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APPROXIMATION OF $\mathbb{R}^X$ WITH COUNTABLE SUBSETS OF $C_p(X)$ AND CALIBERS OF THE SPACE $C_p(X)$

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Abstract: Suppose that X is a Tychonoff space and every $f \in \mathbb{R}^X$ is an accumulation point for some countable $A \subset C_p(X)$. Then $\psi(X) = \omega$ and $\tau = cf(\tau) > \omega$ implies τ is a caliber of $C_p(X)$. The main result of this paper : If a space X can be mapped continuously and injectively onto a metrizable space, then every regular uncountable cardinal is a caliber of $C_p(X)$. An example of a space X is constructed for which $\overline{(C_p(X))}_\omega = \mathbb{R}^X$ but there exists no continuous injection $f:X \longrightarrow Y$ as soon as $\chi(Y) = \omega$.

Key words: ω-closure, caliber, Šanin property, pseudocharacter, pointwise convergence, countable approximation.

Classification: 54A25, 54C40, 54D60

All spaces are assumed to be Tychonoff. If X is a space, then $\mathcal{T}(X)$ is its topology, $\mathcal{T}^*(X) = \mathcal{T}(X) \setminus \{\emptyset\}$ and $\mathcal{T}(x,X)$ is the family of all open neighbourhoods of the point $x \in X$. By $Y^X(C_p(X,Y))$ is denoted the set of all (continuous) mappings from X to Y endowed with the topology of pointwise convergence. Let X be a space and $A \subset X$. The ω-closure $(\overline{A})_\omega$ of A in X is the set $\bigcup\{\overline{B} : B \subset A$ and $|B| = \omega\}$ where the bar denotes the closure in X.

Let $C_p(X) = C_p(X, \mathbb{R}) \subset \mathbb{R}^X$. It is well known that $C_p(X)$ is dense in $\mathbb{R}^X$. We are going to study the situation, when the ω-closure of $C_p(X)$ is equal to $\mathbb{R}^X$. It occurs, for example, when $C_p(X)$ is separable (and hence so is $\mathbb{R}^X$), or if X is discrete. In both cases the pseudocharacter of the space X is countable. Our first observation is the following

1. Proposition. If $\overline{(C_p(X))}_\omega = \mathbb{R}^X$ then $\psi(X) = \omega$.

Proof. Take any $x \in X^d = \{y \in X: \{y\} \notin \mathcal{T}(X)\}$. The function $\chi_x \in \mathbb{R}^X$ with $\chi_x(x) = 1$ and $\chi_x(X \setminus \{x\}) = \{0\}$ can be approximated by a sequence $s = \{f_n : n \in \omega\} \subset C_p(X)$ i.e. $\chi_x \in \bar{s} \setminus s$. Observe that $F = \bigcap \{f_n^{-1} f_n(x) : n \in \omega\} = \{x\}$. In fact, if $y \in F \setminus \{x\}$ then there is an $n \in \omega$ such that $f_n(y) < \frac{1}{2}$, $f_n(x) > \frac{1}{2}$ and therefore $f_n(y) \neq f_n(x)$ in contradiction with the definition of the set F. As F is a G_δ-set in X we have $\psi(x,X) = \omega$. Of course, the pseudocharacter of any $x \in X \setminus X^d$ is countable, so $\psi(X) = \omega$.

It is not difficult to see that the countable pseudocharacter of a space X in no way implies $\overline{(C_p(X))}_\omega = \mathbb{R}^X$. A.V. Arhangel´skii and D.B. Šahmatov proved that there are even metrizable spaces in which it is impossible to approximate all real-valued functions by countable subsets of continuous functions.

Recall that a cardinal τ is a caliber of a space Z (notation $\tau \in \mathrm{Cal}(Z)$) if for every $\gamma \subset \mathcal{T}^*(Z)$ with $|\gamma| = \tau$ there is a subfamily $\gamma_1 \subset \gamma$ such that $\bigcap \gamma_1 \neq \emptyset$ and $|\gamma_1| = \tau$. A space Z is called Šanin space, or the space in which the Šanin condition holds, iff every uncountable regular cardinal is a caliber of Z.

