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Abstract: We show that every countable commutative semigroup admits a productive representation in the class of countable Hausdorff spaces with countable weight. As a consequence, we obtain a countable Hausdorff space X with countable weight, homeomorphic to $X\times X\times X$ but not to $X\times X$.

Key words: Countable Hausdorff space.

Classification: 54B10, 54G15

I. Preliminaries and the Main Theorem. Let $(S,+)$ be a commutative semigroup, $\mathcal{K}$ be a category with finite products, $\mathcal{C}$ a class of its objects. A collection

$$\{X(s)\mid s\in S\}$$

of objects of $\mathcal{C}$ is called a productive representation of $(S,+)$ in $\mathcal{C}$ if

(i) for every $s_1,s_2\in S$, $X(s_1)\times X(s_2)$ is isomorphic to $X(s_1+s_2)$,

(ii) if $s_1\neq s_2$, then $X(s_1)$ is not isomorphic to $X(s_2)$.

The field of problems which commutative semigroups have productive representations in which categories generalizes some problems investigated e.g. by S. Ulam [17], A. Tarski [10],[11], W. Hanf [3], B. Jónsson [4],[5], A.L.S. Corner [2], J. Ketonen [6], R.S. Pierce [9] and others. For example, if the represented semigroup $(S,+)$ is a cyclic group $c_2=\{0,1\}$ of order 2 (i.e. $1+1=0$) and

$\{X(0), X(1)\}$ is its productive representation, then $X=X(1)=$ $=X(1+1+1)$ is isomorphic to $X^3 = X\times X\times X$ but not isomorphic to $X^2 = X\times X \simeq X(1+1) = X(0)$.

In [15], a survey of results concerning productive representations of commutative semigroups in classes of topological spaces was presented and six open problems concerning this topic were formulated. Let us mention that some of them have been already solved, namely Problem 1 in [7], Problem 2 in [16] and Problem 4 in [8]. Here, we solve Problem 5 aboud productive representations in the class of countable spaces with countable weight. Problems 3 and 6 of [15] remain open.

Let us recall here the situation concerning classes of countable topological spaces. If a countable metrizable space X is homeomorphic to X^3, then it is homeomorphic to X^2, by [13]. On the other hand,

every countable commutative semigroup has a productive representation in the class of all countable paracompact spaces.

This is proved in [14]. The construction in [14] uses an infinite collection of pairwise incomparable ultrafilters (in the Rudin-Keisler order) on a countable set and the constructed representing spaces are far from having countable weight. The result concerning countable spaces with countable weight is much weaker. By [15], every countable commutative semigroup has a productive representation in the class of all countable T_1-spaces with countable weight.

In this assertion, T_1-spaces cannot be replaced by T_3-spaces because a T_3-space with countable weight is metrizable and, as mentioned above, the group c_2 has no productive representation in the class of countable metrizable spaces. Problem 5 of [15] is to fill up the gap between T_1-spaces and T_3-spaces. The aim of the present paper is to prove the following

Main Theorem. Every countable commutative semigroup has a productive representation in the class of all countable T_2-spaces with countable weight.

Let us sketch the contents of the next parts of the paper. In II, we introduce the notion of irregularity degree of a topological space and investigate its basic properties. By means of this new topological invariant we prove in III and IV the above Main Theorem. In III, we construct the representing spaces, in IV we prove that they really form a productive representation in the class of all Countable Hausdorff Spaces with Countable Weight (let us use the name CHSCW for this class). In the part IV, we present some strengthenings and generalizations of the Main Theorem.

II. The irregularity degree id.

II.1. The inductive definition of the irregularity degree of a topological space P (similar in its form to the definition of ind - the small inductive dimension) is as follows ($\bar{A}$ denotes the closure of A).

$\mathrm{id}\ \emptyset = -1$

If $x \in P$ then

$\mathrm{id}_P\, x \le n \equiv$ for every neighbourhood $\mathcal{U}$ of x in P there exists a neighbourhood $\mathcal{V}$ of x in P such that $\mathrm{id}(\bar{\mathcal{V}} \setminus \mathcal{U}) \le n - 1$;

$\mathrm{id}\, P \le n \equiv$ for every $x \in P$, $\mathrm{id}_P\, x \le n$;

$\mathrm{id}_P\, x = n \equiv \mathrm{id}_P\, x \le n$ and non $(\mathrm{id}_P\, x \le n - 1)$;

$\mathrm{id}\, P = n \equiv \mathrm{id}\, P \le n$ and non $(\mathrm{id}\, P \le n - 1)$.

Since in our constructions in III and IV we are interested only in spaces with finite irregularity degree, we simply put

$\mathrm{id}\, P = \infty \equiv$ for no natural number n, $\mathrm{id}\, P \leq n$.

Observation. $\mathrm{id}\, P \leq 0 \Longleftrightarrow P$ is regular.

The proofs of the following lemmas are straightforward inductions; the cases $n = 0$ and $n = \infty$ are usually trivial, so we shall indicate in each case the induction step.

II.2. Lemma. If $Q \subseteq P$, then $\mathrm{id}\, Q \leq \mathrm{id}\, P$.

