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A NOTE ON DENSE SUBSPACES OF DYADIC COMPACT SPACES

M. G. TKAČENKO

Abstract: We answer Arhangel'skiĭ's question by the following theorem: Let S be a dense subspace of some dyadic compact space X such that the tightness of S is countable and the lower $\aleph_0$-closure of S coincides with X. Then X is separable.

Some generalizations of this result are given.

Key words and phrases: Dyadic compact space, dense subspace, the tightness, the lower $\aleph_0$-closure, τ-approximative space, $\varkappa$-adic space.

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Recently A.V. Arhangel'skiĭ put the following question: for which cardinals τ there exists a subspace M of Tychonoff cube I^{τ} such that M is of countable tightness and the lower $\aleph_0$-closure of M in I^{τ} coincides exactly with I^{τ} ? Obviously, for each cardinal $\tau \leq 2^{\aleph_0}$ we can choose a suitable M to be a countable dense subspace of I^{τ}. It is shown here that there is no any subspace M of I^{τ} with the above properties for $\tau > 2^{\aleph_0}$. An analogous situation takes place for subspaces of dyadic compact spaces. These results follow from Theorem 1 which is proved in § 1.

In the second section we strengthen our results to the case of q-adic compact spaces (see Definition 3) and show that Theorem 1 holds for any compact space which is a continuous image of a dense subspace of some product $\prod_{\alpha \in A} X_{\alpha}$ with $d(X_{\alpha}) \leq \tau$ for each

$\alpha \in A$. We put also some questions closely related with the theme of the paper.

The following notations are used: $\exp(\tau) = 2^{\tau}$, $\exp_2(\tau) = \exp(\exp(\tau))$ and so on. If S is a subset of X and τ is an infinite cardinal, we put $[S]_{\tau} = \bigcup\{[T]_X : T \subseteq S \text{ and } |T| \leq \tau\}$ and say that $[S]_{\tau}$ is a lower τ-closure of S in X. An intersection of any family γ of open subsets of X with $|\gamma| \leq \tau$ is called a $G_{\delta,\tau}$-subset of X. All spaces are assumed to be completely regular (if not mentioned otherwise).

§ 1. The main result

Lemma 1. Let X be a regular space, $S \subseteq X$ and $X = [S]_{\tau}$. Then $nw(X) \leq |S|^{\tau}$ and $|X| \leq |S|^{\tau} \cdot \exp_2(\tau)$.

Proof. The family $\mathcal{B} = \{[A] : A \subseteq S \text{ and } |A| \leq \tau\}$ forms a network for X (cf. [1, Th.2]) and $|\mathcal{B}| \leq |S|^{\tau}$. The inequality $|X| \leq |S|^{\tau} \cdot \exp_2(\tau)$ follows from the fact that the power of the closure $[A]$ does not exceed $\exp_2(\tau)$ for any subset $A \subseteq S$ with $|A| \leq \tau$ (see [2, Th. 2.4]).

Let X be a space, M dense in X and τ an infinite cardinal. Let us consider the following sentence:

$\varphi(X,\tau,M) \Leftrightarrow$ if h is any continuous mapping of X onto a space Y of the weight τ and N is any subset of h(M) with $|N| \leq \tau$, then $M \cap h^{-1}(N)$ is not dense in M.

Lemma 2. Suppose we are given X, M and τ as above. If, in addition, $\ell(X) \leq \tau$ and $\varphi(X,\tau,M)$ holds, then there exists a continuous mapping f of X onto a space Y of weight $\leq \exp(\tau)$ such that $\varphi(Y,\tau,\tilde{M})$ holds for $\tilde{M} = f(M)$.

Proof. If $w(X) \leq \exp(\tau)$, there is nothing to prove. So suppose $w(X) > \exp(\tau)$. Fix an arbitrary continuous mapping f_0 of X onto a space Y_0 of weight $\leq \exp(\tau)$. Now let $\alpha < \tau^+$ and for every $\beta < \alpha$ a continuous mapping f_β of X onto the space Y_β of weight $\leq \exp(\tau)$ be defined. If α is a limit ordinal, we put $f_\alpha = \Delta\{f_\beta : \beta < \alpha\}$, the diagonal product of mappings f_β's. Then f_α is a continuous mapping of X onto the subspace Y_α in the product $\prod_{\beta < \alpha} Y_\beta$, hence $w(Y_\alpha) \leq \exp(\tau)$.

