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ZEROS FOR STRONGLY ACCRETIVE SET-VALUED MAPPINGS

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Abstract: Let D be an open subset of a Banach space X, and let B(X) denote the family of all nonempty, bounded and closed subsets of X. Suppose $T:D \longrightarrow B(X)$ is a continuous (with respect to the Hausdorff metric) and strongly accretive mapping. It is shown that if for some $z \in D: t(x - z) \notin T(x)$ for x in the boundary of D and $t < 0$, it is sufficient to guarantee that T has a zero in $\bar{D}$. Several implications of this result are considered, particularly on a lccalized version of it.

Key words and phrases: Strongly accretive mappings, locally c-strongly accretive mappings, zeros.

Classification: 47H10

Let X be a Banach space, D a nonempty subset of X, and let B(X) denote the family of all nonempty, bounded and closed subsets of X supplied with the Hausdorff metric H (defined below). A mapping $T:D \to B(X)$ is said to be strongly accretive if for some $k < 1$ and for each $x,y \in D$, $u \in T(x), v \in T(y)$:

$$(1) \qquad (\lambda - k)\|x - y\| \leq \|(\lambda - 1)(x - y) + u - v\|$$

for all $\lambda > k$; while T is said to be accretive if (1) holds for $k=1$. This latter class was introduced independently in 1967 by F.E.Browder [2] and by T.Kato [6] and their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces is well-known (see, for example, [3], [4], [6] or [11]). The theory of accretive operators has been closely related with the existence of fixed points for nonexpansive mappings, which is clearly reflected by the fact that T is accretive if and only if the mapping I - T is pseudo-contractive, a class of mapping which, in the single-value case, includes all nonexpansive mappings.

In a recent paper [12], the author showed the existence of a unique fixed point for strongly pseudo-contractive mappings (a much wider class than contractions) under a condition weaker than the Leray-Schauder type, introduced by Kirk-Morales [8]. Particularly it can be derived the following result from Theorem 1 of [12].

Theorem M. Let X be a Banach space, D an open subset of X, and T a continuous strongly accretive mapping from $\overline{D}$ into X satisfying for some $z \in D$:

$$T(x) \neq t(x - z) \text{ for } x \in \partial D \text{ and } t < 0.$$

Then T has a unique zero in $\overline{D}$.

Theorem M has been used (see [9]) to obtain a number of results concerning the existence of zeros for continuous and accretive single-valued mappings. In view of this, it appears to be important to investigate whether or not the above result holds for set-valued mappings. In fact, we are able to answer this question positively in Theorem 1. Our approach relies on ideas already developed in [14] for single-valued mappings, combined with a recent theorem of Kirk [7] (see below). In the interest of attaining a certain degree of generality, we study a localized version of Theorem 1 via refining arguments of Kirk and the author in [9] and [13]. We also obtain some consequences of the main result which improve the recent theorems of Downing [5]. Finally we obtain a domain invariance theorem for the class of mappings so-called c-strongly accretive.

Theorem K (Kirk,[7]). Let X be a Banach space and D an open subset of X. Suppose $T:D \longrightarrow B(X)$ is continuous (relative to the Hausdorff metric) and strongly accretive. Then $T(D)$ is open in X.

Throughout this paper we use $\bar{D}$ and ∂D to denote, respectively, the closure and the boundary of D, and for $u,v \in X$ we use $seg[u,v]$ to denote the segment $\{tu + (1 - t)v : t \in [0,1]\}$. Also, for a subset A of X, we use $|A|$ to denote $\inf\{\|x\| : x \in A\}$. Finally, for a Banach space X, the mapping $J:X \longrightarrow 2^{X^*}$ denotes the usual normalized duality mapping:

$$J(x) = \{j \in X^* : \|j\| = \|x\|, \langle x,j\rangle = \|x\|^2\}.$$

Following Assad and Kirk [1] we define the Hausdorff metric H as follows: if $r > 0$ and $E \in B(X)$, let

$$V_r(E) = \{x \in X : dist(x,E) < r\}.$$

Then for $A,B \in B(X)$ we define

$$H(A,B) = \inf\{r : A \subset V_r(B) \text{ and } B \subset V_r(A)\}.$$

We shall also make use of the following lemma, which is noted in [1].