Proof. Use the inequality

$$\overline{\mathcal{V} \cap Q}^{Q} \setminus (\mathcal{U} \cap Q) \subseteq \overline{\mathcal{V}}^{P} \setminus \mathcal{U} .$$

II.3. Lemma. If $P = P_1 \cup P_2$ and P_1, P_2 are closed, then

$$\mathrm{id}\, P = \max \{\mathrm{id}\, P_1, \mathrm{id}\, P_2\}.$$

Proof. It suffices to show that $\mathrm{id}_P\, x \leq \max \{\mathrm{id}\, P_1, \mathrm{id}\, P_2\}$ for each $x \in P$. For $x \in P_i \setminus P_{3-i}$ it follows readily that $\mathrm{id}_P\, x = \mathrm{id}_{P_i}\, x$ so we take $x \in P_1 \cap P_2$. Now, use the fact that if $\mathcal{V}_i$ is a P_i-neighbourhood of x for $i=1,2$, then $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is a P-neighbourhood of x and if $\mathcal{U}$ is any other neighbourhood then

$$\overline{\mathcal{V}}^{P} \setminus \mathcal{U} \subseteq (\overline{\mathcal{V}}_1^{P_1} \setminus (\mathcal{U} \cap P_1)) \cup (\overline{\mathcal{V}}_2^{P_2} \setminus (\mathcal{U} \cap P_2)).$$

II.4. Lemma. Let x be a point of P such that

$$\mathrm{id}_P\, x = \mathrm{id}\, P = n < \infty .$$

Then for every neighbourhood $\mathcal{U}$ of x

$$\mathrm{id}\, \overline{\mathcal{U}}^{P} = n.$$

Proof. Straightforward.

II.5. Proposition. Let $P = P_1 \times P_2 \neq \emptyset$. Then

$$\mathrm{id}\, P = \mathrm{id}\, P_1 + \mathrm{id}\, P_2.$$

Proof. First observe that if $\mathcal{V}_i, \mathcal{U}_i \subseteq P_i$ are open for $i=1,2$ and $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, then

$$\mathcal{V}\setminus\mathcal{U}\subseteq((\overline{\mathcal{V}}_1^{P_1}\setminus\mathcal{U}_1)\times\overline{\mathcal{V}}_2^{P_2})\cup(\overline{\mathcal{V}}_1^{P_1}\times(\overline{\mathcal{V}}_2^{P_2}\setminus\mathcal{U}_2)).$$

Then $\leq$ follows in a straightforward way and $\geq$ follows, fixing $x_i\in P_i$ with $\mathrm{id}_{P_i}x_i=\mathrm{id}\,P_i$ for $i=1,2$ and then, using II.4, showing that $\mathrm{id}_P(x_1,x_2)\geq\mathrm{id}\,P_1+\mathrm{id}\,P_2$.

II.6. Example. Let $m\geq 1$ be a natural number. Let us define a space Z_m as follows: Let $\{D_i\mid i=0,\dots,m\}$ be a pairwise disjoint system of countable dense subsets of the interval $(0,1\rangle$ of the real numbers, let B be the set of all rational numbers in the interval $\langle -1,0\rangle$. Put

$$Z_m=B\cup\bigcup_{i=0}^{m}D_i.$$

Let τ_m be the Euclidean metric on $Z_m\subseteq\langle -1,1\rangle$, denote $K_{z,\varepsilon}=\{x\in Z_m\mid \tau_m(x,z)<\varepsilon\}$. The topology of Z_m is defined such that

if $z\in D_i$, then its local base is $\{K_{z,\varepsilon}\cap(\bigcup_{j=i}^{m}D_j)\mid\varepsilon>0\}$;

if $z\in B\setminus\{0\}$, then its local base is $\{K_{z,\varepsilon}\mid\varepsilon>0\}$;

the local base of 0 is $\{(-\varepsilon,0\rangle\cup((0,\varepsilon)\cap D_m)\mid\varepsilon>0\}$.

Observation. Z_m is an element of the class CHSCW and

a) if $z\in D_i$, then $\mathrm{id}_{Z_m}z=i$;

b) $z\in B\setminus\{0\}$ iff z has a clopen (= closed-and-open) neighbourhood, which is a regular space;

c) $\mathrm{id}_{Z_m}0=m$; moreover, 0 is the unique point z of Z_m with the following property:

id $z>0$ and any neighbourhood of z contains a clopen (in Z_m) subset which is a regular space.

III. The basic constructions.

III.1. First, let us describe a method which has been used several times for constructions of productive representations. We start from a collection $\mathcal{X}=\{X_k\mid k\in\omega\}$ of spaces, where ω

denotes the set of all nonnegative integers. For every $f \in \omega^{\omega}$, we put

$$X(f) = \prod_{k \in \omega} X_k^{f(k)} ,$$

i.e. each $X_k^{f(k)}$ is the product of $f(k)$ copies of X_k (if $f(k) = 0$, then $X_k^{f(k)}$ is a one-point space) and $X(f)$ is the product of all $X_k^{f(k)}$, $k \in \omega$. Then, clearly,

$X(f) \times X(g)$ is homeomorphic to $X(f+g)$.

Denote by $\mathbb{U}$ the semigroup of all countable infinite subsets A of $\omega^{\omega} \setminus \{\mathbb{O}\}$ (where $\mathbb{O}$ is the function which maps the whole ω to 0) with the operation + defined by

$A + B = \{f+g \mid f \in A, g \in B\}$.

For every $A \in \mathbb{U}$ denote by $X(A)$ the coproduct (= a disjoint union as clopen subsets) of $\aleph_0$ copies of each $X(f)$ with $f \in A$. Then, clearly,

$X(A) \times X(B)$ is homeomorphic to $X(A+B)$ for all $A, B \in \mathbb{U}$.

If the starting collection $\mathcal{X} = \{X_k \mid k \in \omega\}$ is constructed such that the following implication is fulfilled,

$X(A)$ is homeomorphic to $X(B) \Longrightarrow A = B$,

then $\{X(A) \mid A \in \mathbb{U}\}$ is a productive representation of $\mathbb{U}$. And, by [12], every countable commutative semigroup can be embedded into $\mathbb{U}$.

