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COUNTEREXAMPLE TO THE REGULARITY OF WEAK SOLUTION OF THE QUASILINEAR PARABOLIC SYSTEM

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Abstract: The example of the quasilinear parabolic system is given for which there exists a bounded solution of boundary value problem (with Lipschitz continuous initial and boundary data) having the discontinuity developed in some $t>0$.

Key words: Quasilinear parabolic systems, boundary value problem, regularity.

Classification: 35K35

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1. Introduction. Using standard elliptic counterexamples we can easily construct the quasilinear parabolic system with a bounded weak solution which is not Hölder continuous. Namely, we can consider the discontinuous solution $u=u(x)$ of the elliptic system as a stationary solution of a corresponding parabolic system. In this case, each point of discontinuity is invariant with respect to the variable t. Thus, in general, the regularity for quasilinear parabolic systems (with the number of spatial variables $n\geq 3$) does not take place and the partial regularity results (see e.g. [4],[5],[6]) play the important role.

A more subtle question to be answered is whether some bounded weak solution of the parabolic system could start as a smooth one and develop the discontinuity in some moment $t>0$. The first example giving the positive answer was constructed by M. Struwe [1]. He considered the systems of the diagonal form

(1) $u^i_t - D_\alpha(a^{\alpha\beta}(t,x)D_\beta u^i) = f^i(t,x,u,D_x u)$, $i=1,\dots,3$.

(Here $x=[x_1,\dots,x_3]$, $u=[u^1,\dots,u^3]$, $u^i_t = \partial u^i/\partial t$, $D_\alpha u^i = \partial u^i/\partial x_\alpha$ and $D_x u = \{D_\alpha u^i\}^3_{\alpha,i=1}$. Throughout the whole paper, repeated indices are summed over 1,2,3.)

The coefficients $a^{\alpha\beta}$ are supposed to be bounded and measurable with

(2) $a^{\alpha\beta}(t,x)\,\xi_\alpha\xi_\beta \geqq \lambda|\xi|^2$ a.e..

Function f has the quadratic growth in p:

(3) $|f^i(t,x,u,p)| \leqq a|p|^2 + b$, $i=1,\dots,3$.

Struwe's example possesses the bounded weak solution $u=u(t,x)$ on

(4) $Q = (0,\infty)\times B$ (B is a unit ball in R_3)

which is Lipschitz continuous on the parabolic boundary Γ of Q and discontinuous just on the half-line $\{[t,x]; t\geqq 1,\ x=0\}$.

As it was shown in [2],[3], each bounded weak solution of the system (1)-(3) is Hölder continuous on Q if

(5) $a\|u\|_{L_\infty}\lambda^{-1} < 1$.

In Struwe's counterexample the condition (5) is strongly violated - the left hand side in (5) is much bigger than 1.

In our paper we give the positive answer to the problem for the system

(6) $u^i_t - D_\alpha(A^{ij}_{\alpha\beta}(t,x,u)D_\beta u^j) = 0$, $i=1,\dots,3$,

(7) $A^{ij}_{\alpha\beta}(t,x,u)\,\xi_{\alpha i}\xi_{\beta j} \geqq \mu|\xi|^2$, $\forall\xi\in R_3\times R_3$, $(\mu>0)$.

The example is given in Section 3, meanwhile Section 4 contains necessary calculations. In Section 5 we return to the system (1). We construct for each $\varepsilon > 0$ a system of this type which has the solution with the discontinuity developed in some $t > 0$ and for which

(8) $a \|u\|_{L_\infty} \lambda^{-1} < 1,5(1+\varepsilon)$.

This result gives certain approximation to the Struwe's hypothesis that the loss of regularity properties of initial data is possible if only $a \|u\|_{L_\infty} \lambda^{-1} \geq 1$. Using numerical calculations we conjecture that for our system (constructed for $\varepsilon > 0$ sufficiently small) holds

(8') $a \|u\|_{L_\infty} \lambda^{-1} < 1,21$.

2. Notations. Definitions. Auxiliarities. In this section, besides the definition of the weak solution we summon the properties of some functions E, F, φ, q used as the coefficients in the example constructed in the next sections.

Denote for $T > 0$

(9) $Q_T = \{[t,x]; t \in (0,T), x \in B\}$,

(10) $W_2^{0,1}(Q_T) = \{u \in L_2(Q_T);\ D_\alpha u \in L_2(Q_T),\ \alpha = 1,\dots,3\}$.

