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CARTESIAN CLOSED CATEGORIES, QUASITOPOI AND TOPOLOGICAL UNIVERSES

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Abstract: For a concrete, topological category $\mathcal{K}$ over a suitable base category, the interrelationship of the concepts in the title is investigated. $\mathcal{K}$ is cartesian closed iff regular sinks are finitely productive. $\mathcal{K}$ is a quasitopos iff regular sinks are universal. For categories over Set with constant maps, the latter are precisely the topological universes. These can also be described as categories of sieves for Grothendieck topologies.

Key words: Cartesian closed category, quasitopos , topological universe, regular sink.

Classification: 18D15, 18B25

Introduction. We study concrete categories $\mathcal{K}$ over an arbitrary base-category $\mathcal{X}$, we call $\mathcal{K}$ topological if it is finally complete and fibre-small. Many convenient properties are lifted from $\mathcal{X}$ to topological categories over $\mathcal{X}$, e.g. completeness, cocompleteness, factorization systems, etc. Others are in general not lifted, and we investigate conditions for their lifting: cartesian closedness, universality of colimits, quasitopos structure. For cartesian closedness, the problem has two levels: (a) when is $\mathcal{K}$ cartesian closed? (b) when is $\mathcal{K}$ concretely cartesian closed, i.e., with power-objects formed on the level of $\mathcal{X}$? For suitable base categories $\mathcal{X}$, necessary and sufficient conditions for both problems are:

(1) A topological category is cartesian closed iff regular sinks are finitely productive.

(2) A topological category has concrete powers iff final sinks are finitely productive.

Analogously, we characterize topological categories which are quasitopoi:

(3) A topological category is a quasitopos iff regular sinks are universal.

(4) A topological category is a quasitopos with concrete powers iff final sinks are universal.

Categories with the latter property are called universally topological. We give a constructive description:

(5) Universally topological categories are precisely the categories of γ-closed structured sieves for Grothendieck topologies γ.

For concrete categories $\mathcal{K}$ over Set, we study c-categories, i.e., those in which each constant function is a morphism, and topological universes, which are c-categories with universal final epi-sinks. Every concrete category over Set has a c-modification, and topological universes are precisely the c-modifications of universally topological categories.

I. Cartesian closed topological categories. Recall that a category is said to be cartesian closed if it has finite products, and for each object A the endofunctor A $\times$- has a right adjoint. The values of the right adjoint are called power-objects, and are denoted by [A,B]; the corresponding couniversal map is called evaluation, and is denoted by

$$\text{eval}: A \times [A,B] \longrightarrow B.$$

For example, Set is cartesian closed: [A,B] can be chosen as the set of all maps from A to B, and evaluation is the map defined by eval(x,f) = f(x) for $x \in A$ and $f: A \to B$. Also the terminal category T is (trivially) cartesian closed.

If a topological category over $\mathcal{X}$ is cartesian closed, then $\mathcal{X}$ is cartesian closed (see [8]) and hence, we restrict our attention to cartesian closed base-categories. A crucial concept for the investigation of cartesian closedness is that of a regular sink.

I.1. Definition ([8]). A sink $(A_i \xrightarrow{a_i} A)_{i \in I}$ is called regular if there exists a coproduct $\coprod_{j \in J} A_j$ ($J \subset I$ is a set) such that the canonical morphism $[a_j] : \coprod A_j \longrightarrow A$ is a regular epimorphism. A category is said to have regular sink factorizations if it is cocomplete and for each $(A_i \xrightarrow{a_i} A)$ there exists a monomorphism $m: A' \longrightarrow A$ and a regular sink $(A_i \xrightarrow{b_i} A')$ with $a_i = m \circ b_i$.

I.2. Examples. (i) In Set, regular sinks are precisely the epi-sinks, i.e., sinks $(A_i \xrightarrow{a_i} A)$ such that $A = \bigcup a_i[A_i]$. Set has regular sink factorizations.

(ii) T has (trivially) regular sink factorizations.

(iii) Every topological category $\mathcal{K}$ over a base-category with regular sink factorizations has regular sink factorizations. Regular sinks in $\mathcal{K}$ are precisely the final sinks with regular underlying sinks.

Thus, regular sinks are just final epi-sinks in case $\mathcal{X}$ is balanced, in particular for $\mathcal{X}$ = Set, and they are the sinks $(A_i \longrightarrow A)$ with $A = \operatorname{Sup} A_i$ in case $\mathcal{X} =$ [illegible].

I.3. Definition. Regular sinks are finitely productive provided that for each regular sink $(A_i \xrightarrow{a_i} A)$ and each object B, the sink $(B \times A_i \xrightarrow{id_B \times a_i} B \times A)$ is regular.

I.4. Remarks. (1) Regular sinks are finitely productive iff the product

$$\mathcal{A} \times \mathcal{B} = (A_i \times B_j \xrightarrow{a_i \times b_j} A \times B \mid A_i \xrightarrow{a_i} A \text{ in } \mathcal{A} \text{ and } B_j \xrightarrow{b_j} B \text{ in } \mathcal{B})$$

of regular sinks $\mathcal{A}$ and $\mathcal{B}$ is regular.

(2) The finite productivity of final sinks (or colimits, or coproducts, or regular epis) is defined by replacing "regular sinks" by "final sinks" (etc.) in the above definition.

(3) It is easy to see that in a cocomplete category, colimits are finitely productive iff

(a) coproducts are finitely productive, i.e., we have canonical isomorphisms $\coprod_{i \in I} (B \times A_i) \cong B \times \coprod_{i \in I} A_i$, and

(b) regular epimorphisms are finitely productive.