Now consider the case $\alpha = \beta + 1$. Theorem 2.2 of [3] implies that the power of the set $C(Y_\beta)$ of all continuous real-valued functions defined on Y_β does not exceed $w(Y_\beta)^{\ell(Y_\beta)}$. However, $\ell(Y_\beta) \leq \ell(X) \leq \tau$ and $w(Y_\beta) \leq \exp(\tau)$, hence $|C(Y_\beta)| \leq \exp(\tau)$. It is easy to check that the family $\mathcal{N}_\beta$ of all continuous mappings of Y_β to I^τ has the power $\leq |C(Y_\beta)|^\tau$, so $|\mathcal{N}_\beta| \leq \exp(\tau)$. In particular, $|\mathcal{M}_\beta| \leq \exp(\tau)$, where $\mathcal{M}_\beta$ is the family of those $h \in \mathcal{N}_\beta$, for which $w(h(Y_\beta)) = \tau$.

Let $h \in \mathcal{M}_\beta$. Then $|h(Y_\beta)| \leq \exp w(h(Y_\beta)) = \exp(\tau)$, and $|hf_\beta(M)| \leq \exp(\tau)$. We put $\lambda_h = \{N \subseteq hf_\beta(M) : |N| \leq \tau\}$. Obviously, $|\lambda_h| \leq \exp(\tau)$. Using the assertion $\varphi(X, \tau, M)$, we can find, for each $N \in \lambda_h$, an open subset $O_{N,h}$ of X such that $O_{N,h} \cap f_\beta^{-1}h^{-1}(N) \cap M = \emptyset$. For each $N \in \lambda_h$ we fix also a continuous function $t_{N,h} : X \longrightarrow 0,1$ which is equal to 1 at some point of the set $O_{N,h}$ and vanishing outside $O_{N,h}$. Let t_h be the diagonal product of the mappings $t_{N,h}$ with $N \in \lambda_h$. Then $w(t_h(X)) \leq \exp(\tau)$ for $|\lambda_h| \leq \exp(\tau)$. Finally we put f_α to be equal the diagonal product of the mapping f_β and the family of mappings $\{t_h : h \in \mathcal{M}_\beta\}$. It is clear that $w(f_\alpha(X)) \leq \exp(\tau)$ for $|\mathcal{M}_\beta| \leq \exp(\tau)$. This completes our recursive construction.

Let us put $Y = f(X)$ and $\tilde{M} = f(M)$, where $f = f_{\tau^+}$. We claim that f, Y and $\tilde{M}$ are so as required. Indeed, let p be any continuous mapping of Y onto a space Z of weight τ and $N \subseteq p(\tilde{M})$, $|N| \leq \tau$. One can assume that Z is a subspace of I^τ. For each

$\alpha < \tau^+$ there exists (only one) continuous mapping $h_\alpha : Y \longrightarrow Y_\alpha$ such that $f_\alpha = h_\alpha \circ f$. Obviously, $h_\alpha \prec h_\beta$ whenever $\beta < \alpha < \tau^+$, $id_Y = \varprojlim \{ h_\alpha : \alpha < \tau^+ \}$ and $\ell(Y) \leq \tau$. Applying Theorem 2 of [4] we find an ordinal $\beta < \tau^+$ such that $h_\beta \prec p$, i.e. there exists a continuous mapping $h: Y \longrightarrow Z$ with $p = h \circ h_\beta$. Consequently $h \in \mathcal{M}_\beta$ and the definition of the set $O_{N,h}$ implies that $O_{N,h} \cap f_\beta^{-1} h^{-1}(N) \cap M = \emptyset$, or $O_{N,h} \cap f^{-1} p^{-1}(N) \cap M = \emptyset$. By the choice of the function $t_{N,h}$ we have $t_{N,h}^{-1}(J) \cap f^{-1} p^{-1}(N) \cap M = \emptyset$, where $J = (0,1]$. Further, $f \prec f_\alpha \prec t_{N,h}$, where $\alpha = \beta + 1$, hence there exists a continuous mapping $s: Y \rightarrow [0,1]$ such that $t_{N,h} = s \circ f$. Thus $f^{-1} s^{-1}(J) \cap f^{-1} p^{-1}(N) \cap M = \emptyset$ which implies $s^{-1}(J) \cap p^{-1}(N) \cap f(M) = \emptyset$. So $\varphi(Y, \tau, \bar{M})$ holds because $s^{-1}(J)$ is a non-empty open subset of Y.