Let Q be given as (4) and let

(11) $\Gamma = [(0,\infty) \times \partial B] \cup [\{0\} \times B]$

be its parabolic boundary. Suppose that the coefficients $A^{ij}_{\alpha\beta} = A^{ij}_{\alpha\beta}(t,x,u) : Q \times R_3 \longrightarrow R$ $(i,j,\alpha,\beta = 1,\dots,3)$ of the system (6) are bounded, continuous on R_3 as the functions of u for almost all $[t,x] \in Q$ and measurable on Q as the functions of $[t,x]$ for all $u \in R_3$. Let further

(12) u_0 be given Lipschitz continuous function on Γ.

Definition. The function $u : Q \longrightarrow R_3$ which is bounded and measurable and such that for all $T > 0$ u belongs to the space

$W_2^{0,1}(Q_T)$ is said to be a weak solution of the boundary value problem for the system (6) with the boundary condition u_0 if

(i) for all $\psi \in C^{\infty}(\bar{Q})$ with the compact support in $Q \cup \cup[\{0\} \times B]$ holds

(13) $$\int_Q [u^i \psi_t^i - A_{\alpha\beta}^{ij} D_\beta u^j D_\alpha \psi^i] dt\, dx = - \int_{\{0\} \times B} u_0^i(0,x)\, \psi^i(0,x)\, dx,$$

(ii) $u(t,\cdot) = u_0(t,\cdot)$ in the sense of traces for almost all $t \in (0,\infty)$.

Remark. Similarly we could define a weak solution of the boundary value problem for the system (1).

Define for $\xi \in (0,\infty)$

(14) $$E(\xi) = \int_0^{\xi} e^{-\tau^2}\, d\tau ,$$

(15) $$F(\xi) = \frac{E(\xi) - \xi e^{-\xi^2}}{\xi^2} ,$$

(16) $$q(\xi) = \frac{F(\xi)}{E(\xi)} ,$$

(17) $$\varphi(\xi) = 2E(\xi) - F(\xi).$$

Denote for the function $f:(0,\infty) \to R$

(18) $$f(0) = \lim_{\xi \to 0_+} f(\xi),\ f(\infty) = \lim_{\xi \to \infty} f(\xi).$$

Lemma 1. For the functions E, F, q and φ we have

(19) $$E(0) = F(0) = \varphi(0) = 0,\ q(0) = \frac{2}{3},\ \lim_{\xi \to 0_+} \frac{E(\xi)}{\xi} = 1,$$

(20) $$E(\infty) = \frac{\sqrt{\pi}}{2},\ F(\infty) = 0,\ \varphi(\infty) = \sqrt{\pi} ,\ q(\infty) = 0,$$

(21) All the functions E, F, φ and q are continuous and bounded on $(0,\infty)$,

(22) $E'(\xi) = e^{-\xi^2}$, $F'(\xi) = 2(e^{-\xi^2} - \frac{F(\xi)}{\xi})$,

$$\varphi'(\xi) = \frac{2F(\xi)}{\xi}, \quad \varphi(\xi) - \xi\varphi'(\xi) = 2E - 3F.$$

Proof. All formulas can be established by means of elementary calculus.

Lemma 2. For the function q holds

(23) $q((0,\infty)) = (0,\frac{2}{3})$.

Proof. Try to find α (>0) such that

(24) $q(\xi) < \alpha$ for all $\xi \in (0,\infty)$.

The last inequality takes place iff

(25) $H(\xi) \equiv E(\xi) - \xi e^{-\xi^2} - \alpha\xi^2 E(\xi) < 0$, $\xi \in (0,\infty)$.

But $H(0) = 0$ and

(26) $H'(\xi) = \xi^2 e^{-\xi^2}(2 - \alpha) - 2\alpha\,\xi\,E(\xi) =$

$$= \xi^2 e^{-\xi^2}(2 - 3\alpha) + 2\alpha\,\xi\,(\xi e^{-\xi^2} - E(\xi)).$$

Setting $\alpha = \frac{2}{3}$ in (26) we can see that the first term equals zero meanwhile the negativeness of the second term is obvious. So we have proved that $q(\xi) < \frac{2}{3}$ for all $\xi \in (0,\infty)$. This together with the non-negativeness of q, its continuity and the relations (19)-(21) gives the assertion of the lemma.

