Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 1, 83--86

Persistent URL: `http://dml.cz/dmlcz/106431`

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27,1 (1986)

APPROXIMATE SYMMETRIC DERIVATIVE AND MONOTONICITY

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Abstract: It is proved that, if f is a measurable function on the real line with the lower approximate symmetric derivative nonnegative, then it is essentially nondecreasing on some interval.

Key word: Approximate symmetric derivative

Classification: 26A24

This note gives a partial answer to the following problem: If f is a continuous function on an interval I and if the (lower) approximate symmetric derivative is nonnegative, is f necessarily nondecreasing?

Though several authors have presented incorrect proofs of the positive answer (cf. [2],[3],[5]; for a survey see [4]), the problem remains open, even in the case $f_{ap}^{(1)} = 0$ everywhere on I. Our partial answer is given in the following statement.

Theorem: If f is a measurable function defined on an open interval I and with $\underline{f}_{ap}^{(1)} \geq 0$ on I, then there is an open interval J included in I such that f is nondecreasing on the set of those points of J at which it is approximately continuous.

Recall that $\underline{f}_{ap}^{(1)}(x) = \text{ap} \liminf_{t \to 0} (f(x+t) - f(x-t))/2t.$

To prove the theorem, we need the following lemma.

Lemma: Suppose that f is a measurable function defined on a bounded interval (c,d), $r > s$ are real numbers, $0 < h < (d-c)/2$,

$$|\{x; c < x < c+2h \text{ and } f(x) > r\}| > 3h/2 \text{ and}$$

$$|\{x; d-2h < x < d \text{ and } f(x) < s\}| > 3h/2.$$

Then there is a nonempty open subset G of $(c+h, d-h)$ with

$$|\{t; 0 < t < h \text{ and } f(x-t) > f(x+t)\}| > h/9$$

for every x in G.

<u>Proof of the theorem</u>: Suppose first that $\underline{f}_{ap}^{(1)} > 0$ on I. Using the Baire Category Theorem, we can find an open interval $J = (a,b)$ contained in I and $\delta > 0$ such that we have $(a-\delta, a+\delta) \subset I$ and the set

$$E = \{x \in J; |\{t; 0 < t < h \text{ and } f(x-t) > f(x+t)\}| < h/9$$
$$\text{for every } h \in (0,\delta)\}$$

is dense in J. We prove that f is nondecreasing on the set of those points of J at which it is approximately continuous. Assume, on the contrary, that $a < c < d < b$, f is approximately continuous at c as well as at d and that $f(c) > f(d)$. Then there is h from $(0, \min(\delta, (d-c)/4))$ such that

$$|\{x; c < x < c+2h \text{ and } f(x) > 2/3 \cdot f(c) + 1/3 \cdot f(d)\}| > 3h/2$$

and

$$|\{x; d < x < d-2h \text{ and } f(x) < 1/3 \cdot f(c) + 2/3 \cdot f(d)\}| > 3h/2.$$

But this obviously contradicts the previous lemma.

To prove the general case, we use the above result to infer that the function $x \to f(x)+x$ is nondecreasing on the set of points of approximate continuity belonging to the open interval $J = (a,b)$. Hence there is a function g on J such that $g = f$ a.e., and $x \to$ $\to g(x)+x$ is nondecreasing. Whenever $a < c < d < b$, we get

$$(g(d)+d) - (g(c)+c) \geq \int_c^d (g'(x)+1)dx =$$
$$= \int_c^d \underline{f}_{ap}^{(1)}(x)dx + (d-c) \geq (d-c),$$

hence g is nondecreasing on J, which implies our statement since $f(x) = g(x)$ whenever x is in J and f is approximately continuous at x.

<u>Proof of the lemma</u>: Let $g = (r+s)/2$ and $E = \{x; c < x < d$ and $f(x) \geq q\}$ and $F = (c,d) - E$. We define the function

$$g: x \mapsto |\{t; 0 < t < h \text{ and } f(x-t) \geq q > f(x+t)\}| = |(x,x+h) \cap F \cap$$
$$\cap (2x - E)| = \int_x^{x+h} \chi_E(t) \quad (2x-t)dt.$$

Consider the difference

$$|g(x+\delta) - g(x)| \leq 2|\delta| + \int_{x+\delta}^{x+h} |\chi_E(2(x+\delta)-t) - \chi_E(2x-t)| \, dt \leq$$
$$\leq 2|\delta| + \int_x^{x+h} |\chi_{E'}(t) - \chi_{E'}(t-2\delta)| \, dt \ ; \ E' = 2x - E$$

(E' is a measurable set of a finite measure). The last integral tends to zero as δ goes to 0 (this easy fact is mentioned, for example, in [1], part VI.8, proof of Thm. 20), hence g is continuous on $(c+h,d-h)$, so it is sufficient to find $x \in (c+h,d-h)$ such that $g(x) > h/9$.

Since $|(x-h,x) \cap E| > 3h/2 - h = h/2$ if $x \in [c+h,c+2h]$ and $|(x,x+h) \cap E| < 2h - 3/2h = h/2$ if $x \in [d-2h,d-h]$, the number

$$z = \sup \{x \in [c+h,d-h]; \ |E \cap (x-h,x)| \geq h/2\}$$

is well-defined and belongs to the interval $[c+2h,d-2h]$. Therefore,

$$|E \cap (z-h,z)| \geq h/2 \text{ and also}$$
$$|F \cap (z,z+h)| \geq h/2.$$

Thus, using the substitution $x = z+u-v$, $y = z+u+v$, we get

$$h^2/4 \leq |\{(x,y) \in (z-h,z) \times (z,z+h); \ f(x) \geq q > f(y)\}| =$$
$$= 2 \int_{-h/2}^{h/2} |\{v \in (|u|,h-|u|); \ f(z+u-v) \geq q > f(z+u+v)\}| \, du \leq$$
$$\leq 2 \int_{-h/2}^{h/2} |\{v \in (0,h); \ f(z+u-v) \geq q > f(z+u+v)\}| \, du.$$

Consequently, there is $u \in (-h/2,h/2)$ such that

$$g(z+u) = |\{v \in (0,h);\ f(z+u-v) \geq q > f(z+u+v)\}| \geq h/8 > h/9$$

R e f e r e n c e s

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(Oblatum 26.3. 1985)