2. Proposition. Let X be a space and $\widetilde{(Y)}_\omega = X$. If X is a Šanin space, then so is Y.

Proof. Take any $\gamma \subset \mathcal{T}^*(Y)$ such that $|\gamma| = \tau = \mathrm{cf}(\tau) > \omega$. Choose a family $\mu \in \mathcal{T}^*(X)$, $\mu = \{V_U : U \in \gamma\}$ and $V_U \cap Y = U$ for every $U \in \gamma$. There is a subfamily $\mu_1 \subset \mu$ of power τ with nonempty intersection. Pick an $x \in \bigcap \mu_1$ and a sequence

$s = \{y_n : n \in \omega\}$ approximating x. For every $U \in \mu_1$ there is a $y(U) \in s \cap U$. Therefore $|\{U \in \mu_1 : y(U) = y_n\}| = \tau$ for some $n \in \omega$. Hence the family γ has the order τ at the point y_n i.e. there are τ elements of γ containing y_n. This completes our proof.

3. Corollary. For every space X if $\overline{(C_p(X))_\omega} = \mathbb{R}^X$ then $C_p(X)$ is a Šanin space.

4. Theorem. If X is a metrizable space, then $C_p(X)$ is a Šanin space.

Proof. Let $\gamma \subset \mathcal{T}^*(C_p(X))$ and $|\gamma| = \tau = cf(\tau) > \omega$. The family $\mathcal{E} = \{M(x_1,\dots,x_n; O_1,\dots,O_n) = \{f \in C_p(X) : f(x_i) \in O_i,\ i = 1,\dots,n\} : x_i \in X,\ O_i \subset \mathbb{R}$ are intervals with rational endpoints, $i = 1,\dots,n\}$ is a base of the space $C_p(X)$. The elements of $\mathcal{E}$ will be called standard open sets of $C_p(X)$. It is evident that we can assume that $\gamma \subset \mathcal{E}$. It follows from $\tau = cf(\tau) > \omega$ that there is an $n \in \omega \setminus \{0\}$, rational nonempty intervals $O_1,\dots,O_n$ such that $|\{U \in \gamma : U = M(x_1^U,\dots,x_n^U; O_1,\dots,O_n)\}| = \tau$. So it is sufficient to prove our theorem in case every element $U \in \gamma$ is the set $M(x_1^U,\dots,x_n^U; O_1,\dots,O_n)$ for some $x_1^U,\dots,x_n^U \in X$. Let $K(U) = x_1^U,\dots,x_n^U$. Fix a metric ρ on the space X generating $\mathcal{T}(X)$.

Using the Δ-lemma (see, e.g. [1, p. 12]) choose a subfamily $\gamma_1 \subset \gamma$ such that

(1) $|\gamma_1| = \tau$;

(2) there is a finite $K \subset X$ with $K = K(U) \cap K(V)$ for every $U, V \in \gamma_1$, $U \neq V$;

(3) $\rho(K, K(U) \setminus K) > \delta$ for some $\delta > 0$ and all $U \in \gamma_1$.

Let $K = \{x_1,\dots,x_k\}$ (it might happen that $k = 0$ i.e. $K = \emptyset$). It is possible to guarantee after renumerations of the sets K(U) and choosing $\gamma_2 \subset \gamma_1$ with $|\gamma_2| = \tau$ that every $U \in \gamma_2$ will equal a set

$$M(x_1,\dots,x_k,x_1^U,\dots,x_m^U:\ O_1^*,\dots,O_k^*,O_{k+1}^*,\dots,O_{k+m}^*)$$

for some $x_1^U,\dots,x_m^U \in X$ and $m = n-k$. Put $L(U) = \{x_1^U,\dots,x_m^U\} = K(U)\setminus K$. It follows from (2) that $L(U)\cap L(V) = \emptyset$ for different $U,V \in \gamma_2$. Let $L_i^{\gamma_2} = \{x_i^U : U \in \gamma_2\}$, $i \le m$. Consider the set $H = \{i \in \{1,\dots,m\}: :s(L_i^{\gamma_2}) < \tau\}$. Since in metric spaces X and for our τ every discrete subset B of cardinality τ contains a closed in X subset $A\subset B$ of cardinality τ, we can find $\gamma_3 \subset \gamma_2$ of power τ for which the following conditions are satisfied:

(4) $\rho(\cup\{L_i^{\gamma_3}: i \in H\}, \cup\{L_i^{\gamma_3}: i\in\{1,\dots,m\}\setminus H\}) > 0$;

(5) $w(\overline{\cup\{L_i^{\gamma_3}: i \in H\}}) < \tau$;

(6) $L_i^{\gamma_3}$ is closed and discrete in X for $i \in \{1,\dots,m\}\setminus H$.

Pick $i_1,\dots,i_\ell \in \{1,\dots, m\}$ such that $H = \{i_1,\dots,i_\ell\}$ and consider the family of standard open sets $\{W_U = M(x_1,\dots,x_k,x_{i_1}^U,\dots,x_{i_\ell}^U;$ $O_1^*,\dots,O_k^*,O_{k+i_1}^*,\dots,O_{k+i_\ell}^*) : U \in \gamma_3\}$ of the space $C_p(Y)$ where $Y = \overline{K\cup L_{i_1}^{\gamma_3}\cup\dots\cup L_{i_\ell}^{\gamma_3}}$. We conclude from (5) that $nw(C_p(Y)) = nw(Y) < \tau$, [2]. Thus there is an $f \in C_p(Y)$ and $\gamma_4 \subset \gamma_3$ of power τ for which $f \in \cap\{W_U : U \in \gamma_4\}$. The set $K\cup\overline{\cup\{L_{i_p}^{\gamma_3}: p \le \ell\}}$ is C-embedded in X, so there is a $g \in C_p(X)$ such that $g|Y = f$ and $g(x_i^U) \in O_{i+k}^*$ for every $i \in \{1,\dots,m\}\setminus H$ and $U \in \gamma_4$. Therefore $g \in \cap\{U: :U \in \gamma_4\}$ and we are done.

5. Corollary. If there is a one-to-one mapping $f:X\to Y$ of a space X onto a metrizable space Y, then $C_p(X)$ is a Šanin space.

Proof. The dual mapping $f^*:C_p(Y)\to C_p(X)$ which takes an $h\in C_p(Y)$ to $h\circ f \in C_p(X)$ is an embedding and $f^*(C_p(Y))$ is dense in

$C_p(X)$. Now 5 follows from well known facts about calibers [2].

6. Example. There exists a space X such that $\overline{(C_p(X))_\omega} = \mathbb{R}^X$ and there is no one-to-one continuous mapping of X onto a space of countable character.

Construction. Let $\tau > 2^\omega$. We are going to construct a space X with the following properties:

(7) $X = \cup\{X_i : i \in \omega\}$ where X_i is closed and discrete in X;

(8) $c(X) = \omega$;

(9) $\overline{(C_p(X))_\omega} = \mathbb{R}^X$;

(10) $|X| > \tau$:

We shall need the following

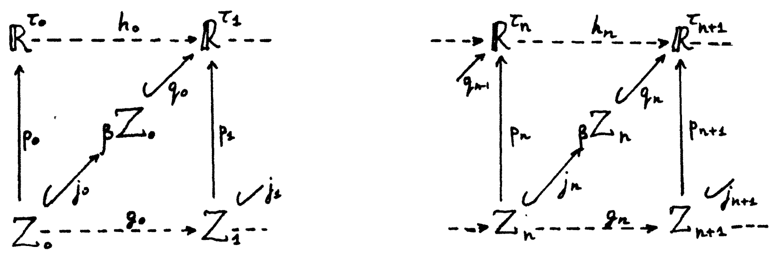
7. Lemma. For any space X we have $\overline{(C_p(X))_\omega} = \mathbb{R}^X$ iff $\overline{(C_p(X))_\omega} \supset \{0,1\}^X$.