The following notion is prompted by Arhangel´skiĭ´s paper [5].

Definition 1. A space X will be called τ-approximative if for any continuous image Y of X with $w(Y) \leq \exp(\tau)$ the inequality $d(Y) \leq \tau$ holds.

It is interesting to recognize how wide the class of τ-approximative spaces is.

Assertion 1. Any product $X = \prod_{\alpha \in A} X_\alpha$ of spaces X_α with $d(X_\alpha) \leq \tau$ is λ-approximative for each $\lambda \geq \tau$.

Proof. Let $\lambda \geq \tau$ and f be any continuous mapping of X onto a space Y of weight $\leq \exp(\lambda)$. Then the Gleason´s theorem implies that there exist a subset $B \subseteq A$ with $|B| \leq \exp(\lambda)$ and a continuous mapping $g: X_B = \prod_{\alpha \in B} X_\alpha \longrightarrow Y$ such that $f = g \circ \pi_B$; here π_B is the natural projection. The density of the product $X_B = \prod_{\alpha \in B} X_\alpha$ does not exceed λ because $d(X_\alpha) \leq \tau \leq \lambda$ for every $\alpha \in B$ and

$|B| \leq \exp(\lambda)$. Since g is continuous, $d(Y) \leq \lambda$.

Assertion 2. The property of being τ-approximative space is preserved by continuous mapping.

It is not difficult to indicate an inner condition on a space X which implies τ-approximativeness of X. The idea of the following definition is taken from [5].

Definition 2. Let τ be an infinite cardinal and S a subset of X. We shall say that X weakly suppresses S if for any subset T of S with $|T| \leq \exp(\tau)$ there exists a subset A of X such that $T \subseteq [A]_X$ and $|A| \leq \tau$.

Assertion 3. Suppose that X contains a dense subset S which is weakly suppressed by X. Then X is τ-approximative.

Lemma 3. Let f be a continuous mapping of X onto Y, $S \subseteq X$, $[S]_\tau = X$ and $T \subseteq f(S)$. If $t(S) \leq \tau$ and $S \cap f^{-1}(T)$ is dense in S, then $[T]_\tau = Y$.

The following theorem is the main result of the paper.

Theorem 1. Suppose we are given a dyadic compact space X and a subspace $M \subseteq X$ such that $[M]_\tau = X$ and $t(M) \leq \tau$. Then $d(X) \leq \tau$.

Proof. We assume that $d(X) > \tau$. Then the cardinal $\lambda = w(X)$ satisfies the inequality $\lambda > \exp_n(\tau)$ for every $n \in N^+$. Indeed, otherwise $\exp_k(\tau) < \lambda \leq \exp_{k+1}(\tau)$ for some $k \geq 0$. Assertion 1 implies immediately that $k \geq 1$ because $d(X) > \tau$. Applying Assertion 1 once more we find a dense subspace S of X such that $|S| \leq \exp_k(\tau)$. As $S \subseteq [M]_\tau$, so there exists a subset $N \subseteq M$ such that $S \subseteq [N]$ and $|N| \leq \exp_k(\tau)$. Clearly N is a dense subset of M and of X, too. The condition $t(M) \leq \tau$ implies $M \subseteq [N]_\tau$. Using the condition $X = [M]_\tau$ we conclude that $[N]_\tau = [[N]_\tau]_\tau \supseteq [M]_\tau = X$. Hence Lemma 1 implies

$nw(X) \leq |N|^{\tau} \leq (\exp_k(\tau))^{\tau} = \exp_k(\tau)$. It contradicts the fact that $\lambda > \exp_k(\tau)$. Thus $\lambda \geq \exp_{\omega}(\tau)$.

Now let us prove that $\varphi(X, \mu, M)$ holds with $\mu = \exp_2(\tau)$. Assuming the contrary we fix some continuous mapping h of X onto the compact space Z of weight μ and a subset $N \subseteq h(M)$ such that $|N| \leq \mu$ and $M \cap h^{-1}(N)$ is dense in M. Then Lemma 3 implies that $[N]_{\tau} = Z$. Applying Lemma 1 we get: $|Z| \leq |N|^{\tau} \cdot \exp_2(\tau) = \exp_2(\tau) = \mu$. However Z is dyadic as a continuous image of the compact dyadic space X, and $w(Z) = \mu$, $cf(\mu) > \exp(\tau) > \aleph_0$. Consequently there exists a continuous mapping of Z onto I^{μ} (see [7]), so $|Z| \geq \exp(\mu)$. This contradiction proves our assertion about $\varphi(X, \mu, M)$.

