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SUPPORT RESULTS VIA EXCEPTIONAL SETS IN BANACH SPACES

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Abstract: A generalization of the Browder support result involving locally compact exceptional sets is given. As application, an inward surjectivity theorem is formulated.

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Let Y be a Banach space. For each subset Z of Y, let int(Z), cl(Z), bd(Z) denote the interior, closure and boundary of Z respectively; also, for each $y \in Y$, put $]y,Z[= \{y + \lambda(z-y);\ 0 < \lambda < 1, z \in Z\}$. The other notational conventions being standard, let the (closed) part B of Y with nonempty boundary be given. We shall say that the point y in bd(B) is a support point of B provided $]y,S[\cap B = \emptyset$ for some open sphere $S \subset Y \setminus B$, the subset of all such points being denoted spt(B). An important problem pertaining to the geometry of Banach spaces is that of determining the "size" of spt(B) in bd(B). For example, in the convex case (when support point means hyperplane support point in the sense of Bishop and Phelps [2]) spt(B) equals bd(B) provided $\mathrm{int}(B) \neq \emptyset$, the general situation being spt(B) is dense in bd(B) (see the above reference); a nonconvex version of this assertion was established by Browder [3]. Since a structural extension of these results cannot be reached as the counterexample in Phelps [10] shows, the only consistent way of generalizing them is that an exceptional set be admitted in the formulation of the problem; that is, M being a (closed) subset of B, under what conditions it is true that $spt(B)\setminus M$ is (nonempty and, eventually) dense in a subset of bd(B) not too "bad" in comparison with bd(B) itself? As far as we know, the only answer to this question has been indicated by Browder [4, Section 1] in case of Y being infinite dimensional and M locally relatively compact; it is our aim to complete his result both methodologically (the dimension of the space having no effect for the argument) and technically (the class of exceptional sets we shall use being strictly larger than the above one). As application, a reformulation in these terms of the surjectivity Gautier-Isac-Penot statements [6] will be given.

Let in the following B denote a proper closed part of Y (hence $bd(B)\neq\emptyset$). The closed subset M of B will be said to be <u>boundary proper</u> when $bd(B)\setminus M$ is nonempty. It clearly follows by the above remarks that, in such a case

(1) $cl(spt(B)\setminus M) = cl(bd(B)\setminus M)$.

The main point is now to indicate sufficient conditions under which the second member of (1) be nonempty. To this end, we shall admit in the sequel $int(B)\neq\emptyset$ for, otherwise (when $B = bd(B)$) each proper closed subset of B is automatically boundary proper. Letting ebd(B) (the <u>essential boundary</u> of B) stand for the intersection $cl(int(B))\cap bd(B)$ (note that in the case we dealt with, ebd(B) is nonempty since int(B) cannot be closed in Y) we shall say the subset M of B is <u>boundary locally flat</u> when each $z\in M\cap ebd(B)$ has a convex neighborhood $U = U_z$ with the property $]x,M\cap bd(B)\cap U[$ has no interior points for all $x\in int(U\cap B)$. Under these conventions, an appropriate answer to the above posed question is contained in the following

Lemma. Suppose M is closed and boundary locally flat. Then M is boundary proper and

(2) $cl(bd(B)\setminus M)\supset ebd(B)$.

Proof. Let $z\in ebd(B)$ be given. If $z\notin M$, the proof is finished so we may suppose $z\in M$. By the locally flatness assumption, there exists a convex neighborhood U of z such that $]x, M\cap bd(B)\cap U[$ has no interior points for all $x\in int(U\cap B)$. Suppose $y\in int(U\setminus B)$ (not empty, by the definition of bd(B) has been fixed. As $(\lambda, w)\mapsto (1-\lambda)x+\lambda w$ (for the arbitrarily chosen $x\in int(U\cap B)$) is continuous in (0,y), an $\varepsilon > 0$ and an open sphere $W\subset int(U\setminus B)$ around y may be found such that $W_\lambda = (1-\lambda)x+\lambda W$ enters in $int(U\cap B)$ for each λ in $(0,\varepsilon)$. On the other hand, each segment joining x with the points of W must intersect bd(B). This gives $W_\lambda\subset]x, bd(B)\cap \cap U[$, which in turn implies $W_\lambda\subset]x, M\cap bd(B)\cap U[$ in case $bd(B)\setminus M$ is disjoint from U. This fact being impossible, $(bd(B)\setminus M)\cap U$ is not empty and the proof is finished. q.e.d.

Now, by simply adding to this lemma the considerations involved in (1), we get our first main result.

Theorem 1. Letting the proper closed part B of Y with int(B) $\neq\emptyset$, assume $M\subset B$ is both closed and boundary locally flat. Then, the subset of all support points of B not belonging to M is dense in the essential boundary of B.

Let us call the subset M of B boundary locally compact when $M\cap bd(B)$ is locally compact in the usual sense. It is an easy consequence of the Mazur's result [5, ch.V, sect. 2] that each boundary locally compact subset is boundary locally flat provided Y is infinite dimensional. So, Browder's theorem we already quoted is a particular case of the above statement; moreover, observing that the union of a boundary locally flat subset of B and a closed (hence proper) subset of int(B) is again boundary locally flat, the inclusion between these results is strictly one. Finally, by the fact that, in the convex case, closure equals closure of the interior (supposed to be not empty) it follows from Theorem 1 that for each proper closed convex part B of Y with nonempty interior and each closed boundary locally flat subset M of B, the (hyperplane) support points of B not belonging to M form a dense part of bd(B); in the absence of this assumption, ebd(B) cannot be replaced by bd(B) in our statement as the choice $B = S \cup M$ where S is a closed sphere and M a disjoint from S closed locally compact subset of Y (supposed to be infinite dimensional) shows.