3. Counterexample. Let q and φ be the functions defined by (16),(17). Let $|x|$ be the Euclidean norm of the point x in R_3. For $t<1$ put $\xi = |x|/2\sqrt{1-t}$. The function

(27) $$u^i(t,x) = \begin{cases} \frac{x_i}{|x|} & \text{for } t \geq 1,\ x \in R_3 \setminus \{0\}, \\ \frac{x_i}{|x|}\frac{\varphi(\xi)}{\sqrt{x}} & \text{for } t < 1,\ x \in R_3 \setminus \{0\}, \\ 0 & \text{for } t \in R,\ x = 0 \end{cases}$$

is a weak solution of the boundary value problem for the system

$$(28)\quad u^i_t - D_\alpha[(\delta_{\alpha\beta}\delta_{ij} + d_{\alpha i}d_{\beta j})D_\beta u^j] = 0,\ i=1,\dots,3,$$

$$(29)\quad d_{\alpha i} = \frac{1}{\sqrt{4(a-2)+(6+a)q(4-3q)}}\{-\delta_{\alpha i}[a-2+ +(6+a)\frac{q(\xi)}{2}] - \frac{x_i x_\alpha}{|x|^2}(6+a)(1-\frac{3q(\xi)}{2})\}\ \text{if}\ t<1,$$

$$(30)\quad d_{\alpha i} = \frac{1}{\sqrt{4(a-2)}}\{-\delta_{\alpha i}(a-2) - \frac{x_i x_\alpha}{|x|^2}(6+a)\},\ t \geqq 1,$$

In the domain Q. (Here a is a real parameter. As a boundary function u_0 we take here the trace of the function u given by (27) on the parabolic boundary Γ .)

In the next section we sketch how the system (28)-(30) was deduced. Further we shall prove that for $a > 2$ the operator $-D_\alpha[(\delta_{\alpha\beta}\ \delta_{ij} + d_{\alpha i}d_{\beta j})D_\beta]$ is elliptic. Thus the system (28)-(30) is parabolic in this case. Its coefficients are bounded. They are also continuous except the points of the half-line $\{[t,x];t\geqq 1, x=0\}$ The function u itself is continuous at the same set meanwhile on the parabolic boundary Γ u is Lipschitz continuous.

Summarizing we obtain

<u>Assertion 1</u>. Let $a > 2$. The function u defined as (27) is a bounded weak solution of the boundary value problem in Q for the linear parabolic system (28)-(30) with the Lipschitz continuous data on Γ The coefficients of the system are bounded measurable functions. The solution u develops the discontinuity at the point $[t,x] = [1,0]$.

Let η be the inverse function to φ on $\langle 0,\sqrt{\pi}\rangle$ Denote

$$(31)\quad G(\omega) = \begin{cases} q(\eta(\sqrt{\pi}\,\omega)), & 0 \leqq \omega < 1, \\ 0 & \omega \geqq 1, \end{cases}$$

$$(32)\quad M(\omega) = \frac{(6+a)(1-(3/2)G(\omega))}{\omega^2},\ \omega > 0,\ M(0) = \lim_{\omega\to 0} M(\omega).$$

Then the function u defined by (27) is also the weak solution of the quasilinear parabolic system of the type

(33) $w^i_t - D_\alpha (A^{ij}_{\alpha\beta}(u) D_x w^j) = 0$

with the coefficients

(34) $A^{ij}_{\alpha\beta} = \delta_{\alpha\beta}\,\delta_{ij} + \nu_{\alpha i}\,\nu_{\beta j}$,

where

(35) $$\nu_{\alpha i} = \frac{1}{\sqrt{4(a-2)+(6+a)G(|u|)(4-3G(|u|))}} \times$$

$$\times \{ -\delta_{\alpha i}\,[a-2 + (6+a)\frac{G(|u|)}{2}] - M(|u|)\, u_i u_\alpha \}.$$

Indeed, for u given by (27) we have $|u| = \varphi(\xi)/\sqrt{\pi}$ $\nu_{\alpha i}(u) = d_{\alpha i}(t,x)$ (cf. (29),(30)).

Assertion 2. Let $a > 2$. The function u defined as (27) is a weak solution of the boundary value problem in Q for the quasilinear parabolic system of the type (33) with the coefficients given by (34)-(35) and with the Lipschitz continuous data on Γ The coefficients are bounded and continuous on R_3.