Now with the use of Lemma 2 we fix a continuous mapping f of X onto some compact space Y of weight $\leq \exp(\mu)$ such that the assertion $\varphi(Y, \mu, \overline{M})$ holds with $\overline{M} = f(M)$. Then $[\overline{M}]_{\tau} = Y$ because f is continuous. Assertions 1 and 2 imply jointly that the compact dyadic space Y is μ-approximative. Consequently there exists a dense subspace S of Y with $|S| \leq \mu$. As S is contained in the lower τ-closure of the set $\overline{M}$, there exists a subset $T \subseteq \overline{M}$ such that $S \subseteq [T]$ and $|T| \leq \mu$. Clearly T is a dense subset of $\overline{M}$ (and of Y). There exists a continuous mapping h of Y onto a compact space Z of weight μ (see Lemma 4 of [8] or Th. 5 of [9]). Thus we have $|h(T)| \leq \mu$ and $\overline{M} \cap h^{-1}h(T) \supseteq T$ is dense in $\overline{M}$. It contradicts $\varphi(Y, \mu, \overline{M})$.

Corollary 1. Let X be a compact dyadic space and S a subspace of X such that $X = [S]_{\aleph_0}$ and $t(S) \leq \aleph_0$. Then X is separable.

§ 2. Some generalizations and questions. An examination of the proof of Theorem 1 shows that the dyadicity of a compact space X was used only partially. In fact we used the following properties of the space X:

(a) λ-approximativeness of X for any $\lambda \geq \tau$;

(b) any continuous image $Y = f(X)$ of weight $\mu = \exp_2(\tau)$ has a power $> \mu$.

Question 1. Does Theorem 1 remain valid if the property of dyadicity of the (compact) space X is replaced by the property (a) only?

Note that the class of q-adic compact spaces defined by L. Shirokov in [10] satisfies conditions (a) and (b) with some stock. This class contains all dyadic compact spaces and is closed under the product operation (with any number of factors), taking a closed G_δ-subspace and invariant under continuous mappings. Moreover, this class has the following remarkable properties:

1) if a compact space X is a continuous image of a dense subspace of some q-adic compact space, then X is q-adic, too;

2) a compact space covered by a countable family of closed q-adic subspaces is q-adic;

3) for any q-adic compact space X and $x \in X$, $\overline{\pi\chi}(x,X) = \chi(x,X)$ ($\overline{\pi\chi}$ denotes the hereditary π-character);

4) if X is a compact q-adic space, $\lambda = w(X)$ and $cf(\lambda) > \aleph_0$, then X can be continuously mapped onto I^λ.

The last property implies, in particular, that every q-adic compact space satisfies condition (b). An analogous assertion about condition (a) follows from the definition of q-adic compact space and Theorem 4 of [10].

$\underline{\text{Definition 3}}$. A compact space X is called q-adic provided that there are a cardinal $\tau \geq \aleph_0$, a subspace $M \subseteq 2^{\tau}$ and continuous mappings $f: M \xrightarrow{onto} X$, $\bar{f}: p2^{\tau} \longrightarrow X$ such that the restriction of $\bar{f}$ to $\pi^{-1}(M)$ is equal to $f \circ \pi$ restricted to $\pi^{-1}(M)$. Here $p\,2^{\tau}$ is the absolute of 2^{τ}, $\pi: p2^{\tau} \longrightarrow 2^{\tau}$ is the natural mapping of $p2^{\tau}$ onto 2^{τ} and 2 is the discrete two-point space.

Theorem 4 of [10] states that the cardinal τ in the definition 3 can be replaced by w(X). Thus any q-adic compact space X of weight $\tau \leq \exp(\lambda)$ is a continuous image of $p2^{\tau}$. However the mapping $\pi: p2^{\tau} \longrightarrow 2^{\tau}$ is irreducible and $d(2^{\tau}) \leq \lambda$, hence $d(p2^{\tau}) \leq \leq \lambda$ and $d(X) \leq \lambda$. This implies easily that any compact q-adic space is λ-approximative for every $\lambda \geq \aleph_0$ (recall that a continuous image of a compact q-adic space is compact and q-adic, too).