Lemma 1. If $A,B \in B(X)$ and $x \in A$, then for each positive number α there exists $y \in B$ such that

$$\|x - y\| \leq H(A,B) + \alpha .$$

In what follows we shall frequently appeal to the following facts.

Lemma 2. Let D be a subset of a Banach space X with $0 \in D$, and let $T:D \longrightarrow B(X)$ be a strongly accretive mapping. Then:

(i) the set $E = \{x \in D : tx \in T(x) \text{ for some } t < 0\}$ is bounded.

(ii) If $\{x_n - u_n\}$ is a bounded sequence in X for $u_n \in T(x_n)$, $t_n \to t$ with $t_n \in [0,1]$, and $z_n = (1 - t_n)x_n + t_n u_n \to y$, then $\{x_n\}$ is a Cauchy sequence.

Proof. (i) Let $tx \in T(x)$ for some $t < 0$. Select $u \in T(x)$ such that $tx = u$ and thus (1) implies

$$(1 - t - k)\|x\| \leq \| - tx + u - v\| = \|v\|$$

for all $v \in T(0)$. Since $t < 0$, it follows that

$$\|x\| \leq |T(0)|/(1 - k).$$

(ii) Let $u_n \in T(x_n)$. Then by choosing $\lambda = t_n^{-1}$ in (1) we obtain

$$(t_n^{-1} - k)\|x_n - x_m\| \leq \|(t_n^{-1} - 1)(x_n - x_m) + u_n - u_m\|$$

yielding

$$(1 - t_n k)\|x_n - x_m\| \leq \|(1 - t_n)(x_n - x_m) + t_n(u_n - u_m)\|$$

$$\leq \|z_n - z_m\| + |t_n - t_m| \, \|x_m - u_m\| .$$

Therefore $\{x_n\}$ is a Cauchy sequence.

Lemma 3. Let C be a closed subset of a Banach space X and let $T: C \to B(X)$ be continuous. Suppose $h_t(x) = (1 - t)x + tT(x)$ for $t \in (0,1]$ and $z_n \in h_{t_n}(x_n)$ where $z_n \to z$, $t_n \to t_0 > 0$ and $x_n \to x_0$. Then $z \in h_{t_0}(x_0)$.

Proof. Let $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that

(2) $\qquad H(T(x_n), T(x)) < \varepsilon/2t_0$ for all $n \geq N$.

Since $z_n \in h_{t_n}(x_n)$, we may choose $u_n \in T(x_n)$ so that $z_n = (1 - t_n)x_n + t_n u_n$. Moreover, by Lemma 1, we may select $v_n \in T(x_0)$ satisfying

(3) $\qquad u_n - v_n = H(T(x_n), T(x_0)) + \varepsilon/2t_0$.

Let $w_n = (1 - t_0)x_0 + t_0 v_n$ for each n, then

$$\|z_n - w_n\| = \|(1 - t_n)x_n + t_n u_n - [(1 - t_0)x_0 + t_0 v_n]\|$$
$$\le |1 - t_n|\,\|x_n - x_0\| + |t_0 - t_n|\,\|x_0 - u_n\| + t_0\,\|u_n - v_n\|.$$

By making use of (2) and (3), we get

(4) $\|z_n - w_n\| \le |1 - t_n|\,\|x_n - x_0\| + |t_0 - t_n|\,\|x_0 - u_n\| + \varepsilon$

for all $n \ge N$. By letting $n \to \infty$ in (4) and observing that $\{u_n\}$ is bounded, we conclude

$$\limsup_{n\to\infty} \|w_n - z\| \le \varepsilon .$$

Since ε is arbitrary and $w_n \in h_{t_0}(x_0)$ for all n, the sequence $\{w_n\}$ converges to z, hence $z \in h_{t_0}(x_0)$.

We begin with a special case of our main result.

<u>Proposition 1</u>. Let X be a Banach space, D an open subset of X, and let $T:\overline{D} \to B(X)$ be a continuous and strongly accretive mapping. Suppose that T maps bounded sets into bounded sets and satisfies for some $z \in D$:

(5) $t(x - z) \notin T(x)$ for $x \in \partial D$ and $t < 0$.