4. Calculations. In the course of this section we often use Lemmas 1, 2 from Section 2 without mentioning it explicitly. Let us recall

(36) $$\xi = \frac{|x|}{2\sqrt{1-t}} \quad \text{for } t < 1.$$

a) Properties of the solution u.

Lemma 3. (i) The function u given by the formula (27) is continuous in $R \times R_3$ except the points of the set

(37) $M = \{[t,x];\ t \geq 1,\ x=0\}$.

(ii) For each $T > 0$ u belongs to $W^{0,1}_2(Q_T)$.

(iii) The function u is Lipschitz continuous on the set Γ

given by (31).

Proof. Ad (i). Discontinuity in the points of M is obvious. The continuity in all other points of $R \times R_3$ can be easily obtained realizing that

$$(38)\quad \text{For } x \neq 0 \ \lim_{t \to 1-} u^i(t,x) = \frac{x_i}{|x|} \frac{1}{\sqrt{\pi}} \lim_{\xi \to \infty} \varphi(\xi) = \frac{x_i}{|x|},$$

$$(39)\quad \text{For } t < 1 \ \lim_{|x| \to 0} u^i(t,x) = 0$$

and either convergences are locally uniform.

Ad (ii). According to (i) and the boundedness of φ we have $u \in L_\infty$. For $x \neq 0$ we can calculate

$$(40)\quad D_\alpha u^i = \begin{cases} \frac{1}{|x|} [\delta_{\alpha i} - \frac{x_\alpha x_i}{|x|^2}], & t \geq 1, \\ \frac{1}{\sqrt{\pi}\ |x|} [\delta_{\alpha i} \varphi(\xi) - \frac{x_\alpha x_i}{|x|^2} (\varphi(\xi) - \xi \varphi'(\xi))], & t < 1. \end{cases}$$

As the functions in the squared brackets are bounded and measurable and the estimate $|D_\alpha u^i(t,x)| \leq C|x|^{-1}$ a.e. is valid we obtain that $D_\alpha u^i \in L_2(Q_T)$.

Ad (iii). The derivatives $D_\alpha u^i$ are bounded and continuous on Γ. To check this fact it suffices to consider the points on Γ where the formula defining u changes and the point [0,0]. There we get

$$(41)\quad \lim_{t \to 1-} D_\alpha u^i = \frac{1}{|x|} [\delta_{\alpha i} - \frac{x_\alpha x_i}{|x|^2}],$$

$$(42)\quad D_\alpha u^i(0,0) = \lim_{|x| \to 0} D_\alpha u^i(0,x) = \frac{2}{3\sqrt{\pi}} \delta_{\alpha i}.$$

Further, we calculate

$$(43)\quad u^i_t = \begin{cases} 0 & \text{for } t \geq 1, \\ \frac{2}{\sqrt{\pi}} \frac{x_i}{|x|^3} \varphi'(\xi) \xi^3 & \text{for } t < 1. \end{cases}$$

For $x \neq 0$ we have

(44) $$\lim_{t \to 1-} u_t^i(t,x) = 2 \frac{x_i}{|x|^3} .$$

So u_t^i is bounded on the set $(0,\infty) \times \partial B$. From the boundedness of first derivatives of u on corresponding subsets of Γ it follows that u is Lipschitz continuous.

b) Sketch of the deduction of the parabolic system with given solution. Modifying the method used by J. Souček in the case of elliptic systems we try to find the parabolic system with given solution u in the form

(45) $$w_t^i - D_\alpha [(\delta_{\alpha\beta} \delta_{ij} - \frac{\tilde{d}_{\alpha i} \tilde{d}_{\beta j}}{(\tilde{d}, D_x u)}) D_\beta w^j] = 0,$$

setting

(46) $$\tilde{d}_{\alpha i} = D_\alpha u^i - b_{\alpha i} .$$

Substituting into the system (45) u (given solution) for w and $\tilde{d}_{\alpha i}$ from (46) we obtain the condition for $b_{\alpha i}$, namely,

(47) $$u_t^i = D_\alpha b_{\alpha i} .$$

So for each definite choice of u we are to find reasonable $b_{\alpha i}$ which satisfy the condition (47).

c) Deduction of the parabolic system with the solution u given by (27). We execute the more interesting part concerning the case $t < 1$. Look for $b_{\alpha i}$ in the form