Let in the following the (proper or not) closed part B of Y be given. We shall say the subset M of B is strongly locally flat (respectively, locally flat when in addition $int(B) \neq \emptyset$) provided for each $z \in M$ (respectively, for each $z \in M \cap cl(int(B))$) there exists a convex neighborhood $U = U_z$ of z with the property $]x, M \cap U[$ has no interior points for all $x \in int(U) \cap B$ (respectively, for all $x \in int(U \cap B)$). In the same context we let H(B)(y) indicate (for each $y \in B$) the translate inward set of y with respect to B as introduced in Halpern and Bergman [7] that is, the subset of all combinations $\lambda^{-1}(z-y)$ with $0 < \lambda \leq 1$ and $z \in B$. Now, as a completion of Theorem 1, our second main result is

Theorem 2. Suppose there exists a proper strongly locally flat (respectively, a locally flat (hence proper) when $int(B) \neq \emptyset$) closed subset M of B with the property

(3) $H(B)(y)$ is dense in Y for each $y \in B \setminus M$.
Then $B = Y$.

Proof. Suppose by contradiction B is a proper (closed) part of Y. It immediately follows by the above lemma plus the remarks concerning (1) that in either case (modulo int(B)) $spt(B) \setminus M$ is not empty. Let y be any point of this subset; there exists by definition an open sphere S of $Y \setminus B$ such that $]y,S[\cap B = \emptyset$. This shows $y+H(B)(y)$ is disjoint from S and (3) will be violated. This ends the argument. q.e.d.

An interesting situation treatable by this procedure is to be described as follows. Let us define after Penot [9] the Bouligand tangent cone of y with respect to B as the (closed) subset $K(B)(y)$ of all $w \in Y$ appearing as limits of the sequences $(\lambda_n^{-1}(z_n-y))$ with (λ_n) in $(0,1]$ converging to zero and (z_n) in B converging to y or, equivalently, the subset of all $w \in Y$ for which $\liminf \lambda^{-1} \operatorname{dist}(y+\lambda w,B) = 0$ as $\lambda \to 0+$. In these terms, a sufficient condition for (3) to be valid being

(4) $K(B)(y) = Y$ for each $y \in B \setminus M$

one may conclude Theorem 2 is an exceptional set extension of the main result of Gautier Isac and Penot [6]. As a variant of this construction we let $J(B,\varepsilon)(y)$ indicate, for each $\varepsilon > 0$, the subset (in $Y_0 = Y \setminus \{0\}$) of all elements $\lambda^{-1}(z-y)$ with $\lambda > 0$, $z \in B$, $0 < \|z-y\| < \varepsilon$, and $J(B)(y)$ (the asymptotic direction set of y with respect to B under Browder's terminology [4, Introduction]) the intersection over $\varepsilon > 0$ of $cl^0(J(B,\varepsilon)(y))$ where cl^0 means the closure modulo Y_0 : in other words, $w \in J(B)(y)$ provided it is a limit of the sequence $(\lambda_n^{-1}(z_n-y))$ with (z_n) in B tending to y and (λ_n) a sequence in $(0,\infty)$ which, from this fact must converge to zero (eventually on a subsequence) and this shows $w \in K(B)(y)$. It immediately follows by a standard connectedness argument that (4) will be surely fulfilled when

(4)′ $J(B)(y) = Y_0$ for each $y \in B \setminus M$

is to be accepted, and this tells us Theorem 4′ of Browder (see the first section of the above reference) basic to the considerations he developed in that context, may be also deemed as a particular case of Theorem 2 (the intervention of a connected open set of Y in place of Y itself having no effect for the substance of the argument); we have to remark at this moment that, in addition to being reductible to (4), condition (4)′ requires the (superfluous modulo (4)) assumption each $y \in B \setminus M$ be an accumulation point of B, which makes Browder′s construction of J(B)(y) (with the use of "cl^o" in the detriment of "cl") to have neither a theoretical nor a practical justification. In particular, taking B = T(X) where T is a multifunction from the Banach space X into Y and introducing the upper Dini derivative at the point $(x,y) \in \Gamma(T)$ (the graph of T) in the direction $a \in X$ as the subset DT(x,y)(a) of all $b \in Y$ appearing as limits of the sequences $(\lambda_n^{-1}(z_n - y))$ with (λ_n) in (0,1] converging to zero and (z_n) in Y with $z_n \in T(x + \lambda_n a_n)$, $n \in \mathbb{N}$ (for some sequence (a_n) (with $a_n \longrightarrow a$) in X) converging to y, condition (4) follows at once from

(5) $DT(x,y)(X) = Y$ for each $y \in T(X) \setminus M$ and some $x \in T^{-1}y$

if we take Proposition 1 of Gautier Isac and Penot (see the above reference) into account; as a consequence, the corresponding version of Theorem 2 extends the normal solvability of Browder′s results (exposed in the introductory part of the above quoted paper) based on Gâteaux derivatives and comprising those of Pokhozhayev [11], as well as the Isac surjectivity results [8] based on DeBlasi derivatives. For a number of related viewpoints concerning this problem we refer to Altman [1].

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