Proof. We must prove only the "if" part of the lemma. Establish first that the set $\mathbb{Q}^X$ is ω-dense in $\mathbb{R}^X$ where $\mathbb{Q} \subset \mathbb{R}$ is the set of all rational numbers. (For an arbitrary $f \in \mathbb{R}^X$ and $m \in \omega \setminus \{0\}$ let $f_m(x) = \frac{n}{m}$ iff $\frac{n}{m} \leq f(x) \leq \frac{n+1}{m}$, $n \in \mathbb{Z}$, $x \in X$. It is clear that $\{f_m : m \in \omega \setminus \{0\}$ converges to f.) Let us approximate $\mathbb{Q}^X$ with countable subsets of $C_p(X)$. Let $Z = \{z_1f_1 + \ldots + z_nf_n : z_i \in \mathbb{Q}, f_i \in \{0,1\}^X\}$. It is evident that $\overline{(Z)}_\omega \supset \mathbb{Q}^X$ and in view of $(C_p(X))_\omega \supset Z$ we have $\mathbb{R}^X \subset (\overline{\mathbb{Q}^X})_\omega \subset (\overline{Z})_\omega \subset \overline{(C_p(X))_\omega}$ and the lemma is proved.

Let $\tau_0 = \tau$ and $\tau_{n+1} = 2^{2^{\tau_n}}$ for $n \in \omega$. Consider the spaces $\mathbb{R}^{\tau_n}$, $n \in \omega$. Denote by Z_n the discrete space of cardinality 2^{τ_n} and fix a bijection $p_n : Z_n \to \mathbb{R}^{\tau_n}$, $n \in \omega$. For every $n \in \omega$ let $j_n : Z_n \to \beta Z_n$ be the natural embedding of Z_n into its Čech-Stone compactification βZ_n.

It follows from $w(\beta Z_n) = \tau_{n+1}$ that there is an embedding

$q_n\colon \beta Z_n \longrightarrow \mathbb{R}^{\tau_{n+1}}$. The diagram below might be of help in grasping the construction.



Here $h_n = q_n \circ j_n\ p_n^{-1}$ and $g_n = p_{n+1}^{-1} \circ q_n\ j_n$ so our diagram is commutative. For $m < n$ let $h_m^n = h_{n-1} \circ \ldots \circ h_m\colon \mathbb{R}^{\tau_m} \longrightarrow \mathbb{R}^{\tau_n}$.

Let $T_0 = Z_0$, $X_{n+1} = Z_{n+1} \setminus p_{n+1}^{-1}(q_n(\beta Z_n))$ for all $n \in \omega$ and $X = \bigcup\{X_n : n \in \omega\}$.

Fix a point $x \in X_n$ and $k \in \omega \setminus \{0\}$. Say that a set $U \subset X$ belongs to the family $\mathcal{B}_x^k$ iff there exists a sequence $S_x^U = \{\langle A_i, V_i, f_i U_i\rangle : i \geq n + k\}$ with the following properties:

(11) $A_{n+k} = \{h_n^{n+k}(p_n(x))\}$;

(12) $A_i \subset \mathbb{R}^{\tau_i}$, $f_i \in C_p(\mathbb{R}^{\tau_i}, [0,1])$, $f_i | A_i \equiv 1$;

(13) $V_i = f_i^{-1}((0,1])$, $V_i \cap (q_{i-1} \circ j_{i-1}(Z_{i-1}) \setminus A_i) = \emptyset$;

(14) $A_{i+1} = h_i(V_i)$, $i \geq n+k$;

(15) $U_i = p_i^{-1}(V_i) \cap X_i$;

(16) $U = \{x\} \cup \bigcup\{U_i : i \geq n+k\}$.