Then $0 \in T(\overline{D})$.

<u>Proof</u>. By translating T and D, we may take $z = 0$ in (5). Since the set E (defined in Lemma 2) is bounded, there is no loss of generality in assuming D is bounded.

Let $h_t:\overline{D} \to B(X)$ be defined by $h_t(x) = (1 - t)x + tT(x)$ for each $t \in [0,1]$, and let

$$M = \{t \in [0,1] : 0 \in h_t(x) \text{ for some } x \in D\}.$$

We first observe that $M \ne \emptyset$ (since $0 \in M$). Now we shall show that $\sup M = 1$. To see this, let $\{t_n\}$ be a sequence of M with $t_n \to t$ as $n \to \infty$. Then, for each n, there exists $x_n \in D$ so that $0 \in h_{t_n}(x_n)$. This means, we may select $u_n \in T(x_n)$ for which

$(1 - t_n)x_n + t_n u_n = 0$, implying that $\{x_n\}$ is a Cauchy sequence (by Lemma 2 ii). Hence $x_n \longrightarrow x \in \overline{D}$ and thus by Lemma 3 we conclude that $0 \in (1 - t)x + tT(x)$ and by (5) $x \in D$. Therefore M is closed in $[0,1]$.

Assume now that M is not open. Then there exists $t \in M$ and a sequence $\{t_n\}$ in $[0,1]$ for which $t_n \notin M$ and $t_n \longrightarrow t$. Let $0 \in h_t(x)$ for some $x \in D$ and let $u \in T(x)$ such that $(1 - t)x + tu = 0$. Suppose B is an open ball centered at x contained in D. If we define $y_n = (1 - t_n)x + t_n u$ for each $n \in \mathbb{N}$ then

$$y_n \in h_{t_n}(x) \subset h_{t_n}(B)$$

while $0 \notin h_{t_n}(B)$, which implies the existence of $u_n \in \text{seg}\,[0,y_n] \cap \partial h_{t_n}(B)$. Since h_{t_n} is strongly accretive for $t_n > 0$, it follows that $h_{t_n}(B)$ is open (by Theorem K), while by (1) $h_{t_n}(\overline{B})$ is closed. Hence we conclude that $\partial h_{t_n}(B) \subset h_{t_n}(\partial B)$, yielding to the existence of a point $x_n \in \partial B$ so that $u_n \in h_{t_n}(x_n)$. Since $y_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $u_n \in \text{seg}\,[0,y_n]$, $u_n \longrightarrow 0$ and thus Lemma 2(ii) implies that $\{x_n\}$ is a Cauchy sequence which must converge, say to $\overline{x} \in \partial B$. Therefore by Lemma 3 $0 \in h_t(\overline{x})$ which, since $x \neq \overline{x}$, contradicts the expansiveness of h_t on B, completing the proof.

Since T is strongly accretive on a set iff I - T is strongly pseudo-contractive, the following result is a direct consequence of Proposition 1.

<u>Corollary 1</u>. Let X be a Banach space and K a closed ball in X. Let $T: K \longrightarrow B(K)$ be a continuous and strongly pseudo-contractive mapping. Then there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

We now state the main result of this paper.

Theorem 1. Let X be a Banach space, and D an open subset of X. Suppose $T:\bar{D} \longrightarrow B(X)$ is a continuous and strongly accretive mapping which satisfies for some $z \in D$:

(6) $t(x - z) \notin T(x)$ for $x \in \partial D$ and $t < 0$.

Then there exists $x \in \bar{D}$ with $0 \in T(x)$.

Proof. As before, we may assume D is bounded and z = 0 in (6). Since the mapping $U = I - T$ is continuous at 0, we may choose a closed ball K centered at 0 and $t \in (0,1)$ such that $K \subset D$ and

$$tU:K \longrightarrow B(K).$$

Since tU is also strongly pseudo-contractive, Corollary 1 implies the existence of $x \in K$ such that $x \in tU(x)$, i.e., $0 \in (1 - t)x + tT(x)$.