I.5. Theorem [8]. Let $\mathfrak{X}$ be a cartesian closed category with regular sink factorizations. For each topological category $\mathcal{K}$ over $\mathfrak{X}$, the following conditions are equivalent:

(1) $\mathcal{K}$ is cartesian closed;

(2) for each K in $\mathcal{K}$ the functor $K \times -$ preserves colimits;

(3) regular sinks are finitely productive in $\mathcal{K}$;

(4) coproducts and regular epimorphisms are finitely productive in $\mathcal{K}$.

I.6. Definition. A cartesian closed topological category is said to be concretely cartesian closed if for arbitrary objects K and L, the underlying object of $[K,L]$ is $[|K|,|L|]$, and the evaluation map is the same as that in the base category.

I.7. Theorem. For each topological category $\mathcal{K}$ over a cartesian closed category $\mathfrak{X}$, the following conditions are equivalent:

(1) $\mathcal{K}$ is concretely cartesian closed;

(2) in $\mathcal{K}$, final sinks are finitely productive;

(3) $\mathcal{K}$ is cartesian closed, and each $\mathcal{K}$-morphism with a discrete range has a discrete domain;

(4) $\mathcal{K}$ is cartesian closed, and in $\mathcal{K}$ each product $K \times D$ with a discrete factor D is discrete.

Proof. $(1) \longleftrightarrow (2)$. By the taught-lift theorem of Wyler [16], in each commuting square of functors

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{F} & \mathcal{K} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathfrak{X} & \xrightarrow[F_0]{} & \mathfrak{X} \end{array}$$

(where U denotes the forgetful functor) in which F_0 has a right adjoint G_0, the functor F preserves final sinks iff F has a right adjoint G with $U \circ G = G_0 \circ U$. By applying this result to $F = K \times -$, $F_0 = |K| \times -$ and $G_0 = [|K|, -]$, we obtain the equivalence of (1) and (2).

$(2) \longrightarrow (4)$. Since $(2) \longrightarrow (1)$, $\mathcal{K}$ is cartesian closed. A $\mathcal{K}$-object D is discrete iff it is final for the empty sink, in which case each $K \times D$ is final for the empty sink and hence, $K \times D$ is discrete.

$(4) \longrightarrow (3)$. Let $f: K \longrightarrow D$ be a $\mathcal{K}$-morphism and let D be discrete. Then K is a retract of $K \times D$, which is discrete, and hence, K is discrete.

$(3) \longrightarrow (1)$. For each pair of objects K and L let eval: $K \times [K,L] \longrightarrow L$ be the evaluation in $\mathcal{K}$. We prove that eval: $|K| \times |[K,L]| \longrightarrow |L|$ has the universal property in the base category $\mathfrak{X}$. Let X be an object of $\mathfrak{X}$ and let $f: |K| \times X \longrightarrow |L|$ be an $\mathfrak{X}$-morphism. For the discrete object D with $|D| = X$ we know that $K \times D$ is discrete (because the projection $K \times D \longrightarrow D$ is a $\mathcal{K}$-morphism) and hence, $f: K \times D \longrightarrow L$ is a $\mathcal{K}$-morphism. Consequently there exists a unique $\mathcal{K}$-morphism $\tilde{f}: D \longrightarrow [K,L]$ with

$f = \text{eval} \circ (id_K \times \tilde{f})$. Hence $\tilde{f}: X \longrightarrow |[K,L]|$ is the unique $\mathcal{X}$-morphism with this property. Uniqueness follows from the fact that each $\mathcal{X}$-morphism with domain X is a $\mathcal{K}$-morphism with domain D. Therefore, $|[K,L]|$ is the power-object of $|K|$ and $|L|$.

I.8. Remarks. (i) No assumptions about factorization systems in $\mathcal{X}$ are needed for the preceding result (unlike Theorem I.5).

(ii) Let T_0 denote a discrete $\mathcal{K}$-object with underlying terminal $\mathcal{X}$-object $|T_0|$. The condition in (4) that D discrete implies all $K \times D$ discrete is equivalent to the following:

(*) $K \times T_0$ is discrete for each $\mathcal{K}$-object K.

In fact, if (*) is satisfied and if D is a discrete $\mathcal{K}$-object, then $|D| = |T_0| \times |D|$ in $\mathcal{X}$ clearly implies $D = D \times T_0$ in $\mathcal{K}$ and hence, all products

$$K \times D = K \times (D \times T_0) = (K \times D) \times T_0$$

are discrete.

I.9. Examples. (a) Rel is concretely cartesian closed over Set. T_0 is the singleton set with the empty relation, and each $K \times T_0$ also carries the empty relation. The power object of $A = (X, \alpha)$ and $B = (Y, \beta)$ is $[A,B] = (Y^X, \gamma)$ where $f_1 \gamma f_2$ iff $a_1 \alpha a_2$ implies $f_1(a_1) \beta f_2(a_2)$.

(b) Conv is not concretely cartesian closed. Here T_0 is discrete as well as indiscrete. The underlying object of $[K,L]$ is hom(K,L).

I.10. Definition [8]. A concrete category over Set is a c-category if all constant functions $|K| \longrightarrow |L|$ are morphisms $K \longrightarrow L$. (For topological categories, this is equivalent to T_0 being indiscrete.)

I.11. Proposition [2, 8]. For each cartesian closed category over Set, the following are equivalent:

(i) $\mathcal{K}$ has canonical function spaces, i.e. $|[K,L]| =$ $= \hom(K,L)$ and evaluation is the map $(f,x) \mapsto (f(x)$;

(ii) $\mathcal{K}$ is a c-category.