III.2. In the present paper, we have to modify the above method because the spaces $X(f)$ are usually uncountable. The idea is to choose suitable subspaces, say $Y(f)$'s, such that still

$Y(f) \times Y(g)$ is homeomorphic to $Y(f+g)$.

In our construction, however, the topology on the subset of the product is also modified a little. Thus, let us suppose that the starting collection $\mathcal{X} = \{X_k \mid k \in \omega\}$ of elements of CHSCW has been already constructed (this will be done in III.3) and let us suppose that a semigroup $S \subseteq \mathbb{U}$ is given such that its support

$$\text{supp } S = \bigcup_{A \in S} A$$

is countable (every countable subsemigroup of $\mathbb{U}$ has countable support, of course). Let us describe the spaces $X(f) = \prod_{k \in \omega} X_k^{f(k)}$ in a way more suitable for handling with coordinates. For every $f \in \omega^{\omega} \setminus \{0\}$, denote

$L(f) = \{(k,j) \mid k \in \omega \ , \ j=1,\dots,f(k)\}$

and for every $\ell = (k,j) \in L(f)$ put $\bar{\ell} = k$. Then, clearly,

$X(f) = \prod_{\ell \in L(f)} X_{\bar{\ell}}$.

For every $f,g \in \text{supp } S$, we choose a bijection

$\mu_{f,g} : L(f) \dot{\cup} L(g) \longrightarrow L(f+g)$,

where $L(f) \dot{\cup} L(g)$ denotes the disjoint union of $L(f)$ and $L(g)$. Let us define a map

$\varphi_{f,g} : X(f) \times X(g) \longrightarrow X(f+g)$

by

$\varphi_{f,g}(x,y) = z$,

where for every $\ell \in L(f+g)$, the ℓ-th coordinate z_ℓ of z is precisely the ℓ'-th coordinate of either x or y, with $\ell' = \mu_{f,g}^{-1}(\ell)$, depending on the fact whether ℓ' is either in $L(f)$ or in $L(g)$. Thus $\varphi_{f,g}$ only permutes coordinates (so it is a homeomorphism of $X(f) \times X(g)$ onto $X(f+g)$).

If $f,g \in \text{supp } S$, σ is a finite decomposition of $L(f)$ and σ' is a finite decomposition of $L(g)$ then $\{\mu_{f,g}(Z) \mid Z \in \sigma$ or $Z \in \sigma'\}$ is a finite decomposition of $L(f+g)$; denote it by $\mu_{f,g}(\sigma \dot{\cup} \sigma')$. Conversely, if σ is a finite decomposition of $L(f+g)$, then $\{L(f) \cap \mu_{f,g}^{-1}(Z) \mid Z \in \sigma\}$ and $\{L(g) \cap \mu_{f,g}^{-1}(Z) \mid Z \in \sigma\}$ form finite decompositions of $L(f)$ and $L(g)$; let us denote them by $\mu_{f,g,1}^{-1}(\sigma)$ and $\mu_{f,g,2}^{-1}(\sigma)$.

For every $f \in \text{supp } S$, we define a countable set $\mathcal{D}(f)$ of finite decompositions of the set $L(f)$ as follows (simultaneously for all $f \in \operatorname{supp} S$, by induction):

$\mathcal{D}_0(f) = \{\{L(f) \setminus K\} \cup \{\{k\} \mid k \in K\} \mid K$ is a finite subset of $L(f)\}$,

$\mathcal{D}_{n+1}(f) = \mathcal{D}_n(f) \cup \{\mu^{-1}_{f,g,1}(\sigma) \mid g \in \operatorname{supp} S$ and $\sigma \in \mathcal{D}_n(f+g)\} \cup$

$\cup \{\mu^{-1}_{g,f,2}(\sigma) \mid g \in \operatorname{supp} S$ and $\sigma \in \mathcal{D}_n(f+g)\} \cup$

$\cup \{\mu_{g,h}(\sigma \dot{\cup} \sigma') \mid g,h \in \operatorname{supp} S,\ g+h = f,\ \sigma \in \mathcal{D}_n(g),\ \sigma' \in \mathcal{D}_n(h)\}$,

$\mathcal{D}(f) = \bigcup_{n=0}^{\infty} \mathcal{D}_n(f)$.

Now, let us suppose that the starting collection $\mathcal{X} = \{X_k \mid k \in \omega\}$ has been constructed such that each X_k is an element of CHSCW and, moreover,

a) there is a distinguished infinite subset H in each of them (the same set for all the X_k's) and

b) for each $k \in \omega$, a continuous metric σ_k is given on X_k such that diam $X_k = 1$, all the metrics σ_k, $k \in \omega$, coincide on H (i.e. $\sigma_k(a,b) = \sigma_{k'}(a,b)$ for all $k,k' \in \omega$, $a,b \in H$) and determine the topology of H.

Then we have two topologies on each $X(f)$, namely the product topology p and the topology of uniform convergence of the collection of the metric spaces $\{(X_k, \sigma_k) \mid k \in \omega\}$.

For every $f \in \operatorname{supp} S$ put

$H(f) = \{x \in X(f) \mid$ there exists $\sigma \in \mathcal{D}(f)$ such that x is constant on each $Z \in \sigma$ and, for each $Z \in \sigma$, the value x_ℓ of x at $\ell \in Z$ is in H$\}$,

$Y(f) = \{x \in X(f) \mid$ there exists $y \in H(f)$ such that $x_\ell = y_\ell$ for all $\ell \in L(f) \setminus K$, where K is finite$\}$.

