Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 1, 206--208

Persistent URL: http://dml.cz/dmlcz/106442

linear bounded functionals on X is denoted by X^*.

Let U be the unit ball of X and let $B \subset X^*$ be the polar of U. Let us consider the collection $\mathcal{C} = \{n^{-1}B : n = 1,2,\dots\}$. Then the following condition C1 is fulfilled.

<u>C1</u> For each weak* neighbourhood W of the point $0 \in X^*$ there exists $E \in \mathcal{C}$ with the property $B \cap E \subset W$.

If, moreover, X is an Asplund space then, according to [1, Lemma 3], it holds

<u>C2</u> For each nonempty set $M \subset (1/2)\,B$ and for each $E \in \mathcal{C}$ there exists a relatively weak* open nonempty subset G of the set M so that $G - G \subset E$.

We now define $\mathcal{K}$ to be the class of all Banach spaces X of which duals X^* have the following property: there exist a weak* compact barrel $B \subset X^*$ and a countable collection $\mathcal{C}$ of weak* closed absolutely convex subsets of X^* so that the conditions C1 and C2 are satisfied.

The main result of this note is expressed by the following assertions (i) - (v).

(i) If $X \in \mathcal{K}$ and $T: X \to Y$ is a continuous linear operator with dense range then $Y \in \mathcal{K}$.

(ii) If $Y \in \mathcal{K}$ and $T: X \to Y$ is a continuous linear operator having the property $T^* Y^* = X^*$ then $X \in \mathcal{K}$.

(iii) The Cartesian product of two spaces from $\mathcal{K}$ belongs again to $\mathcal{K}$.

(iv) Every Asplund space and every weakly compactly generated Banach space is in $\mathcal{K}$.

(v) Every space from $\mathcal{K}$ is a weak Asplund space.

From (iv),(i),(ii) and (v) it immediately follows

<u>Theorem</u> (Christensen, Kenderov, [2]). Suppose that X is an Asplund space and $T: X \to Y$ is a continuous linear operator with dense range. Then every closed linear subspace of Y is a weak Asplund space.

In connection with [3] it is stated that $\mathcal{K}$ is a subclass of the class S.

References

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WIDTH COMPACTNESS THEOREM AND WELL-QUASI-ORDERING INFINITE GRAPHS

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Robertson and Seymour [1] introduced the following concept of a (tree-)width:

<u>Definition.</u> A tree-decomposition of a graph G is a couple (T,X), where T is a tree and $X = (X_t, t \in V(T))$ such that

(W1) $\bigcup_{t \in V(T)} X_t = V(G)$.

(W2) Every edge of G has both endpoints in some X_t.

(W3) $X_t \cap X_{t''} \subseteq X_{t'}$ whenever t' is on the path between t and t''

The width of a tree-decomposition is

$$\sup \left(|X_t| - 1 \right)$$

The (tree-)width of a graph G is the least w such that G admits a tree-decomposition of width w.

Originally, Robertson and Seymour introduced this notion for finite graphs only. However, the definition makes sense for infinite graphs as well. Before stating the Width Compactness Theorem let us recall the following famous theorem of Erdös and de Bruin:

Theorem 1: Let G be an infinite graph such that each finite subgraph has chromatic number $\leq k$. Then G itself has chromatic number $\leq k$. Moreover, let for a finite subset A of $V=V(G)$ f_A be an admissible coloring. Then the desired coloring f can be taken in such a way that for each finite $A \subseteq V$ there is a finite $V \supseteq B \supseteq A$ such that

$$f_B \restriction A = f \restriction A.$$

The Width Compactness Theorem is an analogical statement for width.

Width Compactness Theorem: Let G be an infinite graph such that each finite subgraph has width $\leq k$. Then G itself has width $\leq k$.

Although these statements are similar, there are at least two contradistinctions:

1. While the Erdös-de Bruin theorem requires almost full Axiom of Choice, the Width Compactness Theorem holds in $ZF+AC_\omega$ (i.e. in ZF with the axiom of countable choices).
2. While the desired coloring for the Erdös-de Bruin theorem can be taken as a limit of a suitable subsystem of the given partial colorings, this is no longer true for the tree-width.

Let us pass now to applications to well-quasi-ordering theory. In [1] Robertson and Seymour announced the following two theorems:

Theorem 2: Let H be a fixed finite planar graph. Then there is a number k such that each finite graph G which does not contain H as a minor has tree-width at most k.

Theorem 3: Let k be an integer. Then the class of all finite graphs of tree-width at most k is well-quasi-ordered, i.e. for any sequence

$$G_1, G_2, G_3, \ldots$$

of such graphs there are indices $i < j$ such that G_i is a minor of G_j. Let us recall that a graph H is a minor of a graph G if H can be obtained from G by taking subgraphs and contracting edges.

These theorems represent the best known result concerning well-quasi-ordering finite graphs. For infinite graphs the only known result is the following one due to Nash-Williams [2]:

Theorem 4: The class of (infinite) trees is well-quasi-ordered.

We are able to extend the Robertson-Seymour results to infinite graphs in the following way: The Width Compactness Theorem yields Theorem 2′ which is obtained from Theorem 2 by deleting the second occurence of finite. We are also able to prove Theorem 3′, obtained by deleting the word finite in Theorem 3. This is not trivial and requires the following lemma.

Lemma: If G has tree-width $< w$ then G admits a tree-decomposition of width $< w$ which satisfies the following three additional properties:

(W4) For any $t_1, t_2 \in V(T)$ and $k > 0$ either there are k disjoint paths in G, each between X^{t_1} and X^{t_2} or there exists a $t \in V(T)$ on the path between t_1 and t_2 such that $|X^t| < k$.

(W5) If $t \in V(T)$ is of degree one then $X^t = \emptyset$.

(W6) If $\{t, t'\} \in E(T)$ then $|X^t \triangle X^{t'}| = 1$.

(Here $\triangle$ denotes the symmetric difference.)

Theorems 2′ and 3′ imply immediately the following

Main Theorem: Let H be a fixed finite planar graph. Then the class of all (infinite) graphs which do not contain H as a minor is well-quasi-ordered.

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