II. Quasitopoi. Recall that a topos is a category $\mathcal{X}$ satisfying the following conditions:

(1) $\mathcal{L}$ has finite limits and colimits;

(2) $\mathcal{L}$ is cartesian closed;

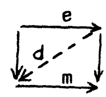
(3) in $\mathcal{L}$ partial morphisms are representable, i.e., for each object A there exists a monomorphism $t_A : A \to A^*$ universal in the following sense: given a partial morphism into A (i.e., a pair consisting of a monomorphism $m : B \to C$ and a morphism $f : B \to A$), there exists a unique pullback

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t_A} \\ C & \dashrightarrow & A^* \end{array}$$

For example, Set is a topos: A^* is a one-point extension of A; also $\mathcal{T}$ is (trivially) a topos.

The topos structure is never lifted from $\mathcal{X}$ to a non-trivial topological category over $\mathcal{X}$ simply because topoi are balanced categories. But a convenient weakening of the axiom (3), studied by Penon [14], gives an interesting concept for topological categories.

A quasitopos is a category $\mathcal{X}$ satisfying (1),(2) and the condition $(3)^*$ obtained from (3) by replacing "monomorphism" by "strong monomorphism". Recall that a monomorphism m is called strong if



with e an epimorphism has a diagonal d. In Set every monomorphism is strong. In a topological category $\mathcal{K}$ over $\mathfrak{X}$, strong monomorphisms are just the initial morphisms with underlying morphism a strong monomorphism in $\mathfrak{X}$.

II.1. <u>Remark</u>. For each object L of a category $\mathcal{L}$ we denote by $\mathcal{L}/L$ the comma-category of all morphisms with the codomain L. Penon proved in [14] that a category $\mathcal{L}$ is a quasitopos iff

(a) $\mathcal{L}$ has finite limits and colimits;

(b) $\mathcal{L}/L$ is cartesian closed for each L in $\mathcal{L}$;

(c) <u>strong subobjects are representable</u>, i.e., there is a strong monomorphism $t:T \longrightarrow \Omega$ with T terminal such that for each strong monomorphism $m:B \longrightarrow C$ there exists a unique pullback

$$\begin{array}{ccc} B & \longrightarrow & T \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ C & \longrightarrow & \Omega \end{array}$$

II.2. <u>Definition</u>. Regular sinks are <u>universal</u> provided for each regular sink $(A_i \xrightarrow{a_i} A)_{i\in I}$ and each morphism $f:B \longrightarrow A$ there exists a regular sink $(B_j \xrightarrow{b_j} B)_{j\in J}$ such that each $f \circ b_j$ factors through some $a_{i(j)}$:

$$\begin{array}{ccc} B_j & \xrightarrow{g_j} & A_{i(j)} \\ {\scriptstyle b_j}\downarrow & & \downarrow{\scriptstyle a_{i(j)}} \\ B & \xrightarrow[f]{} & A \end{array}$$

II.3. <u>Remarks</u>. (i) As in I.4 we define universality of final sinks (or colimits, or coproducts, or regular epis) by replacing, in the above definition, "regular sinks" by "final sinks" (etc.).

(ii) If K has pullbacks then regular sinks are universal iff for each regular sink $(A_i \xrightarrow{a_i} A)_{i \in I}$ and each morphism $f\colon B \longrightarrow A$ the sink $(B_i \xrightarrow{b_i} B)_{i \in I}$ obtained by pointwise pullbacks along f is regular.

$$\begin{array}{ccc} B_i & \xrightarrow{\;g_i\;} & A_i \\ {\scriptstyle b_i}\downarrow & & \downarrow{\scriptstyle a_i} \\ B & \xrightarrow[\;f\;]{} & A \end{array}$$

(Similarly for universal final sinks etc.). Consequently, the universality of regular sinks implies their finite productivity. In fact, if f above is the projection $K \times A \longrightarrow A$ then the resulting sink is $(K \times A_i \xrightarrow{id \times a_i} K \times A)$.

(iii) By replacing in the above definition "morphism f" by "strong monomorphism f", we obtain the condition that regular sinks are hereditary. Analogously we obtain the heredity of final sinks and other types of sinks.

(iv) Let $\mathcal{K}$ be a finitely complete category. Regular sinks are universal iff they are both finitely productive and hereditary. This follows from the fact that each morphism $f\colon B \longrightarrow A$ factors as $f = \pi \circ (id_B, f)$ where $\pi\colon B \times A \longrightarrow A$ is the projection and $(id_B, f)\colon B \longrightarrow B \times A$ is a strong monomorphism (in fact, a section). Pullbacks along f are composed of pullbacks along π followed by pullbacks along (id_B, f).

Analogously with other types of sinks:

universal = finitely productive and hereditary.

II.4. <u>Theorem</u>. Let $\mathfrak{X}$ be a quasitopos with regular sink factorizations. For each topological category $\mathcal{K}$ over $\mathfrak{X}$, the following conditions are equivalent:

(1) $\mathcal{K}$ is a quasitopos;

(2) in $\mathcal{K}$ regular sinks are universal;

(3) in $\mathcal{K}$ colimits are universal;

(4) in $\mathcal{K}$ coproducts and regular epimorphisms are both finitely productive and hereditary;

(5) $\mathcal{K}$ is cartesian closed and has hereditary colimits;

(6) $\mathcal{K}/K$ is cartesian closed for each K in $\mathcal{K}$.