The topology investigated on $Y(f)$ is the infimum of the topologies p and m, i.e. a local base of a point $x \in Y(f)$ is formed by all the sets

$$Y(f) \cap (\mathcal{V} \times \prod_{k \in K} \mathcal{U}_k),$$

where $K \subseteq L(f)$ is finite such that $x_\ell \in H$ for all $\ell \in L(f) \setminus K$, $\mathcal{U}_k$ is a neighbourhood of x_k in X_k for each $k \in K$ and $\mathcal{V}$ is a neighbourhood of $\{x_\ell \mid \ell \in L(f) \setminus K\}$ in the space $\prod_{\ell \in L(f) \setminus K}^{\cdot} (X_\ell, \sigma_\ell)$, where $\prod^{\cdot}$ denotes the product of metric spaces endowed with the metric

$$\sigma(a,b) = \sup_{\ell} \sigma_\ell (a_\ell, b_\ell).$$

Proposition. For each $f \in \operatorname{supp} S$, $Y(f)$ is an element of CHSCW. Moreover, for every $f, g \in \operatorname{supp} S$,

$Y(f) \times Y(g)$ is homeomorphic to $Y(f+g)$.

Proof. Every $Y(f)$ with $f \in \operatorname{supp} S$ is in CHSCW, evidently. The bijection $\wp_{f,g}$ maps $Y(f) \times Y(g)$ precisely onto $Y(f+g)$, this follows from the definition of $\mathcal{D}(f), \mathcal{D}(g), \mathcal{D}(f+g)$; since it only permutes coordinates, it is a homeomorphism.

III.3. We finish this part with the construction of the starting collection $\mathcal{X} = \{X_k \mid k \in \omega\}$ of elements of CHSCW, the system of continuous metrics $\{\sigma_k \mid k \in \omega\}$, σ_k on X_k, and the distinguished subset H of all the X_k's. The proof that this collection really leads to a productive representation of a given semigroup $S \subseteq \mathbb{W}$ (with countable support) will be given in the next part IV.

Let $M = \{M_k \mid k \in \omega\}$ be a pairwise disjoint system of infinite subsets of $\omega \setminus \{0,1\}$. Let us express each M_k as an increasing sequence, i.e. $M_k = \{m_{k,i} \mid i \in \omega\}$, where

$$1 < m_{k,0} < m_{k,1} < m_{k,2} < \dots$$

Let Z_m, B, τ_m, D_j be as in II.6. We define the space X_k by means of the system $\{Z_{m_{k,i}} \mid i \in \omega\}$ of spaces. We multiply them by one-point spaces to make them disjoint, then we form their (disjoint) union and add one point more. Thus,

$$X_k = \{\sigma\} \cup \bigcup_{i=0}^{\infty} \{2^{-i}\} \times Z_{m_{k,i}}.$$

Let us denote

$$E_k = \bigcup_{i=0}^{\infty} (\{2^{-i}\} \times \bigcup_{j=0}^{m_{k,i}} D_j),$$

$$G = \bigcup_{i=0}^{\infty} \{2^{-i}\} \times (B \setminus \{0\})$$

$$H = \{\sigma\} \cup G.$$

Clearly, $H \subseteq X_k$ for all $k \in \omega$. We define the topology of X_k as follows:

each $\{2^{-i}\} \times Z_{m_{k,i}}$ is a clopen subspace of X_k (homeomorphic to $Z_{m_{k,i}}$ as in II.6, under $(2^{-i},x) \rightsquigarrow x$), a local base of σ in X_k is $\{\{\sigma\} \cup \bigcup_{i=j}^{\infty} \{2^{-i}\} \times Z_{m_{k,i}} \mid j \in \omega\}$.

Now, we define the continuous metric σ_k on X_k.

$$\sigma_k(x,y) = \tfrac{2}{3}(2^{-i}+2^{-j}) \text{ whenever } i \neq j,\ x \in \{2^{-i}\} \times Z_{m_{k,i}},$$
$$y \in \{2^{-j}\} \times Z_{m_{k,j}};$$

$$\sigma_k(x,\sigma) = \tfrac{2}{3} \cdot 2^{-i} \text{ whenever } x \in \{2^{-i}\} \times Z_{m_{k,i}};$$

$$\sigma_k(x,y) = \tfrac{1}{3} \cdot 2^{-i} \cdot \tau_{m_{k,i}}(\tilde{x},\tilde{y}) \text{ whenever } \tilde{x},\tilde{y} \in Z_{m_{k,i}} \text{ and}$$
$$x = (2^{-i},\tilde{x}),\ y = (2^{-i},\tilde{y}), \text{ where } \tau_m \text{ is as in II.6.}$$

Then diam $X_k = 1$ and, since each $\tau_{m_{k,i}}$ is a continuous metric on $Z_{m_{k,i}}$, σ_k is really a continuous metric on X_k. Moreover, since every τ_m determines the topology on B (see II.6), σ_k really determines the topology on H and all the metrics σ_k, $k \in \omega$, coincide on H. Finally, let us denote

$$p_{k,i} = (2^{-i},0) \in \{2^{-i}\} \times Z_{m_{k,i}} \subseteq X_k,\ i \in \omega.$$

<u>We conclude</u>: let a semigroup $S \subseteq \mathbb{U}$ with countable support supp S be given, let $\mathcal{X} = \{X_k \mid k \in \omega\}$ be the collection of spaces just constructed; for each $f \in$ supp S, let Y(f) be the space

constructed by means of $\mathcal{X}$ as in III.2 and, for every $A \in S$, let $Y(A)$ be a coproduct of $\aleph_0$ copies of each $Y(f)$ with $f \in A$. Then

α) $Y(A)$ is an element of CHSCW and

β) if $A, B \in S$, then $Y(A) \times Y(B)$ is homeomorphic to $Y(A+B)$.

In the next part IV, we prove the following implication.

