$.

Further we shall prove that $\overline{\mathfrak{A}}_\alpha$ is an elementary substructure of lim $\mathfrak{A}_\alpha$. Since it is evident that $\overline{\mathfrak{A}}_\alpha$ is a substructure of lim $\mathfrak{A}_\alpha$, it remains to verify for each $\varphi \in SFL_{\mathcal{S}}$

(15) $$\varphi^{\overline{\mathfrak{A}}_\alpha} \equiv \varphi^{\mathfrak{A}_{\lim}}.$$

The validity of (15) for clopen formulas of the language $SFL_{\mathcal{S}\cup\overline{A}_\alpha}$ follows from the definition of substructure. Further we shall examine only the nontrivial step of induction and namely that one concerning the existential quantifier. We shall show that

(16) $$((\exists x)\varphi)^{\mathfrak{A}_{\lim}} \Rightarrow ((\exists x)\varphi)^{\overline{\mathfrak{A}}_\alpha}$$

(the converse implication is obvious).

Let $\langle t,\beta\rangle$ be such a couple that $(\varphi(x|\langle t,\beta\rangle))^{\mathfrak{A}_{\lim}}$ and let $\langle u_1,\alpha\rangle, \dots, \langle u_k,\alpha\rangle$ be all individuals in φ . Put $\gamma \geq \alpha, \beta$. Since (see formula (10)) for i = 1,...,k $\langle F_{\alpha,\gamma}(u_i),\gamma\rangle \; I_{\lim} \langle u_i,\alpha\rangle$ and $\langle F_{\beta,\gamma}(t),\gamma\rangle \; I_{\lim} \langle t,\beta\rangle$, we have (owing to validity of ($*$) in lim $\mathfrak{A}_\alpha$) that $(\varphi(\langle F_{\beta,\gamma}(t),\gamma\rangle, \langle F_{\alpha,\gamma}(u_i),\gamma\rangle))^{\mathfrak{A}_{\lim}}$ holds. By induction hypothesis we know (for i = 1,...,k) that $(\varphi(\langle F_{\beta,\gamma}(t),\gamma\rangle, \langle F_{\alpha,\gamma}(u_i),\gamma\rangle))^{\overline{\mathfrak{A}}_\gamma}$ is valid. Hence (using the definition of interpretation) we obtain

(17) $$((\exists x)\varphi(x, \langle F_{\alpha,\gamma}(u_i),\gamma\rangle))^{\overline{\mathfrak{A}}_\gamma}$$

Since $\mathfrak{A}_\gamma$ and $\overline{\mathfrak{A}}_\gamma$ are isomorphic structures, we receive from (17) that it is true $(\exists x)\varphi(x, \langle F_{\alpha,\gamma}(u_i),\gamma\rangle)^{\mathfrak{A}_\gamma}$. But

$F_{\alpha,\gamma}$ is an embedding; therefore the formula $((\exists x)\varphi(x,u_i))^{\mathcal{U}_\alpha}$ holds. Recall now that $\mathcal{U}_\alpha \approx \overline{\mathcal{U}}_\alpha$. This concludes the proof.

The next theorem shows under what assumptions the structure $\lim \mathcal{U}_\alpha$ is $\mathcal{S}$-saturated.

Theorem 3. Let $\mathcal{S}$ be a countable system of classes. Let the coding class K satisfy the following condition: for each countable subclass K_0 of K it is true

$$(\exists \eta \in K)(\forall \delta \in K_0)\ \delta \leq \eta$$

(where K is directed by " $\leq$ "). Let, moreover, the formula

(18) $(\forall \alpha \in K)(\exists \beta \in K)(\beta > \alpha \ \&\ \mathcal{U}_\beta$ is $\mathcal{S}$-saturated)

hold. Then $\lim \mathcal{U}_\alpha$ is an $\mathcal{S}$-saturated structure.

Proof. Let $\{\varphi_n; n \in FN\}$ be a consistent sequence of formulas. They contain only a countable amount of constants of the form $\langle t_i, \alpha_i \rangle$, where $t_i \in A_{\alpha_i}$ (more precisely $t_i \in (A_{\alpha_i})_m$). The assumption on K_0 implies that there is γ such that for each α_i it is true $\gamma \geq \alpha_i$. The formula (18) asserts the existence of $\delta > \gamma$ for which $\mathcal{U}_\delta$ is $\mathcal{S}$-saturated.

Since $\{A_\alpha; \alpha \in K\}$ is an elementary system of structures, we have that $F_{\alpha_i,\delta}(t_i) \in A_\delta$ (see point 2) of the definition) for each α_i. It follows from the definition of I_{lim} that $\langle t_i, \alpha_i \rangle\ I_{lim} \langle F_{\alpha_i,\delta}(t_i), \delta \rangle$. We can therefore assume that all $\langle t_i, \alpha_i \rangle$ belong to A_δ. But $\mathcal{U}_\delta$ is $\mathcal{S}$-saturated and $\mathcal{U}_\delta \precsim \lim \mathcal{U}_\alpha$. Hence there exists such $y \in A_\delta$ that fulfils, at the same time, all formulas of our consistent sequence $\{\varphi_n; n \in FN\}$. From elementary embedding it follows that $\langle y, \delta \rangle$ is the required element (which fulfils all these formulas in $\lim \mathcal{U}_\alpha$). This completes the proof.

Let us still note that if the relation " $\leq$ " on K is a restriction of an Sd-relation, then the assumptions of Theorem 3 can be reformulated into a weaker form. Then it is namely sufficient to require for K to be directed by " $\not\leq$ " and not to be cofinal with any of its countable subclasses - see [V], ch. I, § 4

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