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TWO EXAMPLES OF PSEUDO-RADIAL SPACES

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Abstract: Using an Ostaszewski-type construction , we prove in ZFC the existence of
a) Hausdorff pseudo-radial space of countable tightness which is not sequential,
b) Hausdorff pseudo-radial space in which tightness and quasi-character differ.

Key words and phrases: Pseudo-radial space, sequential space, tightness, quasi-character.

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Secondary 54G20, 54D55

Introduction. Pseudo-radial or chain-net spaces were introduced by H. Herrlich in 1967 [H]. They are a natural characterization of both linearly ordered and sequential spaces. (See for example [A],[MW].) Recall that a space X is pseudo-radial, if for each non-closed $M \subseteq X$ there is some $x \in \overline{M} - M$ and a (countable or transfinite) sequence $\{x_\alpha : \alpha \in \varkappa\} \subseteq M$ converging to x, i.e. each neighbourhood of x contains all x_α's beginning from some α_0 on.

If "there is some $x \in \overline{M} - M$" is replaced by "for each $x \in \overline{M} - M$" in the above definition, then the space is called radial or Fréchet chain-net.

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When investigating the cardinality properties of pseudo-radial and radial spaces, A.V. Arhangel'skiĭ, R. Isler and G. Tironi introduced a new cardinal invariant, so-called quasi-character, as follows.

$q\chi(X) = \min \{\tau : (\forall A \subseteq X)(\forall x \in \overline{A} - A)(\exists \mathcal{F} \subseteq \mathcal{P}(A))(|\mathcal{F}| \leq \tau \ \&$
$\& \ x \in \overline{\bigcup \mathcal{F}} \ \& \ (\forall F \in \mathcal{F})(x \notin \overline{F}))\}$.

They also proved that for T_1 radial spaces, $q\chi(X) = t(X)$, leaving the case of pseudo-radial spaces open. The best result in this direction says that $q\chi(X) = t(X)$ for a pseudo-radial space X provided t(X) is a successor cardinal and GCH is assumed ([AIT]).

We shall construct assuming ZFC only a Hausdorff pseudo-radial space Z with $\omega = q\chi(X) < t(X)$.

Using essentially the same construction, we shall also disprove the old conjecture that a pseudo-radial Hausdorff space with countable tightness is necessarily sequential. Here, of course, a plenty of counterexamples was published by various authors before ([O],[JKR] to mention few), but - as far as we know - all of them depended on some additional axiom of set-theory.

The construction. Let $\varkappa_0$ be a cardinal number, define by induction $\varkappa_{n+1} = 2^{\varkappa_n}$, $\varkappa = \sup \{\varkappa_n : n \in \omega\}$. Equip each $\varkappa_n$ with the discrete topology and denote by M the Tychonoff product $\prod_{n \in \omega} \varkappa_n$. Then M is a complete metric zero-dimensional space, $w(M) = \varkappa$, $|M| = 2^{\varkappa} = \varkappa^{\omega}$. Further, if $C \subseteq M$ and $|\overline{C}| > \varkappa$, then $|\overline{C}| = 2^{\varkappa}$. The last assertion needs, perhaps, a proof.

Denote by A_n the set $\{\xi \in \varkappa_n : |\pi_n^{-1}[\{\xi\}] \cap \overline{C}| > \varkappa\}$. Then $\overline{C} = \bigcup_{n \in \omega} \pi_n^{-1}[\varkappa_n - A_n] \cap \overline{C} \cup \prod_{n \in \omega} A_n \cap \overline{C}$. Since for each $n \in \omega$, $|\pi_n^{-1}[\varkappa_n - A_n] \cap \overline{C}| \leq \varkappa_n \cdot \varkappa = \varkappa$, we have $|\prod_{n \in \omega} A_n \cap \overline{C}| > \varkappa$. But this means that for each $\tau < \varkappa$ there is some $n \in \omega$ with $|A_n| > \tau$, otherwise $|\prod_{n \in \omega} A_n \cap \overline{C}| < \varkappa$ would contradict the assumption

$|\bar{C}| > \varkappa$. So we have proved the following:

If $\tau < \varkappa$, and if $|\bar{C}| > \varkappa$, then there is some $n \in \omega$ such that

$|\{\xi \in \varkappa_n : |\pi_n^{-1}[\{\xi\}] \cap \bar{C}| > \varkappa\}| > \tau$.