(*) $\left\{ \begin{array}{l} \text{if } A,B \in S \text{ and } Y(A) \text{ is homeomorphic to a clopen sub-} \\ \text{space of } Y(B), \text{ then } A \subseteq B. \end{array} \right.$

This will complete the proof of the Main Theorem because every countable commutative semigroup is isomorphic to a subsemigroup of $\mathbb{W}$, by [12].

IV. <u>The recognizing of A from</u> Y(A). In this part, we show that the set $A \in S$ of sequences can be recognized from the topological structure of the space $Y(A)$. We present the definitions 1 - 4 below and prove that $F(Y(A)) = A$.

IV.1. <u>Definition 1</u>. Let P be a topological space. We say that $x \in P$ is <u>essential</u> in P iff $id_P\, x \geq 1$ and any neighbourhood of x contains a clopen subset of P, which is a regular space. We say that x is <u>distinguished</u> if it is essential in P and there exists its neighbourhood $\mathcal{V}$ such that if $y \in \mathcal{V}$ is essential in P, then $id_P\, y = id_P\, x$.

<u>Definition 2</u>. Let P be a topological space. For every $x \in P$, we define

$q(x) = \{ m \in \omega \mid$ every neighbourhood of x contains a distinguished point y with $id_P\, y = m \}$.

<u>Definition 3</u>. Let $M_k = \{ m_{k,0}, m_{k,1}, \dots \}$ be as in III.3. Let P be a topological space. For every $x \in P$ and every $k \in \omega$ define $c_x(k) \subseteq \omega$ and $g_x(k) \in \omega \cup \{\omega\}$ by

$j \in c_x(k)$ iff for every $m \in \omega$ and every neighbourhood $\mathcal{U}$

of x there exists $z \in \mathcal{U}$ such that card $q(z) = j$ and $q(z) \subseteq$ $\subseteq M \setminus \{0,1,\dots,m\}$,

$g_x(k) = \sup o_x(k)$.

Remark. For every topological space P and every $x \in P$, we have defined a function $g_x\colon \omega \longrightarrow \omega \cup \{\omega\}$. Let us write

$$g_y < g_y$$

iff $g_y(k) \leq g_x(k)$ for all $k \in \omega$ and $g_y \neq g_x$.

Definition 4. Let P be a topological space. Put

$V(P) = \{x \in P \mid$ there exists a neighbourhood $\mathcal{U}$ of x such that $g_y < g_x$ for every $y \in \mathcal{U} \setminus \{x\}\}$,

$F(P) = \{g_x \mid x \in V(P)\}$.

Remark. As mentioned above, we are going to prove that for every $A \in S$,

$$F(Y(A)) = A.$$

(More precisely, ω is never a value of g_x for any $x \in V(Y(A))$, hence g_x can be regarded as a function $\omega \longrightarrow \omega$ and in this sense $F(Y(A)) = A$.)

This will imply (*) in III.3, as we show below.

IV.2. First, we discuss essential and distinguished points.

Observations. a) If Z_m is as in II.6, then 0 is its unique essential point, hence it is its distinguished point.

b) Since each copy of $Y(f)$ with $f \in A$ is a clopen subspace of $Y(A)$, $x \in Y(f)$ is essential in $Y(f)$ and $id_{Y(f)}\ x = n$ iff x is essential in $Y(A)$ and $id_{Y(A)}\ x = n$. Hence x is distinguished in $Y(f)$ iff it is distinguished in $Y(A)$ (more precisely, $x \in (Y(f))_j$, where $(Y(f))_j$ is a copy of $Y(f)$ in $Y(A) = \coprod_{j \in \omega, f \in A} (Y(f))_j$, $\coprod$ denoting the coproduct).

Let $L(f)$ and $\bar{\ell}$ be as in III.2, let X_k, B_k, G, $p_{k,i}$ be as in III.3.

<u>Lemma A.</u> Let $x \in Y(f)$ be such that all its coordinates x_ℓ, $\ell \in L(f)$, are in G. Then x is not essential in Y(f).

<u>Proof.</u> If $x_\ell \in G$ for $\ell \in L(f)$, then id x = 0 hence x is not essential.

<u>Lemma B.</u> Let x be in Y(f) and there exists $t \in L(f)$ such that the t-th coordinate x_t of x is in $E_{\bar t}$. Then x is not essential in Y(f).

<u>Proof.</u> Let us suppose $x_t \in E_{\bar t}$. Since $E_{\bar t}$ is open in $X_{\bar t}$,

$$Y(f) \cap (E_{\bar t} \times \prod_{\ell \in L(f) \setminus \{t\}} X_{\bar\ell})$$

is a neighbourhood of x in Y(f) which does not contain a clopen regular subspace because $E_{\bar t}$ does not contain a clopen regular subspace.

<u>Lemma C.</u> Let $K \subseteq L(f)$ be non-empty and finite, let the coordinates of a point $x \in Y(f)$ fulfill the following:

x_ℓ is in G for all $\ell \in L(f) \setminus K$,

$x_\ell = p_{\bar\ell, i(\ell)}$ for all $\ell \in K$ (for a suitable $i(\ell) \in \omega$).

Then x is essential in Y(f) and

$$\mathrm{id}_{Y(f)}\, x = \sum_{\ell \in K} m_{\bar\ell, i(\ell)}.$$

<u>Proof.</u> Every neighbourhood of x in Y(f) contains a neighbourhood of the form

$$Y(f) \cap (\mathcal{V} \times \prod_{\ell \in K} \mathcal{U}_\ell),$$

where $\mathcal{U}_\ell$ is a neighbourhood of $x_\ell = p_{\bar\ell, i(\ell)}$ in $X_{\bar\ell}$ and $\mathcal{V}$ is a subspace of the metrizable space $\mathop{\boxdot}_{\ell \in L(f) \setminus K} (G)_\ell$ (where $\boxdot$ is as in III.2). Consequently every neighbourhood of x in Y(f) contains a clopen regular subspace (by II.6) and $\mathrm{id}_{Y(f)}\, x = \sum_{\ell \in K} m_{\bar\ell, i(\ell)} > 1$, (by II.5).