(48) $$b_{\alpha i} = \frac{1}{\sqrt{\pi}} (\frac{\delta_{\alpha i}}{|x|} P(\xi) + \frac{x_\alpha x_i}{|x|^3} Q(\xi)), \quad (\xi \text{ given by (36)}).$$

After simple calculations we get

(49) $$D_\alpha b_{\alpha i} = \frac{x_i}{|x|^3} \frac{1}{\sqrt{\pi}} [(\xi P'(\xi) - P(\xi)) + (\xi Q'(\xi) + Q(\xi))].$$

The functions P and Q are proposed to be of the form

(50) $P(\xi) = aE(\xi) + gF(\xi)$, $Q(\xi) = cE(\xi) + dF(\xi)$,

$(a, g, c \text{ and } d \in R)$.

Differentiating we obtain

$$(51) \quad \begin{aligned} \xi P'(\xi) &= (a+2g)\, \xi e^{-\xi^2} - 2gF(\xi), \\ \xi Q'(\xi) &= (c+2d)\, \xi e^{-\xi^2} - 2dF(\xi). \end{aligned}$$

Substitution from (50),(51) to (49) leads to the following form of the condition (47):

$$(52) \quad (c-a)E(\xi) + (c+2d+a+2g)\, \xi e^{-\xi^2} - (d+3g)F(\xi) = \\ = 4E(\xi) - 4\, \xi e^{-\xi^2}.$$

Comparing the coefficients standing by E, F and $\xi e^{-\xi^2}$ we have

$$(53) \quad c-a=4,\ c+2d+a+2g = -4,\ d+3g = 0,$$

from which

$$(54) \quad g = \frac{a}{2} + 2,\ c = 2(\frac{a}{2} + 2),\ d = -3(\frac{a}{2} + 2).$$

This together with (49)-(51) yields

$$(55) \quad b_{\alpha i} = \frac{1}{\sqrt{\pi}} \left[\frac{\delta_{\alpha i}}{|x|} (aE + (\frac{a}{2} + 2)F) + \frac{x_\alpha x_i}{|x|^3} (\frac{a}{2} + 2)(2E - 3F) \right].$$

Rewriting the corresponding part of (40) in the form

$$(56) \quad D_\alpha u^i = \frac{1}{\sqrt{\pi}} \left[\frac{\delta_{\alpha i}}{|x|} (2E - F) - \frac{x_\alpha x_i}{|x|^3} (2E - 3F) \right]$$

we get with use of (46)

$$(57) \quad \tilde{d}_{\alpha i} = \frac{E}{\sqrt{\pi}\, |x|} \left[-\delta_{\alpha i} (a - 2 + (a+6)\frac{q}{2}) - \frac{x_\alpha x_i}{|x|^3} (a+6)(1 - \frac{3q}{2}) \right].$$

Now we can calculate

$$(58) \quad (\tilde{d}, D_x u) = \frac{E^2}{\pi\, |x|^2} \left[4(2-a) + (a+6)q(3q-4) \right].$$

<u>Lemma 4</u>. Let $a > 2$. Then $(\tilde{d}, D_x u) < 0$ and the operator

$$(59) \quad - D_\alpha [(\delta_{\alpha\beta} \ \delta_{ij} - \frac{\tilde{d}_{\alpha i} \tilde{d}_{\beta j}}{(\tilde{d}, D_x u)}) D_\beta]$$

is elliptic. Thus, in this case, the system (45) is parabolic.

Proof. If $(\tilde{d}, D_x u) < 0$, then

$$(60) \quad (\delta_{\alpha\beta} \ \delta_{ij} - \frac{\tilde{d}_{\alpha i} \tilde{d}_{\beta j}}{(\tilde{d}, D_x u)}) \xi_{\alpha i} \xi_{\beta j} = | \xi |^2 - \frac{(\tilde{d}, \xi)^2}{(\tilde{d}, D_x u)} \geqq | \xi |^2 ,$$

so the ellipticity is obvious.

The relation $(\tilde{d}, D_x u) < 0$ for $a > 2$ follows immediately from the fact that $q : (0, \infty) \rightarrow (0, \frac{2}{3})$, as it was proved in Section 2 (Lemma 2)

Substituting now to (45) for $\tilde{d}$, $(\tilde{d}, D_x u)$ from (57) and (58) we obtain in the end the system (28),(29).