The standard branching argument works now: for each $n \in \omega$ and for each $\varphi \in \prod \{\varkappa_i : i \in n\}$ there is a closed $C_\varphi \subseteq \bar{C}$ such that $|C_\varphi| > \varkappa$, $C_\varphi \supseteq C_\psi$ if and only if $\varphi \subseteq \psi$, $C_\varphi \cap C_\psi = \emptyset$ if and only if there is $i \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi$ such that $\varphi(i) \neq \psi(i)$. Indeed, if $\varphi \in \prod_{i \in n} \varkappa_i$ and C_φ is known ($C_\emptyset = \bar{C}$ of course), then there is some $\varkappa_n$ with

$|\{\xi \in \varkappa_m : |\pi_m^{-1}[\{\xi\}] \cap C_\varphi| > \varkappa\}| > \varkappa_n$.

So we can select $C_{\varphi \frown \{\eta\}}$ for $\eta \in \varkappa_n$ to be a member of the disjoint family

$\{\pi_m^{-1}[\{\xi\}] \cap C_\varphi : \xi \in \varkappa_m \ \& \ |\pi_m^{-1}[\{\xi\}] \cap C_\varphi| > \varkappa\}$.

Since, obviously, for each $f \in \prod \{\varkappa_n : n \in \omega\}$, $\bigcap C_{f \upharpoonright n} \neq \emptyset$, we have $|\bar{C}| \geq \varkappa^\omega$ and, by our choice of $\varkappa$, $\varkappa^\omega = 2^\varkappa$

The above are the properties of M which we shall need further. Denote by $\wp$ the metric topology of M and fix some clopen base $\mathcal{B}$ for M, $|\mathcal{B}| = \varkappa$.

Enumerate all subsets of M of cardinality $\varkappa$ the closure of which is of cardinality $2^\varkappa$ as $\{T_\alpha : \alpha < 2^\varkappa\}$ in such a way that each set is listed $2^\varkappa$ times. Then for each T_α select a point $x_\alpha \in \bar{T}_\alpha^{\wp}$ and a convergent sequence S_α, such that $\lim S_\alpha = x_\alpha$, $S_\alpha \subseteq T_\alpha$ and for $\alpha \neq \beta$, $x_\alpha \neq x_\beta$. This is clearly possible for, by the previous choice, each T_α has $2^\varkappa$ accumulation points, so there is still one among them distinct of all x_β, $\beta < \alpha$.

Let $X = \{x_\alpha : \alpha < 2^\varkappa\}$ and denote again by $\wp$ the original topology of M restricted to X.

We shall construct a new topology τ on X in Ostaszewski style. Let $X_\alpha = \{x_\beta : \beta < \alpha\}$ for $\alpha < 2^\varkappa$. Define $\tau_\varkappa$ to be the discrete topology on $X_\varkappa$. Suppose (X_α, τ_α) have been defined for all $\alpha < \beta$ where $\beta < 2^\varkappa$. The inductive assumptions are as follows:

(i) For each $\alpha < \gamma < \beta$, (X_α, τ_α) is a subspace of (X_γ, τ_γ).

(ii) For each $\alpha < \gamma < \beta$, X_α is an open subset of (X_γ, τ_γ).

(iii) Each (X_α, τ_α) is first-countable, locally compact, locally countable.

(iv) The topology τ_α is finer than $\wp \upharpoonright X_\alpha$, for each $\alpha < \beta$.

If β is a limit cardinal, let $\tau_\beta = \bigcup\{\tau_\alpha : \alpha < \beta\}$. Obviously (X_β, τ_β) again satisfies (i) - (iv).

If $\beta = \alpha + 1$, we are to find a neighbourhood basis of x_α. There are two possibilities:

If $|S_\alpha \cap X_\alpha| < \omega$, let x_α be isolated in X_β, i.e. a neighbourhood basis of x_α is $\{x_\alpha\}$.

If $|S_\alpha \cap X_\alpha| = \omega$, select some clopen base of x_α in $(M, \wp)$, say $\{B_0 \supsetneq B_1 \supsetneq \ldots \supsetneq B_n \supsetneq \ldots\}$ such that for each n, $S_\alpha \cap X_\alpha \cap (B_n - B_{n+1}) \neq \emptyset$, and select $y_n \in S_\alpha \cap X_\alpha \cap (B_n - B_{n+1})$.

Since, by our assumption, τ_α is finer than $\wp$, $B_n - B_{n+1}$ is an open neighbourhood of y_n, so we can find a countable compact neighbourhood of y_n, say U_n, with $U_n \subseteq B_n - B_{n+1}$. Fix this choice of U_n's and define the neighbourhood base at x_α as

$\{\{x_\alpha\} \cup \bigcup\{U_n : n \geq k\} : k \in \omega\}$.

It is again clear that (i) - (iv) hold for (X_β, τ_β).

As might be expected, the desired topology τ for X is $\bigcup_{\alpha < 2^\varkappa} \tau_\alpha$.