<u>Proposition.</u> Let x be in Y(f). Then x is distinguished in

Y(f) iff precisely one coordinate of x is equal to $p_{k,i}$ and all the others are in G.

<u>Proof</u>. a) Let x be in Y(f), $x_\ell \in G$ for all $\ell \in L(f) \setminus \{t\}$, $x_t = p_{\bar{t},i}$. Put

$$\mathcal{U}_\ell = G \text{ for all } \ell \in L(f) \setminus \{t\}$$
$$\mathcal{U}_t = \{2^{-i}\} \times Z_{m_{\bar{t},i}}.$$

Then $\mathcal{U} = Y(f) \cap \prod_{\ell \in L(f)} \mathcal{U}_\ell$ is a neighbourhood of x in Y(f). By Lemmas A-C, $y \in \mathcal{U}$ is essential iff $y \in Y(f)$, $y_\ell \in G$ for all $\ell \in L(f) \setminus \{t\}$ and $y_t = p_{\bar{t},i}$. Then id $y = m_{\bar{t},i} =$ id x, hence x is a distinguished point of Y(f).

b) Conversely, let x be a distinguished point of Y(f). Since x is essential, none of its coordinates are in some E_k, by Lemma B. Hence all its coordinates are in $H = \{\sigma\} \cup G$, except, possibly, finitely many which are equal to some $p_{k,i}$'s. First, we prove that no coordinate of x can be equal to σ. Thus, let us suppose that there exists $t \in L(f)$ such that $x_t = \sigma$. Then every neighbourhood of x contains infinitely many essential points z with id z all distinct. In fact, we can choose $z_t = p_{\bar{t},i}$ with sufficiently large i and, since no coordinate of x is in E_k and G is dense in each $X_k \setminus E_k$, we can find z_ℓ in G arbitrarily close to x_ℓ for all $\ell \in L(f) \setminus \{t\}$ such that $z = \{z_\ell \mid \ell \in L(f)\}$ is in Y(f) and sufficiently close to x. Then z is an essential point of Y(f) with id $z = m_{\bar{t},i}$. And x is an accumulation point of all such z's with all larger i's, so x cannot be a distinguished point of Y(f). Thus, if x is a distinguished point of Y(f), then there exists $K \subseteq L(f)$ finite such that

$x_\ell \in G$ for all $\ell \in L(f) \setminus K$,

$x_\ell = p_{\bar{\ell},i(\ell)}$ for all $\ell \in K$ (and suitable $i(\ell) \in \omega$).

By Lemma A, K is non-empty. Let us suppose that card $K > 1$. Then id $x = \sum_{\ell \in K} m_{\bar{\ell},i(\ell)}$ but every its neighbourhood contains an essential point y with id $y = m_{\bar{t},i(t)}$ for $t \in K$. In fact, if

$y_\ell = x_\ell$ for all $\ell \in (L(f)\setminus K)\cup\{t\}$ and y_ℓ is in G and sufficiently close to x_ℓ for all $\ell \in K\setminus\{t\}$, then really y is an essential point with id $y = m_{\bar{t},i(t)}$; if card $K > 1$, then id $y \neq$ id x, which is a contradiction. Consequently card $K = 1$.

IV.3. Now, we investigate the invariant q(x) from Definition 2.

<u>Observation.</u> If Q is a clopen subspace of P and $x \in Q$, then $q_P(x) = q_Q(x)$, evidently. Hence for every $f \in A$ and $x \in Y(f)$,

$$q_{Y(A)}(x) = q_{Y(f)}(x).$$

<u>Lemma.</u> Let x be in Y(f), m be in ω. Then $m \in q(x)$ iff no coordinate of x is in any E_k and at least one coordinate of x is equal to $p_{k,i}$ with $m_{k,i} = m$.

<u>Proof.</u> If a coordinate of $x \in Y(f)$ belongs to some E_k, then x has a neighbourhood containing no essential point so that $q(x) = \emptyset$. Hence if $q(x) \neq \emptyset$, no coordinate of x is in any E_k. If no coordinate of x is equal to $p_{k,i}$ with $m_{k,i} = m$, then x has a neighbourhood containing no distinguished point y with id $y = m$, this follows from IV.2 Proposition; hence $m \notin q(x)$. Conversely, let us suppose that at least one coordinate of x is equal to $p_{k,i}$ with $m_{k,i} = m$, say the t-th one, and no coordinate x_ℓ of x is in $E_{\bar{\ell}}$. Since G is dense in each $X_{\bar{\ell}} \setminus E_{\bar{\ell}}$, we can find a distinguished point y sufficiently close to y such that

$y_t = x_t = p_{k,i}$,

$y_\ell \in G$ for all $\ell \in L(f)\setminus\{t\}$,

hence id $y = m$. Thus, $m \in q(x)$.

IV.4. Let us investigate the invariants from Definition 3.

<u>Observation.</u> If Q is a clopen subspace of P and $x \in Q$, then the definition of $a_x(k)$ and $g_x(k)$ with respect to P and with

respect to Q coincide.