Proof. (1) $\longleftrightarrow$ (6). By II.1, it is sufficient to verify that strong monomorphisms are representable in any topological category $\mathcal{K}$. Let $t: T \longrightarrow \Omega$ be the representing strong monomorphism in $\mathfrak{X}$, and let $t: T^* \longrightarrow \Omega^*$ be the corresponding $\mathcal{K}$-morphism with indiscrete objects T^* and Ω^*. Then T^* is terminal in $\mathcal{K}$, and t is a strong monomorphism. For each strong monomorphism $m: B \longrightarrow C$ in $\mathcal{K}$ we have the unique pullback

$$\begin{array}{ccc} |B| & \longrightarrow & T \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ |C| & \xrightarrow[m^*]{} & \Omega \end{array}$$

in $\mathfrak{X}$.

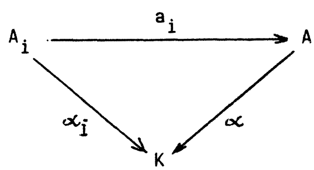
Indiscreteness of Ω^* and initiality of m immediately imply that

$$\begin{array}{ccc} B & \longrightarrow & T^* \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ C & \xrightarrow[m^*]{} & \Omega^* \end{array}$$

is the required unique pullback in $\mathcal{K}$.

(2) $\longleftrightarrow$ (6). This is an immediate consequence of Theorem I.5

via the following simple observations. If $\mathcal{K}$ is topological over $\mathfrak{X}$ then $\mathcal{K}/K$ is topological over $\mathfrak{X}/|K|$. Regular sinks in $\mathcal{K}/K$ are precisely those sinks



$(i \in I)$

for which the sink $(A_i \xrightarrow{a_i} A)_{i\ I}$ is regular in $\mathcal{K}$. For each object (A,α) of $\mathcal{K}/K$, the functor $(A,\alpha)\times - : \mathcal{K}/K \longrightarrow \mathcal{K}/K$ is given by the formation of pullbacks along α in $\mathcal{K}$.

The implications $(2) \longrightarrow (5) \longrightarrow (4) \longrightarrow (3) \longrightarrow (2)$ are clear.

II.5. Remark. The equivalence of (1) and (6) does not require regular factorizations in $\mathfrak{X}$.

II.6. Examples. (a) The category Conv of convergence spaces is a quasitopos.

(b) The category of (compact T_2)-generated topological spaces fails to be a quasitopos although coproducts are universal and regular epimorphisms are finitely productive.
Regular epimorphisms are not hereditary.

In fact, no non-trivial topological subcategory of the category of topological spaces is a quasitopos, see [7]. In contrast, there are quasitopoi of uniform spaces, see [18].

(c) The category Mer of merotopic spaces [12] fails to be a quasitopos although regular epimorphisms are universal and coproducts are hereditary. Coproducts are not finitely productive.

(d) A concretely cartesian closed topological category over Set need not be a quasitopos. For example, let $\mathcal{K}$ have objects $(X,0)$ for all sets X and $(X,1)$ for all non-empty sets X, let

$f:(X,k) \longrightarrow (X',k')$ be a $\mathcal{X}$-morphism iff $f:X \longrightarrow X'$ is a map and $k \leq k'$. Then $\mathcal{X}$ is topological over Set. A sink $((X_i,k_i) \xrightarrow{f_i} (X,k))$ is final iff $k = \text{Sup } k_i$. Hence final sinks are productive, i.e., $\mathcal{X}$ is concretely cartesian closed over Set. Moreover regular epimorphisms in $\mathcal{X}$ are hereditary but $\mathcal{X}$ does not have hereditary coproducts: if $a \neq b$, the coproduct of $(\{a\},0)$ with $(\{b\},1)$ is $(\{a\ ,b\},1)$; and its subobject $(\{a\},1)$ is not the coproduct of the intersections $(\{a\},0)$ and $(\emptyset,0)$.

(e) For complete lattices, considered as topological categories over $\mathcal{T}$, quasitopoi are precisely the locales, i.e., the cartesian closed ones. Any complete lattice which is not a locale is an example of a concrete category with universal regular epimorphisms and hereditary coproducts in which coproducts are not finitely productive.

(f) The quasi-category of quasi-topological spaces of Spannier is illegitimate, see [9], and hence not topological.

II.7. Definition. A topological category with universal final sinks is said to be universally topological.

II.8. Theorem. For each topological category $\mathcal{X}$ over a quasitopos, the following conditions are equivalent:

(1) $\mathcal{X}$ is universally topological;

(2) $\mathcal{X}$ is a quasitopos with concrete powers;

(3) $\mathcal{X}$ is a quasitopos and each $\mathcal{X}$-morphism with a discrete range has a discrete domain;

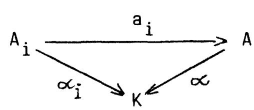
(4) $\mathcal{X}/K$ is concretely cartesian closed over $\mathfrak{X}/|K|$ for each K in $\mathcal{X}$.

Proof. (1) $\longrightarrow$ (2). See II.5 and I.11.

(2) $\longrightarrow$ (3). See I.11.

(3) $\longrightarrow$ (4). By I.11, it suffices to verify that in each $\mathcal{K}/K$ morphisms with discrete range have discrete domain. This follows from the fact that an object $A \xrightarrow{\alpha} K$ is discrete in $\mathcal{K}/K$ iff A is discrete in $\mathcal{K}$ ..