Clearly, (X, τ) is first-countable, locally compact, locally countable. The next property, being crucial, has to be proved: if C is closed in the topology τ for X, then either $|C| \leq \varkappa$ or

$|C| = 2^{\varkappa}$.

Indeed, suppose $|C| > \varkappa$. Then $|\overline{C}^{\rho}| = 2^{\varkappa}$ and, since $w(M) = \varkappa$, there is a subset $T \subseteq C$, $|T| = \varkappa$, such that $\overline{T}^{\rho} = \overline{C}^{\rho}$. In particular, $|\overline{T}^{\rho}| = 2^{\varkappa}$.

Since $|T| = \varkappa$, there is some $\gamma < 2^{\varkappa}$ such that $T \subseteq X_{\gamma}$. The set T appears $2^{\varkappa}$ times in our list, and in each occurence $\alpha > \gamma$ with $T_{\alpha} = T$, the point x_{α} belongs to $\overline{T}^{\tau}$. So $|\overline{T}^{\tau}| = 2^{\varkappa}$ and since C was assumed to be closed in τ, $C \supseteq \overline{T}^{\tau}$.

Having passed the difficult part of the construction, choose a point ∞ not belonging to X and define $Z = X \cup \{\infty\}$ with the neighbourhood base at ∞ consisting of all sets $\{\infty\} \cup (X - A)$, where $A \subseteq X$, A is closed in τ, $|A| \leq \varkappa$. The space Z is Hausdorff This is trivial, since each point of X has a countable compact neighbourhood.

The space Z is pseudo-radial. Indeed, let $W \subseteq Z$, $\overline{W}^{Z} \neq W$. If there is some $x \in X$, $x \in \overline{W}^{Z} - W$, then there is a convergent sequence in W with x as its limit, by the first-countability of (X, τ). Otherwise $\overline{W}^{Z} - W = \{\infty\}$, hence W is closed in (X, τ) and ∞ is its accumulation point. According to our definition of topology on Z, $|W| > \varkappa$, hence $|W| = 2^{\varkappa}$ and for each neighbourhood U of ∞, $|W - U| \leq \varkappa$. So any subset of W of cardinality $\varkappa^{+}$ converges to ∞.

The tightness of Z equals $\varkappa$. Indeed, if $W \subseteq X$ and $\infty \in \overline{W}^{Z}$ then $|W| > \varkappa$. There is a set $T \subseteq W$, $|T| = \varkappa$ such that $\overline{T}^{\rho} \supseteq W$. But this implies that $|\overline{T}^{\rho}| = 2^{\varkappa}$, therefore $|\overline{T}^{\tau}| = 2^{\varkappa}$, too. But then $\infty \in \overline{T}^{Z}$, therefore $t(Z) \leq \varkappa$. (Other points than ∞ are, of course, uninteresting.) On the other hand, $t(Z) \geq \varkappa$ for the trivial reason that if $W \subseteq X$, if $|W| < \varkappa$ then $|\overline{W}^{\rho}| < \varkappa$, too, so $\{\infty\} \cup (X - \overline{W}^{\tau})$ is a neighbourhood of ∞ disjoint with W.

It remains to consider two special cases.

1. Let $\varkappa_0 = 2$. In this case, the starting metric space is

nothing else than the Cantor set and the final space Z is pseudo-radial, Hausdorff and $t(Z) = \omega$.

Z is not sequential. Consider $\overline{X}^Z - X$. This set contains the point ∞ only, and there is no sequence $\{s_n : n \in \omega\}$ converging to ∞ : notice that $\{s_n : n \in \omega\}$ should be a closed discrete subset of (X, τ) then, but in this case, $\{\infty\} \cup (X - \{s_n : n \in \omega\})$ is a neighbourhood of ∞ disjoint with it.

2. Let $\varkappa_0 = \omega$. We have $\varkappa > \omega$ in this case, and Z is pseudo-radial, Hausdorff and $t(Z) = \varkappa$.

Yet $q\chi(Z) = \omega$. This is clear if one considers points from X by the 1^{st} countability of (X, τ).

Let us discuss the case $W \subseteq X$, $\infty \in \overline{W}^Z$. Since $t(Z) = \varkappa$, there is some $T \subseteq W$, $|T| = \varkappa$, $\infty \in \overline{T}^Z$. Making use of the fact that $\varkappa$ is a singular cardinal, find some $T_n \subseteq T$ such that $T = \cup \{T_n : n \in \omega\}$, and for each n, $|T_n| < \varkappa$. Then for each n, $|\overline{T}_n^Z| < \varkappa$, too, so $\infty \notin \overline{T}_n^Z$. So $q\chi(Z) = \omega$.

Added in proof. After this paper was completed we learned from I. Juhász that he and W. Weiss found independently examples of pseudo-radial spaces with similar properties. We do not know any details of their proof.

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