Let $h_t:\bar{D} \longrightarrow B(X)$ be defined by $h_t(x) = (1 - t)x + tT(x)$ for each $t \in (0,1]$, and let

$$M = \{t \in (0,1]: 0 \in h_t(x) \text{ for some } x \in D\}.$$

Observe that h_t is strongly accretive and M is a nonempty set with sup $M > 0$ (by the above argument). To complete the proof it suffices to show, successively, that sup M = 1 and $1 \in M$.

Suppose $t_o = \sup M < 1$. Let $\{t_n\}$ be a sequence of M with $t_n \longrightarrow t_o$ as $n \longrightarrow \infty$, and let $x_n \in D$ be such that $0 \in (1 - t_n)x_n + t_nT(x_n)$. Choose $u_n \in T(x_n)$ so that $(1 - t_n)x_n + t_nu_n = 0$. Since D is bounded and $\{t_n\}$ is bounded away from zero, the sequence $\{x_n - u_n\}$ is bounded. Thus by Lemma 2(ii) $\{x_n\}$ is a Cauchy sequence, implying $x_n \longrightarrow x_o \in \bar{D}$. It follows that, by Lemma 3, $0 \in (1 - t_o)x_o + t_oT(x_o)$ and by (6) $x_o \in D$, proving $t_o \in M$.

Since by assumption $t_o < 1$, we select a sequence $\{t_n\}$ in the open interval $(t_o,1)$ such that $t_n \longrightarrow t_o^+$. Since $t_n \notin M$ for each n, the argument given in Proposition 1 leads to the same type of contradiction. Therefore $t_0 = 1 \in M$.

The single-valued version of Theorem 1 can be easily derived from Theorem 1 of the author [13] in more general setting. Actually, if T is a single-valued mapping from $\overline{D}$ into X, Theorem 1 remains valid for the much wider class of locally strongly pseudo-contractive mappings.

Theorem 2. Let X be a Banach space and D a bounded open subset of X. Suppose $T:\overline{D} \longrightarrow B(X)$ is a continuous and accretive mapping satisfying for some $z \in D$:

(7) $\qquad t(x - z) \notin T(x)$ for $x \in \partial D$ and $t < 0$.

Then $\inf\{|T(x)| : x \in \overline{D}\} = 0$.

Proof. Let $T_n:\overline{D} \longrightarrow B(X)$ be defined by $T_n(x) = (1/n)(x - z) + T(x)$, for each $n \in \mathbb{N}$. Then T_n is a continuous strongly accretive mapping which also satisfies (7). Then, by Theorem 1, there exists $x_n \in \overline{D}$ so that $0 \in T_n(x_n)$ for each n. Since $\{x_n\}$ is bounded it follows that $|T(x_n)| \longrightarrow 0$ as $n \longrightarrow \infty$, concluding that $\inf\{|T(x)| : x \in \overline{D}\} = 0$.

We should note that in [5], Downing has shown Theorem 2 under the additional assumptions that T is lipschitzian and it takes values in P(X), i.e., if $x \in X$ and $A \in P(X)$, there exists a point $a \in A$ with $\|x - a\| = \inf\{\|x - y\| : y \in A\}$.

Next, we extend a theorem of Kirk and Schöneberg [10] to a set-valued mapping, and we also improve Theorem 2.1 of [5], which is also an extension of the aforementioned theorem of [10].

Theorem 3. Let D be a bounded open subset of a Banach space X, and let $T:\overline{D} \longrightarrow B(X)$ be continuous and accretive. Suppose there exists $z \in D$ such that

(8) $|T(z)| < |T(x)|$ for all $x \in \partial D$.

Then $\inf\{|T(x)| : x \in \bar{D}\} = 0$. If in addition, $\bar{D}$ has the fixed point property with respect to (single-valued) nonexpansive self-mappings, then $0 \in T(\bar{D})$.