(4) $\longrightarrow$ (1). For each morphism $f: K' \longrightarrow K$ in $\mathcal{K}$, the functor $(K',f)\times - : \mathcal{K}/K \longrightarrow \mathcal{K}/K$ has a concrete right adjoint and hence, it preserves final sinks in $\mathcal{K}/K$ by Wyler's taught lift theorem [16]. The functor $(K',f)\times -$ is given by the formation of pull-backs along f in $\mathcal{K}$. Furthermore, final sinks in $\mathcal{K}/K$ are precisely those sinks



for which the sink $(A_i \xrightarrow{a_i} A)$ is final in $\mathcal{K}$ and hence, pull-backs along any $\mathcal{K}$-morphism f preserve final $\mathcal{K}$-sinks.

II.9. *Definition* [13]. A *topological universe* is a topological c-category with universal final epi-sinks.

II.10. *Remark*. Since each epi-sink in Set is regular, topological universes are precisely those topological c-categories which are quasitopoi.

The category of convergence spaces is a topological universe.

II.11. *Definition* [8]. By the c-*modification* of a topological category $\mathcal{K}$ is meant the concrete subcategory $\mathcal{K}_c$ of $\mathcal{K}$, consisting of those objects K for which all maps from a terminal $\mathcal{K}$-object into K are $\mathcal{K}$-morphisms.

II.12. *Example*. The c-modification of the category of relations is the category of reflexive relations.

Each c-modification of a universally topological category is a topological universe, see [8]. Conversely:

II.13. <u>Theorem</u>. A concrete category over Set is a topological universe iff it is a c-modification of a universally topological category.

Proof. (A) Let $\mathcal{K}$ be a topological universe. Denote by $\hat{\mathcal{K}}$ the category whose objects are all pairs (X,K) where X is a set and K is a $\mathcal{K}$-object with $|K| \subset X$, and whose morphisms $f:(X,K) \longrightarrow (X',K')$ are those maps $f: X \longrightarrow X'$ for which $f(|K|) \subset |K'|$ and the corresponding restriction $\hat{f}$ of f is a $\mathcal{K}$-morphism $\hat{f}: K \longrightarrow K'$. $\hat{\mathcal{K}}$ is a concrete category over Set. A terminal object of $\hat{\mathcal{K}}$ is $(\{0\},T)$, where T is terminal in $\mathcal{K}$ and $|T| = \{0\}$. Then $\mathcal{K} = \hat{\mathcal{K}}_c$, provided that each $\mathcal{K}$-object K is identified with the $\hat{\mathcal{K}}$-object $(|K|,K)$:

(a) If $(X,K) \in \hat{\mathcal{K}}_c$ then for each $x \in X$ we have a map $f:\{0\} \longrightarrow X$ defined by $f(0) = x$. Since $f:(\{0\},T) \longrightarrow (X,K)$ is a morphism, we conclude $x \in |K|$. Thus, $(X,K) = (|K|,K) \in \mathcal{K}$;

(b) For each $K \in \mathcal{K}$ and each map $f:\{0\} \longrightarrow K$ we know that $f:T \longrightarrow K$ is a $\mathcal{K}$-morphism and hence $f:(\{0\},T) \longrightarrow (|K|,K)$ is a $\hat{\mathcal{K}}$-morphism. Thus, $(|K|,K) \in \hat{\mathcal{K}}_c$.

It remains to verify that $\hat{\mathcal{K}}$ is universally topological. Note that each $\hat{\mathcal{K}}$-object (X,K) is the coproduct of (|K|,K) and the discrete object (X - |K|,D) (where D denotes the initial $\mathcal{K}$-object). For each $\hat{\mathcal{K}}$-structured sink $((X_i,K_i) \xrightarrow{f_i} X)_{i \in I}$ the final object is (X,K) where $|K| = \bigcup_{i \in I} f_i(|K_i|)$, and K is the final object in $\mathcal{K}$ of the $\mathcal{K}$-structured sink of restricted maps $(K_i \xrightarrow{\hat{f}_i} |K|)_{i \in I}$. Thus, $\hat{\mathcal{K}}$ is topological (since fibres are obviously small in $\hat{\mathcal{K}}$). Let $g:(X',K') \longrightarrow (X,K)$ be an arbitrary morphism, and let us form the pointwise pullbacks:

$$\begin{array}{ccc} (X_i',K_i') & \xrightarrow{\quad g_i \quad} & (X_i,K_i) \\ {\scriptstyle h_i}\big\downarrow & & \big\downarrow{\scriptstyle f_i} \\ (X',K') & \xrightarrow[\quad g \quad]{} & (X,K) \end{array}$$

Then the restricted morphisms $\hat{g}:K' \longrightarrow K$ and $\hat{f}_i:K_i \longrightarrow K$ have the corresponding pullback

$$\begin{array}{ccc} K_i' & \xrightarrow{\quad \hat{g}_i \quad} & K_i \\ {\scriptstyle \hat{h}_i}\big\downarrow & & \big\downarrow{\scriptstyle \hat{f}_i} \\ K' & \xrightarrow[\quad \hat{g} \quad]{} & K \end{array}$$

in $\mathcal{K}$ (where $\hat{h}_i$ denotes the restriction of h_i, and $\hat{g}_i$ that of g_i). Since the sink $(K_i \xrightarrow{\hat{f}_i} K)$ is regular (because it is a final epi-sink) in $\mathcal{K}$, by II.4 the sink $(K_i' \xrightarrow{\hat{h}_i} K')$ is also regular in $\mathcal{K}$. Moreover $|K'| = \bigcup_{i \in I} f_i(|K_i'|)$ because for each $x \in |K'|$ we have $g(x) \in |K| = \bigcup f_i(|K_i|)$ and hence, there exists $i \in I$ and $y \in |K_i|$ with $g(x) = f_i(y)$. Since pullbacks are concrete in $\mathcal{K}$, we conclude that there is $t \in |K_i'|$ with $\hat{h}_i(t) = x$ and $\hat{g}_i(t) = y$. Consequently, the sink $((X_i',K_i') \xrightarrow{h_i} (X',K'))$ is final in $\hat{\mathcal{K}}$

III. A construction of universally topological categories. The only examples of universally topological categories we have encountered so far were locales (over $\mathcal{T}$) and the category Rel of relations (over Set). We present now a variety of examples obtained by starting with an arbitrary concrete category and extending it to a universally topological one. It turns out that each universally topological category can be obtained in this way. Similar constructions can be found in [4],[8],[13] and [17], new here is the characterization of legitimacy in case we start with a large category.