The sequence S_x^U will be called corresponding to the set U. Let $\mathcal{B}_x = \bigcup\{\mathcal{B}_x^k : k \in \omega \setminus \{0\}\}$. The families $\mathcal{B}_x$ being constructed for all $x \in X$ announce a set $U \subset X$ open (i.e. $U \in \mathcal{T}(X)$) iff for every $x \in U$ there is a $V \in \mathcal{B}_x$ with $V \subset U$.

We can treat the topology $\mathcal{T}(X)$ as satisfactory in case we check the following three properties:

I. For $x,y \in X$, $U^x \in \mathcal{B}_x$, $U^y \in \mathcal{B}_y$ and $z \in U^x \cap U^y$ there exists a $U^z \in \mathcal{B}_z$ such that $U^z \subset U^x \cap U^y$.

II. $(X, \mathcal{T}(X))$ is a T_0-space, which is trivial.

III. The small inductive dimension of $(X, \mathcal{T}(X))$ equals zero.

I. Take $n_x, n_y, n_z, k_z, k_y \in \omega$ with $x \in X_{n_x}$, $y \in X_{n_y}$, $z \in X_{n_z}$, $U^x \in \mathcal{B}_x^{k_x}$, $U^y \in \mathcal{B}_y^{k_y}$ and the sequences

$S_x = \{\langle A_i^x, V_i^x, f_i^x, U_i^x \rangle : i \geq n_x + k_x\}$ and

$S_y = \{\langle A_i^y, V_i^y, f_i^y, U_i^y \rangle : i \geq n_y + k_y\}$ corresponding to the sets U^x, U^y. Choose an arbitrary $U \in \mathcal{B}_z^1$ and let $S_z^U = \{\langle A_i, V_i, f_i, U_i \rangle : i \geq n_z + 1\}$ be its corresponding sequence. Put $A_i^z = A_i \cap A_i^x \cap A_i^y$, $V_i^z = V_i \cap V_i^x \cap V_i^y$, $f_i^z = f_i \cdot f_i^x \cdot f_i^y$ and $U_i^z = U_i \cap U_i^x \cap U_i^y$, for $i \geq n_z + 1$. It is sufficient to prove that $U^z = \{z\} \cup \bigcup\{U_i^z : i \geq n_z + 1\} \in \mathcal{B}_z^1$. Let us establish that $S_z = \{\langle A_i^z, V_i^z, f_i^z, U_i^z \rangle : i \geq n_z + 1\}$ will correspond to U^z. It is obvious that (11) and (12) are fulfilled. Of course $V_i^z = (f_i^z)^{-1}((0,1])$. Let $x^* \in V_i^z \cap (q_{i-1} \circ j_{i-1}(Z_{i-1}))$. Then $x^* \in A_i^x \cap A_i^y \cap A_i$ so the second part of (13) holds, too. It follows from $A_{i+1}^z = A_{i+1} \cap A_{i+1}^x \cap A_{i+1}^y = h_i(V_i) \cap h_i(V_i^x) \cap h_i(V_i^y) = h_i(V_i \cap V_i^x \cap V_i^y) = h_i(V_i^z)$ that (14) takes place as well. The property (15) is fulfilled. Thus, we finished with I.

III. Take a $U^x \in \mathcal{B}_x$ and its corresponding sequence $S_x = \{\langle A_i^x, V_i^x, f_i^x, U_i^x \rangle : i \geq n_x + k_x\}$ where $x \in X_{n_x}$ and $k_x \geq 1$. We may additionally assume (taking a smaller element of $\mathcal{B}_x$ if necessary) that for our S_x the following condition is satisfied:

(17) for every $i \geq n_x + k_x$ there is a $g_i^x \in C_p(\mathbb{R}^{\tau_i}, [0,1])$ with $g_i^x | V_i^x \equiv 0$, $g_i^x | q_{i-1} \circ j_{i-1}(Z_{i-1}) \setminus A_i^x \equiv 1$. Suppose that $y \in X_m \setminus U^x$. Pick an $n \in \omega$ such that $n > \max\{m, n_x + k_x\}$. Let $A_{n+1}^y = \{h_m^{n+1}(p_m(y))\}$. It is clear that $A_{n+1}^x \cap A_{n+1}^y = \emptyset$. If the sets A_i^y

for $n+1\leq i\leq k$ and V_i^y, f_i^y for $n+1\leq i<k$ are chosen so that (12)-(14) and

(18) $A_i^x\cap A_i^y=\emptyset$ for $n+1\leq i\leq k$,

(19) $V_i^x\cap V_i^y=\emptyset$ for $n+1\leq i<k$

take place, let $V_k^y=(g_k^x)^{-1}((0,1])$, $f_k^y=g_k^x$ and $A_{k+1}^y=h_k(V_k^y)$. Now it follows from (17) that (18)-(19) hold if we replace k by k+1.

Once the sets A_i^y, V_i^y, f_i^y are constructed for all $i\geq n+1$, let $U_i^y=p_i^{-1}(V_i^y)\cap X_i$, $i\geq n+1$. The set $U^y=\{y\}\cup\bigcup\{U_i^y: i\geq n+1\}$ is a member of $\mathcal{B}_y$, having $\{\langle A_i^y,V_i^y,f_i^y,U_i^y\rangle : i\geq n+1\}$ as its corresponding sequence. It follows from (19) that $U^y\cap U^x=\emptyset$, so U^x is closed in X and ind X = 0.

We now turn to prove that $c(X)=\omega$. If on the contrary there is an uncountable disjoint family $\gamma_1\subset\mathcal{T}^*(X)$ then there exist different points x_α , $\alpha<\omega_1$ belonging to X_n for some $n\in\omega$ and $k\geq 1$ such that there are $U^\alpha\in\mathcal{B}^k_{x_\alpha}$ with $U_k^\alpha\cap U_k^\beta=\emptyset$ for $\beta\neq\alpha$. But then the family $\gamma=\{p_k(U_k^\alpha): \alpha<\omega_1\}\subset\mathcal{T}^*(\mathbb{R}^{\tau_k})$ and γ is disjoint in contradiction with $c(\mathbb{R}^{\tau_k})=\omega$. Hence $c(X)=\omega$.

Let us prove that $\overline{(C_p(X))_\omega}=\mathbb{R}^X$. It is sufficient by Lemma 7 to show that $\overline{(C_p(X))_\omega}\supset\{0,1\}^X$. Take any $A\subset X$. We must approximate the function χ_A ($\chi_A(A)=\{1\}$, $\chi_A(X\setminus A)=\{0\}$) with a countable subset $S\subset C_p(X)$. Let $A_n=(X_0\cup\ldots\cup X_n)\cap A$. Show that there is an $f_n\in C_p(X)$ with $f_n|(X_0\cup\ldots\cup X_n)=\chi_{A_n}$. Let $A_n^i=A_n\cap X_i$ and $A^{n+1}=\bigcup\{h_i^{n+1}(p_i(A_n^i)):0\leq i\leq n\}$, $B^{n+1}=\bigcup\{h_i^{n+1}(p_i(X_i\setminus A_n^i)):0\leq i\leq n\}$. Then $A^{n+1}\cap B^{n+1}=\emptyset$ and there exists a $g_{n+1}\in C_p(\mathbb{R}^{\tau_{n+1}},[0,1])$ such that $g_{n+1}|A^{n+1}\equiv 1$, $g_{n+1}|B^{n+1}\equiv 0$. Let $V^{n+1}=g_{n+1}^{-1}((1/2,1])$. If the sets A^i, B^i, V^i and functions $g_i\in C_p(\mathbb{R}^{\tau_i},[0,1])$ are constructed for $n+1\leq i\leq k$ so that $V^i=g_i^{-1}((1/2,1])$, $V^i\cap B^i=\emptyset$. Let $A^{k+1}=h_k(V^k)$, $B^{k+1}=h_k(\mathbb{R}^{\tau_k})\setminus A^{k+1}$. Take any $g_{k+1}\in C_p(\mathbb{R}^{\tau_{k+1}},[0,1])$