Lemma. Let x be in $Y(f)$. If some coordinate x_ℓ of x is in $E_{\bar{\ell}}$, then $c_x(k) = \{0\}$ for all $k \in \omega$. Otherwise, $g_x(k)$ is the number of all the coordinates x_ℓ of x, for which simultaneously

$$\bar{\ell} = k \text{ and } x_\ell = \sigma .$$

Proof. If a coordinate x_ℓ of x is in $E_{\bar{\ell}}$, then x has a neighbourhood $\mathcal{U}$ containing no essential point so that $q(z) = \emptyset$ for every $z \in \mathcal{U}$, hence card $q(z) = 0$; consequently $c_x(k) = \{0\}$ for all $k \in \omega$.

Let us suppose that no coordinate x_ℓ of x is in $E_{\bar{\ell}}$. Let $k \in \omega$ be given; we denote by $K \subseteq L(f)$ the set of all $\ell \in L(f)$ such that $\bar{\ell} = k$ and $x_\ell = \sigma$ (hence card $K \leq f(k)$).

a) We prove that card $K \leq g_x(k)$. Let a neighbourhood $\mathcal{U}$ of x and $m \in \omega$ be given. We can find $z \in \mathcal{U}$ with $q(z) \subseteq M_k \setminus \{0, \dots, m\}$ and card $q(z)$ = card K as follows: we choose distinct numbers $m_{k,i(\ell)}$, $\ell \in K$, in $M_k \setminus \{0, \dots, m\}$ such that $p_{k,i(\ell)}$ is sufficiently close to σ and put

$z_\ell = p_{k,i(\ell)}$ for all $\ell \in K$

$z_\ell \in G$ sufficiently close to x_ℓ for all $\ell \in L(f) \setminus K$

(since G is dense in $X_{\bar{\ell}} \setminus E_{\bar{\ell}}$, this is possible) and such that $z = \{z_\ell \mid \ell \in L(f)\}$ is in $Y(f)$. Then $q(z) = \{m_{k,i(\ell)} \mid \ell \in K\}$, by IV.3 Lemma. Since card $q(z)$ = card K, card $K \in c_x(k)$, so that card $K \leq g_x(k)$.

b) To prove the converse inequality let us denote

$\mathcal{U}_\ell = X_{\bar{\ell}}$ whenever either $\bar{\ell} \neq k$ or $x_\ell = \sigma$,

$\mathcal{U}_\ell = \{2^{-i(\ell)}\} \times Z_{m_{\bar{\ell},i(\ell)}}$ whenever $\bar{\ell} = k$ and $x_\ell \in \{2^{-i(\ell)}\} \times$

$\times Z_{m_{\bar{\ell},i(\ell)}}$.

Choose $m > \max \{m_{\ell,i(\ell)} \mid \mathcal{U}_\ell \neq X_{\bar{\ell}}\}$ and put

$$\mathcal{U} = Y(f) \cap \prod_{\ell \in L(f)} \mathcal{U}_\ell .$$

Clearly, $\mathcal{U}$ is a neighbourhood of x and card $q(z) \leq$ card K for every $z \in \mathcal{U}$ with $q(z) \subseteq M_k \setminus \{0,\dots,m\}$. Consequently, $g_x(k) \leq$ $\leq$ card K.

IV.5. Now, we investigate the invariants V(P), F(P) from Definition 4.

Observation. If Q is a clopen subspace of P, then $V(Q) = Q \cap V(P)$ and $F(Q) \subseteq F(P)$.

Proposition. For every $A \in S$ and $f \in A$,

V(Y(f)) consists precisely of the point with all coordinates equal to σ, $F(Y(f)) = \{f\}$ and $F(Y(A)) = A$.

Proof. This follows easily from IV.4 Lemma.

Corollary. If $A, B \in S$ and Y(A) is homeomorphic to a clopen subspace of Y(B), then

$A = F(Y(A)) \subseteq F(Y(B)) = B$.

Thus, we have proved (*) in III.3.

V. Some strengthenings of the Main Theorem.

V.1. The following strengthening can be seen immediately from the proof of the Main Theorem: If S is a commutative semigroup (not necessarily countable) such that there exists an embedding

$$\varphi : S \longrightarrow \mathbb{W}$$

with $\bigcup_{s \in S} \varphi(s)$ countable, then S has a productive representation in the class CHSCW. This has e.g. the following consequences:

a) The additive group (R,+) of all real numbers has a productive representation in CHSCW. (In fact, there exists an embedding $\varphi : (Q,+) \longrightarrow \mathbb{W}$ of the additive group of all rational numbers with $\varphi(q) \cap \varphi(q') = \emptyset$ whenever $q \neq q'$, by [12]. Then $\psi : (R,+) \longrightarrow \mathbb{W}$ defined by

$$\psi(r) = \bigcup_{\substack{q \in Q \\ q \leq r}} \varphi(q)$$

is an embedding of $(R,\dashv)$ into $\mathbb{U}$ and $\bigcup_{r \in R} \psi(r)$ is countable.)

b) There is an $X \in$ CHSCW which has $2^{\aleph_0}$ non-homeomorphic square roots. (In fact, put $S = \exp \omega$ and $s+s' = \emptyset$ for all $s,s' \in S$. Put $S_0 = \{s \in S \mid \text{card } s \leq 1\}$. Then there is an embedding $\varphi : S_0 \longrightarrow \mathbb{U}$ with $\varphi(s) \cap \varphi(s') = \emptyset$ whenever $s \neq s'$, by [12]. Then $\psi : S \longrightarrow \mathbb{U}$, defined by

$$\psi(\emptyset) = \varphi(\emptyset)$$
$$\psi(s) = \bigcup_{n \in s} \varphi(n) \text{ for } s \in S,\ s \neq \emptyset,$$

is an embedding with $\bigcup_{s \in S} \psi(s)$ countable. If $\{X(s) \mid s \in S\}$ is a productive representation of $(S,+)$ in CHSCW, then the space $X = X(\emptyset)$ has $2^{\aleph_0}$ non-homeomorphic square roots.)