III.1. Definition. Let $\mathcal{K}$ be a category.

(1) A sink S in $\mathcal{K}$ is called a sieve provided the following implication holds:

$(A \xrightarrow{f} B \in \mathrm{Mor}\,\mathcal{K}$ and $B \xrightarrow{g} C \in S) \Rightarrow A \xrightarrow{gf} C \in S$.

(2) For any object A the sieve consisting of all morphisms with codomain A is called the full sieve for A.

(3) A collection γ of sieves is called a Grothendieck topology for $\mathcal{K}$ provided it satisfies the following four conditions:

(T1) every full sieve belongs to γ;

(T2) γ is composition-closed, i.e. whenever $(A_i \xrightarrow{a_i} A)_{i\in I}$ belongs to γ and, for each $i \in I$, $(B_j \xrightarrow{b_j} A_i)_{j\in J_i}$ belongs to γ, then $(B_j \xrightarrow{a_i \cdot b_j} A)_{i\in I, j\in J_i}$ belongs to γ;

(T3) γ is pullback-stable, i.e. whenever $f:A \to B$ is a morphism and S is a sieve with codomain B, which belongs to γ, then the sieve $f^{-1}S$, consisting of all morphisms $C \xrightarrow{g} A$ with $f \circ g \in S$, belongs to γ;

(T4) γ contains with any sieve any larger sieve.

(4) If $\mathcal{K}$ is a concrete category over $\mathfrak{X}$ and γ is a Grothendieck topology for $\mathcal{K}$, consisting of final sinks only, then $(\mathcal{K}, \gamma)$ is called a concrete site over X.

III.2. Construction. For each concrete site $(\mathcal{K}, \gamma)$ over $\mathfrak{X}$ we construct a quasicategory over $\mathfrak{X}$ (i.e., our construction is in general non-legitimate; see below) as follows: A structured sink $\mathcal{A} = (K_i \xrightarrow{f_i} X)_{i\in I}$ in $\mathfrak{X}$ is said to be

(a) a structured sieve if for each $K \xrightarrow{f} X$ in $\mathcal{A}$ and each morphism $g:K' \longrightarrow K$ in $\mathcal{K}$, $K' \xrightarrow{f \circ g} X$ is in $\mathcal{A}$;

(b) γ -closed if for each $(K_i \xrightarrow{f_i} K)$ in $\mathcal{A}$ and each $K \xrightarrow{g} X$ in $\mathcal{X}$, such that $K_i \xrightarrow{g \circ f_i} X$ is in $\mathcal{A}$ for all i, also $K \xrightarrow{g} X$ is in $\mathcal{A}$.

We denote by

$$\underline{\mathrm{Siev}}(\mathcal{K}, \gamma)$$

the quasicategory of all γ-closed structured sieves. Morphisms from $\mathcal{A} = (K_i \xrightarrow{f_i} X)_{i \in I}$ to $\mathcal{B} = (L_j \xrightarrow{g_j} Y)_{j \in J}$ are those $\mathcal{X}$-morphisms $f:X \longrightarrow Y$ for which $K_i \xrightarrow{f_i} X$ in $\mathcal{A}$ implies $K_i \xrightarrow{f \circ f_i} Y$ is in $\mathcal{B}$. The forgetful functor of $\underline{\mathrm{Siev}}(\mathcal{K}, \gamma)$ into $\mathcal{X}$ sends $(K_i \xrightarrow{f_i} X)$ to X. We consider $\mathcal{K}$ as a full subcategory of $\underline{\mathrm{Siev}}(\mathcal{K}, \gamma)$ by identifying each object K with the full structured sieve

$$\tilde{K} = (L \xrightarrow{f} |K|)_{f:L \longrightarrow K \text{ in } \mathcal{K}}.$$

Observe that for a γ-closed structured sieve $\mathcal{A} = (K_i \xrightarrow{f_i} X)_{i \in I}$ a structured morphism $K \xrightarrow{f} X$ belongs to $\mathcal{A}$ iff $\tilde{K} \xrightarrow{f} \mathcal{A}$ is a morphism in $\underline{\mathrm{Siev}}(\mathcal{K}, \gamma)$.

III.3. Remark. There are two reasons why the construction above can be illegitimate:

(i) the objects can be proper classes in which case they are not elements of any class;

(ii) the conglomerate of all objects can be too large, for example, it can be in a bijective correspondence to the conglomerate of all subclasses of a proper class.

The latter obstacle is obviously the essential one: in case that the conglomerate of all objects is legitimate, i.e., in a bijective correspondence with some class $\mathcal{C}$, we can overcome (i) by using

$\mathscr{C}$ as the class of objects of a category isomorphic (in an obvious way) to Siev $(\mathcal{K}, \gamma)$. By an abuse of language, we call the latter category Siev $(\mathcal{X}, \gamma)$ whenever the class of all γ-closed sieves is legitimate. In particular, this is the case whenever the quasicategory Siev $(\mathcal{K}, \gamma)$ is fibre-small . We characterize such sites now. Recall that a structured map $A \xrightarrow{a} X$ is a pair consisting of an object A in $\mathcal{K}$ and a morphism $a:|A| \longrightarrow X$ in $\mathfrak{X}$.