Remarks. 1) In the case $n > 3$ we may proceed in the similar way.

2) The idea of the expression of the functions $b_{\alpha i}$ in the form (48),(50) arose in the connection with our attempts to exploit the original Struwe´s counterexample [1] Trying to use it directly, we were not able to remove the discontinuity of the obtained system of the type (1) from the points of the whole hyperplane $\{[t,x];\ t=1,\ x \in R_3\}$.

3) The standard proof of the fact that the function u defined by (27) is a weak solution of the boundary value problem (13) with the coefficients given by (28)-(30) is omitted.

5. Remark to the systems with quadratic growth. Let u be the function defined by (27). After an easy but tedious calculation (which we carry out for $t < 1$ only) we get for arbitrary $A > 0$

$$(59) \quad \frac{u^i_t - A \Delta u^i}{|D_x u|^2} = \frac{x_i}{|x|} (A+1) \frac{1}{2} \Upsilon (\xi),$$

where

$$(60)\quad \Psi(\xi) = \sqrt{\pi}\ \frac{E - \xi e^{-\xi^2}}{E^2 - EF + \frac{3}{4}F^2}$$

So u is a weak solution of the boundary value problem for the system

$$(61)\quad w_t^i - A\,\Delta w^i = \frac{x_i}{|x|}(A+1)\,\frac{1}{2}\ \Psi(\xi)\ |D_x w|^2$$

$$(\equiv f(t,x,w,D_x w))$$

with Lipschitz continuous boundary data on Γ.

So for the right hand side in (61) we have the estimate

$$(62)\quad f(t,x,w,p) \leqq a\ |p|^2$$

with

$$(63)\quad a = (A+1)\,\frac{1}{2}\ \sup\ \Psi(\xi).$$

Because of $\|u\|_{L_\infty} = 1$ and $\lambda = A$ we have

$$(64)\quad a\,\|u\|_{L_\infty}\,\lambda^{-1} = (1 + \frac{1}{A})\,\frac{1}{2}\ \sup\ \Psi(\xi).$$

Provided we had an estimate

$$(65)\quad \frac{1}{2}\ \sup\ \Psi(\xi) = K$$

we could choose for each $\varepsilon > 0$ such $A > 0$ in (61) that

$$(66)\quad a\,\|u\|_{L_\infty}\,\lambda^{-1} < K(1+\varepsilon).$$

Assertion 3. The estimate (65) holds with K = 1,5.

Proof. Using (16) we can rewrite

$$(67)\quad \Psi(\xi) = \frac{\sqrt{\pi}}{1 - q + \frac{3}{4}q^2}\ \frac{E - \xi e^{-\xi^2}}{E^2}$$

Taking account of $q \in (0,\frac{2}{3})$ and $1 - q + \frac{3}{4}q^2 = 3(\frac{1}{2}q - \frac{1}{3})^2 + \frac{2}{3}$ we have

$$(68)\quad \frac{\sqrt{\pi}}{1 - q + \frac{3}{4}q^2} < \frac{3}{2}\sqrt{\pi}.$$

Denoting

(69) $S(\xi) = \dfrac{E(\xi) - \xi e^{-\xi^2}}{E^2(\xi)}$

we get $S(0) = 0$, $S(\infty) = \frac{2}{\sqrt{\pi}}$ and using (15),(16) and Lemma 2:

$$S'(\xi) = 2E^{-3} e^{-\xi^2} \xi^2(E-F) = 2E^{-2} e^{-\xi^2} \xi^2(1-q) > 0$$

for all $\xi > 0$. Thus S increases and

(70) $0 < S(\xi) < \frac{2}{\sqrt{\pi}}$.

From (67)-(70) we obtain

(71) $\Psi(\xi) < \frac{3}{2}\sqrt{\pi}\ \frac{2}{\sqrt{\pi}} = 3, \quad \forall \xi > 0.$

Remark. The estimate (65) holds probably with a $K < 1,21$ as the following calculated values of Ψ suggest.

ξ	0,1	0,5	1,0	1,2	1,87	1,89
$\Psi(\xi)$	0,1767	0,8962	1,7563	2,0268	2,3997	2,4003

ξ	1,91	1,93	2,0	3,0	5,0	15,0
$\Psi(\xi)$	2,4005	2,4003	2,3971	2,2258	2,0807	2,0089

we can conjecture that $\sup \Psi(\xi) < 2,42$.

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