Proof. We first show that (8) implies condition (7): $t(x - z) \notin T(x)$ for $x \in \partial D$ and $t < 0$. Suppose $u = t(x - z)$ for some $u \in T(x)$, $x \in \partial D$ and $t < 0$. Then by choosing $\lambda = 1 - t$ and $k = 1$ in (1) we have

$$-t\|x - z\| \le \|-t(x - z) + u - v\| = \|v\|$$

for each $v \in T(z)$. Since $|T(x)| \le -t\|x - z\|$ and $-t\|x - z\| \le |T(z)|$, we conclude that $|T(x)| \le |T(z)|$ which contradicts (8). Therefore, Theorem 2 implies $\inf\{|T(x)| : x \in \bar{D}\} = 0$. From this latter fact one may assume the existence of $z \in D$ such that

$$|T(z)| < \inf\{|T(x)| : x \in \partial D\}.$$

By Theorem 2.4 of [7], there exists a (single-valued) nonexpansive mapping $f : \bar{D} \longrightarrow D$ whose fixed points are zeros of T. Hence the added assumption on $\bar{D}$ completes the proof.

The following theorem is a localization of Theorem 1. To prove this result, we invoke some lemmas whose proofs are patterned after Kirk-Morales [9] and Morales [13].

Theorem 4. Let X be a Banach space, and D an open subset of X. Suppose $T : \bar{D} \longrightarrow B(X)$ is a continuous and locally strongly accretive mapping on D which satisfies for some $z \in D$:

(9) $t(x-z) \notin T(x)$ for $x \in \partial D$ and $t < 0$.

Then there exists $x \in \bar{D}$ with $0 \in T(x)$.

To prove this theorem we need the following lemmas.

Lemma 4. Let X be a Banach space and D an open subset of X.

Suppose $T:\overline{D} \longrightarrow 2^X$ is a continuous mapping which is locally strongly accretive on D. Suppose also that $t_0 x_0 \in T(x_0)$ for some $x_0 \in D$ and $t_0 < 0$, and suppose for $\sigma_0 > 0$, $B(x_0; \sigma_0) \subset D$. Then:

(a) If $t < 0$ satisfies

$$|t - t_0| \leq \sigma_0 (1 - k)/\|x_0\|, \tag{10}$$

there is a unique point $x_t \in B(x_0; \sigma_0)$ such that $tx_t \in T(x_t)$.

(b) The point x_t in (a) satisfies

$$\|x_t - x_0\| \leq \|x_t\| \; |t - t_0|/(1 - t_0 - k).$$

<u>Proof</u>. Since T is locally strongly accretive on D, there exists a closed ball $B = B(x_0; \sigma_0)$ where T is globally strongly accretive. Suppose $t < 0$ satisfies (10). We shall show that the mapping $T - tI$ satisfies (9) on ∂B (with $z = x_0$). To see this, suppose there exist $s < 0$ and $x \in \partial B$ such that

$$s(x - x_0) \in T(x) - tx.$$

Choose $u_0 \in T(x_0)$ and $u \in T(x)$ so that $u_0 = t_0 x_0$ and $s(x - x_0) = u - tx$. Then by setting $\lambda = 1 - t - s$ in (1) we have

$$(1 - t - s - k)\|x - x_0\| \leq \| - (s + t)(x - x_0) + u - u_0\|$$

$$= \| - (s + t)(x - x_0) + s(x - x_0) + tx - t_0 x_0\|$$

$$= \|x_0(t - t_0)\|$$

from which (using (10)) and the fact that $\|x - x_0\| = \sigma_0$)

$$(1 - t - s - k)\|x - x_0\| \leq (1 - k)\|x - x_0\|.$$

This implies $s > 0$, which is a contradiction. Therefore, by Theorem 1, $T - tI$ has a unique zero x_t in B, i.e., $tx_t \in T(x_t)$.

To prove (b), select $\lambda = 1 - t_0$. The strong accretiveness of T implies

$$(1 - t_0 - k)\|x_t - x_0\| \leq \| - t_0(x_t - x_0) + tx_t - t_0 x_0\|,$$

yielding

$$\|x_t - x_0\| \le \|x_t\| \; |t - t_0|/(1 - t_0 - k).$$

Lemma 5. Let X be a Banach space, D an open subset of X and $T:\bar{D} \longrightarrow 2^X$ a continuous mapping which is locally strongly accretive on D. For $A \subset D$, set $E_A = \{t < 0 : tx \in T(x)$ for some $x \in A\}$ and let $E = \{x \in D : tx \in T(x)$ for some $t < 0\}$. Then

(i) the set E is either empty or the union of nontrivial components, each of which is a continuous image of a subinterval of $(-\infty, 0]$.