Thus we have the following stronger version of Theorem 1.

<u>Theorem 2</u>. Let X be a compact q-adic space. If there exists a subspace $S \subseteq X$ such that $t(S) \leq \tau$ and $X = [S]_{\tau}$, then $d(X) \leq \tau$.

<u>Corollary 2</u>. Let a compact space X be an image of a dense subspace of any product with compact metrizable factors under a continuous mapping. If there exists a subspace $S \subseteq X$ such that $t(S) \leq \aleph_0$ and $X = [S]_{\aleph_0}$, then X is separable.

Obviously Corollary 2 generalizes slightly Corollary 1. However it is possible to prove something stronger. To do this we need the following assertion.

<u>Assertion 4</u>. Let $K = \prod_{\alpha \in A} K_{\alpha}$ be a product of spaces K_{α} with $d(K_{\alpha}) \leq \tau$, S a dense subspace of K and f maps continuously S onto a compact space X. Then X is λ-approximative for any $\lambda \geq \tau$.

Proof. Let us fix a cardinal $\lambda \geq \tau$ and a continuous mapping φ of X onto a compact space Y with $w(Y) \leq \exp(\lambda)$. Put $g = \varphi \circ f$. Then g is a continuous mapping of S onto a compact space Y of weight $\leq \exp(\lambda)$ and $nw(K_\alpha) \leq \exp(d(K_\alpha)) \leq \exp(\lambda)$ for every $\alpha \in A$. Theorem 1 of [11] implies that we can find a subset $B \subseteq A$ with $|B| \leq \exp(\lambda)$ and a continuous mapping $\bar{g}: p_B(S) \longrightarrow Y$ such that $g = \bar{g} \circ p_B|_S$; here p_B is the projection of K onto $K_B = \prod_{\alpha \in B} K_\alpha$. Clearly $\bar{S} = p_B(S)$ is a dense subspace of K_B. Let βS and βK_α be the Stone-Čech compactifications of spaces S and K_α, resp.

The natural embedding $i: \bar{S} \hookrightarrow K_B$ is extended to a continuous mapping $\pi: \beta\bar{S} \xrightarrow{onto} \bar{K}_B$, where $\bar{K}_B = \prod_{\alpha \in B} \beta K_\alpha$. As i is a homeomorphism and $i(\bar{S})$ is dense in K_B so π is irreducible. Further, $|B| \leq \exp(\lambda)$ and $d(\beta K_\alpha) \leq d(K_\alpha) \leq \tau \leq \lambda$ for every $\alpha \in B$ which implies that $d(\bar{K}_B) \leq \lambda$ Consequently $d(\beta \bar{S}) \leq \lambda$. Finally, the mapping $\bar{g}: \bar{S} \longrightarrow Y$ is extendable tc a continuous mapping $G: \beta\bar{S} \xrightarrow{onto} Y$ that gives us the inequality $d(Y) \leq \lambda$.

<u>Assertion 5</u>. Suppose we are given an infinite cardinal τ , a product $K = \prod_{\alpha \in A} K_\alpha$, a dense subspace $S \subseteq K$ and a continuous mapping f of S onto a compact space X of weight λ with $cf(\lambda) > \tau$. Then the following is valid:

(a) if $w(K_\alpha) \leq \tau$ for every $\alpha \in A$ then there exists a continuous mapping of X onto I^λ , in particular, $|X| = \exp(\lambda)$;

(b) if $nw(K_\alpha) \leq \tau$ for every $\alpha \in A$ then $|X| > \lambda$.

It should be noted that Assertion 5(a) is a generalization of the B.A. Efimov and J. Gèrlitz's result concerning continuous mappings of dyadic compact spaces onto Tychonoff cubes (see [7] and [12], resp.). The proof of Assertion 5(a) that I have in mind is based on L. Shirokov's methods [10]. In the next we will use the point (b) of Assertion 5 only, therefore the proof of the point (a) is omitted.

Proof of (b). I. Let us assume first that λ is a regular cardinal. It is sufficient to prove that there exists a closed subset F of X such that $\chi(x,F) = \lambda$ for every point $x \in F$. Indeed, the Čech-Pospíšil's theorem [13] will imply then that $|X| \geq |F| \geq \exp(\lambda)$.