III.4. Definition. Let $(\mathcal{K}, \gamma)$ be a concrete site. For structured maps $A \xrightarrow{a} X$ and $A' \xrightarrow{a'} X$ we define

$$(A,a) \leq_{\gamma} (A',a')$$

provided that there exists a γ-sink $(B_i \xrightarrow{b_i} A)_{i \in I}$ and $\mathcal{K}$-morphisms $b_i':B_i \longrightarrow A'$ $(i \in I)$ with $a \circ b_i = a' \circ b_i'$ $(i \in I)$. Call (A,a) and (A',a') equivalent [in symbols: $(A,a) \equiv_{\gamma} (A',a')$] iff $(A,a) \leq_{\gamma} (A',a') \leq_{\gamma} (A,a)$.

We say that $(\mathcal{K}, \gamma)$ is a structurally small site if for each X in $\mathfrak{X}$ there exists a set of representatives of the equivalence $\equiv_{\gamma}$ on structured maps with range X.

III.5. Proposition. A concrete site $(\mathcal{K}, \gamma)$ is structurally small iff Siev $(\mathcal{K}, \gamma)$ is a fibre-small (and hence a legitimate) category.

Proof. (A) Let Siev $(\mathcal{K}, \gamma)$ be fibre-small . For each structured map $A \xrightarrow{a} X$ consider the structured sink

$$\downarrow(A,a) = \{(A',a') \mid (A',a') \equiv (A,a)\}.$$

By the pullback-stability of γ , it is a structured sieve. By the composition-closedness of γ , it is γ-closed. Since

$$\downarrow(A_1,a_1) = \downarrow(A_2,a_2) \quad \text{iff} \quad (A_1,a_1) \equiv_{\gamma} (A_2,a_2)$$

structural smallness of $(\mathcal{K}, \gamma)$ follows immediately from fibre-smallness of Siev $(\mathcal{K}, \gamma)$.

(B) Let $(\mathcal{X},\gamma)$ be structurally small. For each X in $\mathcal{X}$ we have a set $(A_i \xrightarrow{a_i} X)_{i\in I}$ of representatives as in the above definition. Each γ-closed sieve $\mathcal{A}$ has the property that with any $(A,a)\in\mathcal{A}$ it contains all smaller elements, i.e.,

$$(A,a)\in\mathcal{A} \text{ implies } \downarrow(A,a)\subset\mathcal{A}.$$

Consequently, $\mathcal{A}$ is determined by the set $I(\mathcal{A}) = \{i\in I \mid (A_i,a_i)\in\mathcal{A}\}$ in the sense that for two γ-closed structured sieves on X,

$$I(\mathcal{A}_1) = I(\mathcal{A}_2) \text{ implies } \mathcal{A}_1 = \mathcal{A}_2.$$

Since the collection of all subsets of I is a set, Siev $(\mathcal{X},\gamma)$ is small-fibred.

III.6. Examples. Let γ_0 be the (smallest) Grothendieck topology on $\mathcal{X}$, consisting of all full structured sieves. Then $(\mathcal{X},\gamma_0)$ is structurally small iff for each X in $\mathcal{X}$ the following equivalence has a set of representatives: $A \xrightarrow{a} X$ is equivalent to $B \xrightarrow{b} X$ iff there exist $\mathcal{X}$-morphisms $f:A\longrightarrow B$ with $a = b\circ f$, and $g:B\longrightarrow A$ with $b = a\circ g$. In particular:

(a) (Set, γ_0) is structurally small over Set: $A \xrightarrow{a} X$ is equivalent to $B \xrightarrow{b} X$ iff $a(A) = b(B)$.

(b) As shown in [1], $(\mathcal{X},\gamma_0)$ is not structurally small over Set for the categories $\mathcal{X}$ of metrizable spaces, relations and semigroups. In fact, Siev $(\mathcal{X},\gamma_0)$ is not even legitimate for these categories.

(c) $(\mathcal{X},\gamma_0)$ is structurally small over $\mathcal{T}$ iff $\mathcal{X}$ is small (i.e., a poset).

(d) More generally, for each Grothendieck topology γ on a small concrete category $\mathcal{X}$, $(\mathcal{X},\gamma)$ is structurally small.

III.7. Theorem. For a fibre-small concrete category $\mathcal{X}$ the following conditions are equivalent:

(1) $\mathcal{K}$ is universally topological;

(2) $\mathcal{K} \equiv$ Siev $(\mathcal{K}, \gamma)$ for some Grothendieck topology γ ;

(3) $\mathcal{K}$ is concretely isomorphic to Siev $(\mathcal{L}, \gamma)$ for some structurally small site $(\mathcal{L}, \gamma)$.

Proof. (1) $\longrightarrow$ (2). The collection γ of all final sinks in $\mathcal{K}$ is pullback-stable (since $\mathcal{K}$ is universally topological) and hence, γ is a Grothendieck topology. We are to prove that $K \cong$ Siev $(\mathcal{K}, \gamma)$, i.e., that each γ-closed structured sieve $\mathcal{A} = (A_i \xrightarrow{a_i} X)$ equals the full structured sieve $\widetilde{K}$ of some $\mathcal{K}$-object K. Let K be the final object in $\mathcal{K}$ of the structured sink $\mathcal{A}$ .Then the sink $\mathcal{A}^* = (A_i \xrightarrow{a_i} K)$ is in γ . The structured map $K \xrightarrow{id} X$ has the property that for each $A_i \xrightarrow{a_i} K$ in $\mathcal{A}^*$ we have $A_i \xrightarrow{id \circ a_i} X$ in $\mathcal{A}$; since $\mathcal{A}$ is γ-closed, we conclude that $K \xrightarrow{id} X \in \mathcal{A}$. Then $\widetilde{K} \subseteq \mathcal{A}$ because $\mathcal{A}$ is closed under composition, and hence $\widetilde{K} = \mathcal{A}$.