In addition, if F is any component of E, then

(ii) if $t_0 < 0$ and $t_0 \in E_F$, then the set $G = \{x \in F : tx \in T(x)$ for some $t \in E_F \cap [t_0, 0)\}$ is bounded; and

(iii) if $t_n x_n \in T(x_n)$ with $t_n \longrightarrow t \le 0$ $(t_n \le 0)$ and $\{x_n\} \subset F$, then x_n is a Cauchy sequence.

Proof. (i) is an immediate consequence of Lemma 4.

(ii) Suppose $x_0 \in F$ with $t_0 x_0 \in T(x_0)$, and choose $\sigma > 0$ such that T is globally strongly accretive on the closed ball $B(x_0; \sigma) \subset D$. Let $tx_t \in T(x_t)$, where $x_t \in B(x_0; \sigma) \cap F$ and $t_0 < t < 0$. Then by selecting $\lambda = 1 - t$ in (1) we have

$$(1 - t - k)\|x_t - x_0\| \le \|-t(x_t - x_0) + tx_t - t_0 x_0\|$$

$$= (t - t_0)\|x_0\|,$$

which implies

$$(1 - t - k)\|x_t\| \le (1 - t - k)(\|x_t - x_0\| + \|x_0\|)$$

$$\le (1 - t - k)((t - t_0)/(1 - t - k) + 1)\|x_0\| = (1 - t_0 - k)\|x_0\|.$$

Therefore $\|x_t\| \le \|x_0\|(1 - t_0 - k)/(1 - k)$ for all $x_t \in G$.

(iii) Suppose $t_m < t_n$. Then by Lemma 4 the segment $[t_m, t_n]$ can be covered by a finite number of overlapping subintervals

$\{I_i\}_{i=1}^{r}$ which have the property that for each i and t, $s \in I_i$, the correspondent $x_t, x_s \in F$ satisfy

$$(11) \qquad \|x_t - x_s\| \leq M|t - s|/(1 - k),$$

where $M = \sup\{\|x_t\| : x_t \in F, t_0 \leq t < 0\}$ with $t_0 = \inf\{t_n\}$.

We may now select $s_i \in I_i \cap I_{i+1}$ such that $t_m = s_0 < s_1 < \ldots$ $\ldots < s_{r+1} = t_n$. Then by (11),

$$\|x_{s_i} - x_{s_{i+1}}\| \leq M|s_i - s_{i+1}|/(1 - k), \ i = 0,1,\ldots,r,$$

and thus

$$\|x_m - x_n\| \leq \sum_{i=0}^{n} \|x_{s_i} - x_{s_{i+1}}\| \leq M \sum_{i=0}^{n} |s_i - s_{i+1}|/(1 - k) =$$

$$= M|t_m - t_n|/(1 - k).$$

Therefore $\{x_n\}$ is a Cauchy sequence.

Proof of Theorem 4. Without loss of generality, we may assume $z = 0$ in (9). As it was shown before (see the proof of Theorem 1), there exists $s \in (0,1)$ and a ball B centered at 0 such that the mapping $(1 - t)I + tT$ has a zero in B for each $t \in (0,s)$. Therefore, if we define the set E as in Lemma 5, there exists a component F_0 of E for which $0 \in \overline{F}_0$.

Let $h_t : \overline{D} \longrightarrow B(X)$ be defined by $h_t(x) = (1 - t)x + tT(x)$ for each $t \in (0,1]$, and let

$$M = \{t \in (0,1] : 0 \in h_t(x) \text{ for some } x \in F_0\}.$$

We first note that M is a nonempty set (by the argument mentioned above) having $\sup M > 0$. We shall show successively that $\sup M = 1$ and $1 \in M$.