To prove this fact we put $M = \{x \in X: \chi(x,X) < \lambda\}$. Given the subspace $S \subseteq \prod_{\alpha \in A} K_\alpha$ and the mapping $f:S \xrightarrow{onto} X$ we apply the main reasoning in [11] to the pair of sets $M \subseteq X$ and $N = f^{-1}(M) \subseteq S$. Using the conditions of Assertion 5(b) and the choice of sets M, N one can show that there exist a subset $B \subseteq A$ with $|B| < \lambda$ and a continuous mapping $\bar{f}:p_B(N) \longrightarrow M$ such that $f|_N = \bar{f} \circ p_B|_N$; p_B stands for the natural projection of K onto K_B. As $|B| \leq \tau$ and $nw(K_\alpha) \leq \tau < \lambda$ for every $\alpha \in B$, so $nw(p_B(N)) < \lambda$. This fact and the continuity of $\bar{f}$ imply that $nw(M) < \lambda$.

Let us put $\mu = nw(M)$, $\mu < \lambda$. The set $X \setminus M$ is not empty because $nw(X) = w(X) = \lambda$. Choose a point $p \in X \setminus M$. The inequality $\ell(M) \leq nw(M) = \mu$ implies that there exists a $G_{\delta,\mu}$-subset $\mathcal{U}$ of X such that $p \in \mathcal{U}$ and $\mathcal{U} \cap M = \emptyset$. The space X is regular, hence there exists a closed subset F of X such that $p \in F \subseteq \mathcal{U}$ and $\chi(F,X) \leq \mu$. Then F has the required property. Indeed, if $x \in F$ and $\chi(x,F) < \lambda$, then $\chi(x,X) \leq \chi(x,F) \cdot \chi(F,X) < \lambda \cdot \mu = \lambda$ whence $x \in M \cap F$. However $F \subseteq \mathcal{U}$ and $\mathcal{U} \cap M = \emptyset$, a contradiction.

II. Now we consider the case of singular cardinal λ. One can assume that $\exp(\mu) < \lambda$ for any $\mu < \lambda$. For if $\exp(\mu_0) > \lambda$ for some $\mu_0 < \lambda$, then μ_0^+ is a regular cardinal, $\mu_0^+ < \lambda$ and $\exp(\mu_0^+) > \lambda$. Moreover, we can choose the cardinal μ_0 so that $\tau \leq \mu_0$. Let us fix a continuous mapping φ of X onto a compact space Y of weight μ_0^+. Then $\varphi \circ f$ is a continuous mapping of S onto a compact space Y of the regular weight $\mu_0^+ > \tau$ and the first part of the proof (applied to Y instead of X) implies that $|Y| \geq \exp(\mu_0^+) > \lambda$. Consequently $|X| = |Y| > \lambda$.

Thus we can assume that $\exp(\mu) \leq \lambda$ for each $\mu < \lambda$. Then the strict inequality $\exp(\mu) < \lambda$ holds for each $\mu < \lambda$. Indeed, if $\exp(\mu_0) = \lambda$ for some $\mu_0 < \lambda$, then $\exp(\mu) = \lambda$ for any μ with $\mu_0 \leq \mu < \lambda$. Consequently $\lambda^\mu = (\exp(\mu))^\mu = \exp(\mu) = \lambda$ for any cardinal μ satisfying $\mu_0 \leq \mu < \lambda$. This implies readily that the cardinal λ is regular; that is a contradiction.

For every $\mu < \lambda$, we put $X_\mu = \{x \in X: \chi(x,X) \leq \mu\}$. Theorem 1 of [11] implies that $nw(X_\mu) \leq \mu \cdot \tau < \lambda$ for each $\mu < \lambda$. Consequently the cardinality of the closure of X_μ in X does not exceed $\exp_2(nw(X_\mu)) < \lambda$ for any $\mu < \lambda$, hence $X \setminus [X_\mu] \neq \emptyset$. Let $\theta = cf(\lambda)$ and $\lambda = \sup\{\mu_\alpha : \alpha < \theta\}$, where $\mu_\alpha < \lambda$ for every $\alpha < \theta$. Let also F_α be a closed G_δ-set of X such that Int $F_\alpha \neq \emptyset$ and $F_\alpha \cap [X_{\mu_\alpha}] = \emptyset$, $\alpha < \theta$. It is important to note that every regular cardinal $\mu > \tau$ is a caliber of X. Indeed, let μ be a regular cardinal with $\mu > \tau$. Then μ is a precaliber of each K_α , $\alpha \in A$, because $d(K_\alpha) \leq nw(K_\alpha) \leq \tau$. Consequently μ is a precaliber of the product space $K = \prod_{\alpha \in A} K_\alpha$ (see Th. 4.8 of [2]) and of the dense subspace S of K. Further, precalibers are preserved under continuous mappings onto, hence μ is a precaliber of X. Finally, the notions of "caliber" and "precaliber" coincide in the class of compact spaces. This implies, in particular, that the cardinal $\theta = cf(\lambda)$ is a caliber of X. Consequently there exists a subfamily $\gamma \subseteq \{F_\alpha : \alpha < \theta\}$ with the finite intersection property such that $|\gamma| = \theta$. Put $F = \bigcap \gamma$. Clearly F is a $G_{\delta,\theta}$-set in X and $F \cap X_\mu = \emptyset$ for any $\mu < \lambda$. The argument similar to that of the first part of the proof shows that $\chi(x,F) = \lambda$ for any point $x \in F$. Hence $|X| \geq |F| \geq \exp(\lambda) > \lambda$.