V.2. Let us describe another strengthening of the Main Theorem: Let a space P in CHSCW and a subsemigroup S of $\mathbb{U}$ with countable support be given. Then there exists a productive representation $\{Z(A) \mid A \in S\}$ of S in CHSCW such that P is a retract of each representing space Z(A). In fact, put

$$T = P \times E,$$

where $E \in$ CHSCW is a space such that the points x with id $x > 0$ are dense in it. Define $\mathfrak{X} = \{X_k \mid k \in \omega\}$ as in III.3 and, for every $f \in$ supp S and $A \in S$, define Y(f) and Y(A) by means of $\mathfrak{X}$ as in III.2. Finally, for every $A \in S$, put

$$Z(A) = \coprod_{n \in \omega} T^n \times Y(A).$$

Clearly, P is a retract of Z(A). Since each Y(A) is homeomorphic to a coproduct of $\aleph_0$ copies of itself, we see that

$Z(A) \times Z(B)$ is homeomorphic to $Z(A + B)$.

And we can recognize the set $A \in S$ from the structure of Z(A) as in IV. In fact, no $T^n \times Y(A)$ with $n > 0$ contains essential points (because of the factor E) so that only $T^0 \times Y(A)$ (which is homeomorphic to $Y(A)$ because T^0 is a one-point space) influences the invariants c_X, g_X, so that

$V(Z(A)) = V(Y(A))$ and $F(Z(A)) = F(Y(A))$.

Consequently, if $Z(A)$ is homeomorphic to a clopen subspace of $Z(B)$, then $A \subseteq B$.

<u>Corollary</u>. Every space P in CHSCW is a retract of a space in CHSCW having $2^{\aleph_0}$ non-homeomorphic square roots or of a space $X \in$ CHSCW homeomorphic to X^3 but not to X^2.

V.3. The next strengthening of the Main Theorem is as follows: Given a semigroup $S \subseteq \mathcal{U}$ with countable support and a space P in CHSCW, there are $2^{\aleph_0}$ non-homeomorphic productive representations of S in CHSCW such that each representing space has P as its retract. (We say that $\{Z(A) \mid A \in S\}$ and $\{Z'(A) \mid A \in S\}$ are non-homeomorphic representations if none of the spaces $Z(A)$, $A \in S$, is homeomorphic to any of the spaces $Z'(B)$, $B \in S$.) In fact, the construction of the productive representation presented in III depends on a given pairwise disjoint system $M = \{M_k \mid k \in \omega\}$ of infinite subsets of ω. If we choose $M' = \{M'_k \mid k \in \omega\}$ such that

$$(\bigcup_{k \in \omega} M_k) \cap (\bigcup_{k \in \omega} M'_k) \text{ is finite}$$

then none of the spaces $Z(A)$, $A \in S$, of the productive representation constructed by means of M is homeomorphic to any of the spaces $Z'(B)$, $B \in S$, of the representation constructed by means of M' (this can be seen using the method of IV.) .

V.4. Let us mention the following generalization of the Main Theorem: In [1], J. Adámek and V. Koubek investigate a sum-productive representation of an ordered commutative semigroup $(S, +, \leq)$ in a category $\mathcal{K}$ with finite products and finite coproducts (=sums). It is a collection $\{X(s) \mid s \in S\}$ of objects of $\mathcal{K}$

such that

(i) $X(s_1) \times X(s_2)$ is isomorphic to $X(s_1+s_2)$ for all $s_1, s_2 \in S$;

(ii) $X(s_1)$ is a summand of $X(s_2)$ iff $s_1 \leq s_2$.

For $\mathcal{K}$ = CHSCW, being a summand is precisely being homeomorphic to a clopen subspace.

(Any commutative semigroup (S,+) can be ordered by the discrete order (i.e. any two distinct elements are incomparable.) Then a sum-productive representation $\{X(s), s \in S\}$ in CHSCW fulfils (i) and

if $s_1 \neq s_2$ then neither $X(s_1)$ is homeomorphic to a clopen subspace of $X(s_2)$ nor $X(s_2)$ is homeomorphic to a clopen subspace of $X(s_1)$.)

The semigroup $\mathbb{U} \subseteq \exp \omega^{\omega}$ is an ordered semigroup, it is ordered by inclusion. If $S \subseteq \mathbb{U}$ has countable support, we have constructed its productive representation $\{Y(A) | A \in S\}$ such that ($*$) of III.3 is fulfilled. This means that $\{Y(A) | A \in S\}$ is a sum-productive representation of S, where S inherits its order from $\mathbb{U}$. And, by [1], every countable ordered commutative semigroup $(S,+,\leq)$ can be embedded in $\mathbb{U}$ such that $s_1 \leq s_2$ iff $\varphi(s_1) \subseteq \varphi(s_2)$ (where φ is the embedding). Consequently,

every countable ordered commutative semigroup has a sum-productive representation in CHSCW.

Moreover, also some uncountable ordered commutative semigroups have a sum-productive representation in CHSCW - the existence of an embedding onto an ordered subsemigroup of $\mathbb{U}$ with a countable support is a sufficient condition. One can see e.g. that the embedding $\psi: (R,+) \longrightarrow \mathbb{U}$ from V.1 a) preserves the order so that $\{Y(\psi(r)) | r \in R\}$ is a sum-productive representation of the additive group of all real numbers with their natural order in CHSCW. The strengthenings described in V.2 and V.3 can be done also for sum-productive representations of ordered commutative semigroups.

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