Suppose $t_0 = \sup M < 1$. Let $\{t_n\}$ be a sequence of M with $t_n \longrightarrow t_0$ as $n \longrightarrow \infty$, and let $x_n \in F_0$ be such that $0 \in h_{t_n}(x_n)$. Then by Lemma 5(iii), the sequence $\{x_n\}$ is Cauchy and since F_0 is a closed set in E, $\{x_n\}$ converges to $x_0 \in F_0$. It follows, from Lemma 3, that $0 \in h_{t_0}(x_0)$, proving $t_0 \in M$.

Since by assumption $t_0 < 1$, we may choose a sequence $\{t_n\}$ in the open interval $(t_0, 1)$ such that $t_n \to t_0^+$. Since $x_0 \in D$ (by (9)) and $t_n \notin M$ for each n, we may carry out the proof of Proposition 1, concluding that $t_0 = 1 \in M$. This means there exists $x \in \overline{D}$ for which $0 \in T(x)$.

Our next theorem involves an apparently wider class of strongly accretive mappings. Let $c:[0,\infty) \to [0,\infty)$ be a continuous function having $c(t) > 0$ for each $t \in [0,\infty)$, and let D be a subset of a Banach space X. A mapping $T:D \to 2^X$ is said to be <u>locally c-strongly accretive</u> if for each point $z \in D$ there is a neighborhood N such that for each $x, y \in N$ there exists $j \in J(x - y)$ satisfying

$$(12) \qquad \langle u-v, j \rangle \geq c(\max\{\|x\|, \|y\|\})\|x - y\|^2$$

for $u \in T(x)$ and $v \in T(y)$.

<u>Theorem 5</u>. Let X be a Banach space, D an open subset of X and $T:D \to B(X)$ a continuous locally c-strongly accretive mapping. Then T(D) is open.

<u>Proof</u>. Let $y_0 \in T(D)$. Then there exists $x_0 \in D$ such that $y_0 \in T(x_0)$. Since T is locally c-strongly accretive, we may choose an open ball B centered at x_0 for which (12) holds for all $x, y \in B$. Then the assumptions on c imply

$$\gamma = \inf\{c(\|u\|) : u \in B\} > 0.$$

Now if $u \in T(x)$ and $v \in T(y)$ for $x, y \in B$, then

$$\langle u-v, j \rangle \geq \gamma \|x - y\|^2$$

for some $j \in J(x - y)$. This means T is strongly accretive on B, and thus Theorem K implies T(B) is an open subset of X, completing the proof.

We remark that Theorem 5 extends Theorem 4.1 of Ray and Walker [15] and Theorem 2 of Torrejón [16]. Acutally they show the single-valued version of Theorem 5 under a more restrictive assumptions on the function c (defined above). We should also mention that our proof for single-valued mappings can be obtained via using Theorem 3 of Deimling [4].

Our final theorem is a combination of Theorem 4 with the following coercive condition imposed on the operator T:

(13) $$T^{-1}(K) \text{ is bounded whenever } \bar{K} \text{ is compact.}$$

<u>Theorem 6</u>. Let X be a Banach space and let $T: X \longrightarrow B(X)$ be continuous and c-strongly accretive, satisfying condition (13). Then $T(X) = X$.

<u>Proof</u>. Since by Theorem 5 $T(X)$ is open, it remains to show that $T(X)$ is closed. To see this, let $\{u_n\}$ be a sequence in $T(X)$ such that $u_n \longrightarrow u$. We choose $x_n \in X$ such that $u_n \in T(x_n)$ for each n. By (12) there exists $j \in J(x_n - x_m)$ such that

$$\langle u_n - u_m, j \rangle \geq c(\max\{\|x_n\|, \|x_m\|\})\|x_n - x_m\|^2.$$

Since (13) implies that $\{x_n\}$ is bounded, there is a number $\gamma > 0$ (as in the proof of Theorem 5) for which

$$\langle u_n - u_m, j \rangle \geq \gamma \|x_n - x_m\|^2$$

for all $n, m \in \mathbb{N}$. Hence the sequence $\{x_n\}$ is a Cauchy sequence which must converge to some $x \in X$. Since T is continuous, Lemma 3 (with $t = 1$) implies that $u \in T(x)$.

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