Question 2. Can one improve Assertion 5(b) by showing that I^{λ} is a continuous image of X ?

Let X be a compact space satisfying the conditions of Assertion 4. Then Assertions 4 and 5(b) imply jointly that a space X satisfies the conditions (a) and (b) at the beginning of § 2. Thus we have obtained the following result.

Theorem 3. Let a compact space X be a continuous image of a dense subspace of some product with factors of density $\leq \tau$. If there exists a subspace $S \subseteq X$ such that $t(S) \leq \tau$ and $[S]_{\tau} = X$, then $d(X) \leq \tau$.

Corollary 3. Let a compact space X be a continuous image of a dense subspace of some product of separable spaces. If there exists a subspace S of X such that $t(S) \leq \aleph_0$ and $[S]_{\aleph_0} = X$, then X is separable.

The other generalization of the dyadic compact spaces is the class of $\varkappa$-adic compact spaces, i.e. the class consisting of all continuous images of $\varkappa$-metrizable compact spaces. The classes of $\varkappa$-metrizable and $\varkappa$-adic compact spaces had been introduced by E.V. Ščepin (see [14],[15]). Every $\varkappa$-adic compact space X is π-characteristic in the sense of B.E. Šapirovskiĭ, i.e., for any regular uncountable cardinal τ, the closure in X of the set $M_{\tau} = \{x \in X: \pi\chi(x,X) < \tau\}$ is of weight $< \tau$. Moreover, every $\varkappa$-adic compact space X satisfies the Šanin's condition. It means that any regular uncountable cardinal is a caliber of X. Consequently one can apply Theorem 4 of [16] which implies that every $\varkappa$-adic compact space of weight λ with $cf(\lambda) > \aleph_0$ is continuously mapped onto I^{λ}. Thus any $\varkappa$-adic (and, of course, $\varkappa$-metrizable) compact space satisfies the condition (b).

Question 3. a) Is it true that every $\varkappa$-metrizable compact space is τ-approximative for each $\tau \geq \aleph_0$?

b) Does Theorem 1 hold for $\varkappa$-metrizable compact space X ?

The affirmative answer to Question 3 a) would imply the same answer to Question 3 b). Note that the similar questions concerning with $\varkappa$-adic compact spaces are equivalent to those ones that have been formulated.

Question 4. Is it true that a product of two τ-approximative spaces is τ-approximative? And what about it if the factors are compact?

Let A be an infinite set, M a dense subspace of the Tychonoff cube I^A and τ an infinite cardinal. By analogy with the sentence $\varphi(X,\tau,M)$ we define the new sentence $\Phi(A,\tau,M)$ by putting $\Phi(A,\tau,M) \Leftarrow\Rightarrow$ for any set $B \subseteq A$ with $|B| = \tau$ and $N \subseteq p_B(M)$ with $|N| \leq \tau$, the set $M \cap p_B^{-1}(N)$ is not dense in M; here p_B is the natural projection of I^A onto I^B.

Question 5. Do there exist A, τ and M with $\tau \leq |A|$ such that $\Phi(A,\tau,M)$ holds?

Finally, let us consider the following hypothesis.

(H) If X is a compact space, $M \subseteq X$, $t(M) \leq \tau$ and $X = [M]_\tau$, then $t(X) \leq \exp(\tau)$.

Obviously, the main results of the paper follow immediately from (H).

Question 6. Does the hypothesis (H) hold?

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