(2) $\longrightarrow$ (3) follows from III.5.

(3) $\longrightarrow$ (1). By III.5, Siev $(\mathcal{L}, \gamma)$ is small-fibred. It is sufficient to prove that it has universal final sinks. First, each structured sink $(\mathcal{A}_i \xrightarrow{f_i} X)_{i \in I}$ in Siev $(\mathcal{L}, \gamma)$ has a final object; the sink $\mathcal{A}$ of all $A \xrightarrow{a} X$ for which there exists a γ-sink $(B_j \xrightarrow{b_j} A)_{j \in J}$ such that each $B_j \xrightarrow{a \circ b_j} X$ $(j \in J)$ is equal to $B_j \xrightarrow{f_i \circ a'} X$ for some $i \in I$ and $B_j \xrightarrow{a'} |\mathcal{A}_i|$ in $\mathcal{A}_i$. (It is easy to verify that $\mathcal{A}$ is a structured sieve, since γ is pullback-stable, and that $\mathcal{A}$ is γ-closed, since γ is composition-closed.)

To prove the universality, let $\mathcal{D}$ be a γ-closed structured sieve (with $|\mathcal{D}| = Y$), let $k: \mathcal{D} \longrightarrow \mathcal{A}$ be a morphism in Siev $(\mathcal{L}, \gamma)$ and let S be the sink in Siev $(\mathcal{L}, \gamma)$, consisting of those

$g: \mathcal{B} \longrightarrow \mathcal{D}$ for which $k \circ g$ factors through some f_i:

$$\begin{array}{ccc} \mathcal{B} & \dashrightarrow & \mathcal{A} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_i} \\ \mathcal{D} & \xrightarrow[k]{} & \mathcal{A} \end{array}$$

To show that S is a final sink in Siev $(\mathcal{L}, \gamma)$, let $\mathcal{B} \xrightarrow{d} |\mathcal{D}|$ belong to $\mathcal{D}$ Then $\mathcal{B} \xrightarrow{kd} |\mathcal{A}|$ belongs to $\mathcal{A}$ Hence, by finality of $(\mathcal{A}_i \xrightarrow{f_i} \mathcal{A})_{i \in I}$, there exists a γ-sink $(\mathcal{B}_j \xrightarrow{b_j} \mathcal{B})_{j \in J}$ such that each kdb_j factors through some $f_{i(j)}$:

$$\begin{array}{ccccc} \mathcal{B}_j & \dashrightarrow & & & \mathcal{A}_{i(j)} \\ {\scriptstyle b_j}\downarrow & & & & \downarrow{\scriptstyle f_{i(j)}} \\ \mathcal{B} & \xrightarrow[d]{} & \mathcal{D} & \xrightarrow[k]{} & \mathcal{A} \end{array}$$

By the definition of S, each $db_j : \mathcal{B}_j \longrightarrow \mathcal{D}$ belongs to S. Hence, by the above description of final sinks in Siev $(\mathcal{L}, \gamma)$, S is final.

III.8. Examples. (a) Let $\mathcal{L}$ be the concrete category over Set with single object L, $|L| = \{0,1\}$ and hom $(L,L) = \{id\}$, and let γ_0 be the smallest Grothendieck topology on L. Then Siev $(\mathcal{L}, \gamma_0)$ is the category of relations.

(b) Let $\mathcal{L}'$ be the concrete category over Set with the single object L, $|L| = \{0,1\}$ and hom $(L,L) = \{id, f\}$ where f is the transposition, and let γ_0 be the smallest Grothendieck topology on L . Then Siev $(\mathcal{L}', \gamma_0)$ is the category of symmetric relations.

III.9. Corollary. Topological universes are precisely the concrete categories over Set concretely isomorphic to

$$\text{Siev } (\mathcal{L}, \gamma)_c$$

for structurally small concrete sites $(\mathcal{L}, \gamma)$ over Set. The objects of Siev $(\mathcal{L},\gamma)_c$ are all γ-closed structured epi-sieves.

III.10. Examples. (a) For L resp. L as in III.8, Siev $(\mathcal{L}, \gamma_0)_c$ is the category of reflexive relations, and Siev $(\mathcal{L}', \gamma_0)_c$ is the category of reflexive and symmetric relations.

(b) For $\mathcal{L} = \mathcal{X}$ = Set and γ the topology generated by all finite sinks (i.e., a sieve S belongs to γ iff there exists a finite subset T of S such that every $f \in S$ factors through some $g \in T$), Siev $(\mathcal{L},\gamma)_c$ is the category of bornological spaces [10].

(c) For the category L of finite sets, considered as a concrete category over Set, Siev $(\mathcal{L}, \gamma_0)_c$ is the category of simplicial complexes.

III.11. Remark. Categorical investigations motivated by problems in analysis (duality theory and theory of manifolds) are concerned primarily with cartesian closed topological c-categories (see e.g. Frölicher [5] and Seip [15]) and with topological universes (see Binz [3], Hogbe-Nlend [10] and Nel [13]).

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