with $g_{k+1}|A^{k+1} \equiv 1$, $g_{k+1}|B^{k+1} \equiv 0$ and put $V^{k+1} = g_{k+1}^{-1}((1/2,1])$. Once the sequence $\{\langle A^i, V^i, g_i\rangle : i \geq n+1\}$ is constructed let $U^i = p_i^{-1}(V^i) \cap X_i$ and $U = A_n \cup \bigcup\{U^i : i \geq n+1\}$. By the same reasoning as in III one can prove that U is clopen in X so $\chi_U \in C_p(X)$ and $\chi_U|(X_0 \cup \ldots \cup X_n) = \chi_{A_n}$. Let $f_n = \chi_U$ and check that $S = \{f_n : n \in \omega\}$ approximates χ_A. In fact, if $K \subset X$ is finite, there is an $n \in \omega$ with $K \subset X_0 \cup \ldots \cup X_n$. Then $f_n|K = \chi_{A_n}|K = \chi_A|K$ and all properties of our space are established.

Take any space Y for which there exists a continuous mapping $f: X \xrightarrow{\text{onto}} Y$. If $\chi(Y) = \omega$, then $|Y| \leq 2^{\chi(Y)\cdot c(Y)} = 2^{\omega}$. So f is not injective in view of $\tau > 2^{\omega}$.

8. Example. There is a space X with $\psi(X) > \omega$ and $\tau \in \mathrm{Cal}(C_p(X))$ for every $\tau = \mathrm{cf}(\tau) > \omega$.

Proof. Take a set A of power $\lambda = \omega_{\omega_1}$ and $a_* \notin A$. Let $X = \{a_*\} \cup A$. As to $\mathcal{T}(X)$ it will contain all points of A and $\mathcal{T}(a_*, X) = \{\{a_*\} \cup U : U \subset A, |A \setminus U| < \lambda\}$. For an arbitrary $\tau = \mathrm{cf}(\tau) > \lambda$ we have $\tau > \lambda \geq nw(C_p(X))$ so $\tau \in \mathrm{Cal}(C_p(X))$. If $\tau < \lambda$ and $\gamma = \{U_\alpha : \alpha < \tau\}$ is a family of standard open sets of $C_p(X)$ we may assume that there is an $n \in \omega \setminus \{0\}$ and rational intervals $O_1, \ldots, O_n$ such that $U_\alpha = M(x_1^\alpha, \ldots x_n^\alpha; O_1, \ldots, O_n)$ for all $\alpha < \tau$. Let $K_\alpha = \{x_1^\alpha, \ldots, x_n^\alpha\}$ and $H = \bigcup\{K_\alpha : \alpha < \tau\}$. It is clear that H is closed, discrete and C-embedded in X. As $\mathbb{R}^H$ is a Šanin space, there exists an $f \in \mathbb{R}^H$ such that $f(x_i^\alpha) \in O_i$ for $\alpha < \tau$ and $i \in \{1, \ldots, n\}$. Then $\hat{f} \in \bigcap\{U_\alpha : \alpha < \omega\}$ for any $\hat{f} \in C_p(X)$ with $\hat{f}|H = f$, and this proves that $\tau \in \mathrm{Cal}(C_p(X))$.

9. Remark. Reasoning as in 6 (when proving $c(X) = \omega$) one can prove that the space X from the example 8 is a Šanin space. It follows from (7) that X has a G_δ-diagonal. Thus we have another

answer to J. Ginsburg and R.G. Woods´ question [4]. The space X, being Šanin space, yields a generalization of the result of D.B. Šahmatov [3],[5]. Šahmatov´s example was originally the first answer to the question in [4].

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