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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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MINIMAL ULTRAFILTERS AND MAXIMAL ENDOMORPHIC UNIVERSES

A. TZOUVARAS

Abstract: We prove that there is an ultrafilter on Sd_V which is minimal in the Rudin-Keisler ordering of ultrafilters. Using such an ultrafilter we construct maximal endomorphic universes with standard extensions. Moreover, for such a universe A, the equivalence $\underset{A}{\doteq}$ is the equality.

Key words: Alternative set theory, ultrafilter, Rudin-Keisler ordering, endomorphic universe, standard extension.

Classification: 02K10, 02K99

The existence of an endomorphic universe with a standard extension was established in [S-V]. Specifically, it was proved that if $\mathfrak{M}$ is a non-trivial ultrafilter on Sd_V which contains all supersets of a countable class and d is an element such that o, $\mathfrak{M}$,d are coherent, then there is an endomorphism F such that F, $\mathfrak{M}$, d are coherent, F"V d = V and F"V has a standard extension.

We are going to strengthen this result by showing the existence of an ultrafilter $\mathfrak{M}$, such that if F, $\mathfrak{M}$, d are coherent and F "V [d] = V, then F"V[a] = V for every a $\notin$ F"V.

Since A[a] is the smallest endomorphic universe which contains $A \cup \{a\}$ and $\cup A$ = V, it is evident that F"V is maximal in the usual sense, i.e. it is not included in any proper strictly greater endomorphic universe.

In the first section we show that maximal endomorphic universes exist, while in the second one we get results about the relation $\overset{=}{_A}$, when A is maximal.

§ 1. Minimal Ultrafiletrs. The following definition is a version suitable for our needs of the well-known classical definition (see e.g. [B] p.17, where our minimal ultrafilters are called there "Ramsey").

1.1. Definition. An ultrafilter $\mathfrak{M}$ on Sd_V is called minimal if for every set-definable function F on V, there is a class $X \in \mathfrak{M}$ such that $F \upharpoonright X$ is either constant or one-to-one.

Since we are going to deal with ultrafilters containing sets, the next rather trivial result is in order.

1.2. Lemma. If $\mathfrak{M}$ contains a set, then $\mathfrak{M}$ is minial , iff for every set-function f such that $dom(f) \in \mathfrak{M}$, there is a $u \in \mathfrak{M}$ such that $f \upharpoonright u$ is either constant or one-to-one.

Proof. Suppose $\mathfrak{M}$ is minimal and $dom(f) \in \mathfrak{M}$. Extend f to F on V by setting $F(x) = 0$ for $x \notin dom(f)$. Then $F \upharpoonright X$ is either constant or one-to-one for some $X \in \mathfrak{M}$ and obviously this is the case for $f \upharpoonright X \cap dom(f)$.

Conversely, if F is a set-definable function on V and $u \in \mathfrak{M}$, then $F \upharpoonright u$ is a set-function, hence there is a $v \in \mathfrak{M}$ such that $F \upharpoonright u \cap v$ is either constant or one-to-one.

1.3. Theorem. Let X be a countable class. Then there is a minimal non-trivial ultrafilter $\mathfrak{M}$ on Sd_V such that for every u,

$$X \subseteq u \longrightarrow u \in \mathfrak{M} .$$

Proof. Let $(u_\alpha)_{\alpha \in \Omega}$ be an enumeration of all $u \supseteq X$ and let $(f_\alpha)_{\alpha \in \Omega}$ be an enumeration of all set functions. We are going to construct a sequence of classes $(\mathfrak{M}_\alpha)_{\alpha \in \Omega}$ such that:

(i) For each $\alpha \in \Omega$, $\mathfrak{M}_\alpha$ is an, at most countable, subclass of Sd_V.

(ii) $\alpha < \beta \rightarrow \mathfrak{M}_\alpha \subseteq \mathfrak{M}_\beta$

(iii) $\mathfrak{M}_\alpha$ is closed under finite intersections and $(\forall Z \in \mathfrak{M}_\alpha)$ $(Z \cap X$ is infinite).

(iv) $u_\alpha \in \mathfrak{M}_\alpha$ for every $\alpha \in \Omega$.

(v) For each $\alpha \in \Omega$, if $dom(f_\alpha) \cap Z \cap X$ is infinite for all $Z \in \bigcup \{\mathfrak{M}_\beta ; \beta \in \alpha \cap \Omega\}$, then there is a $u \in \mathfrak{M}_\alpha$ such that either $f_\alpha \upharpoonright u$ is one-to-one or $rng(f_\alpha \upharpoonright u)$ is finite.

Suppose $(\mathfrak{M}_\alpha)_{\alpha \in \Omega}$ has been constructed. Since $\bigcup\{\mathfrak{M}_\alpha ; \alpha \in \Omega\}$ contains no finite set, there is a non trivial ultrafilter $\mathfrak{M}$ extending $\bigcup\{\mathfrak{M}_\alpha ; \alpha \in \Omega\}$. To see that $\mathfrak{M}$ is minimal, after Lemma 2, take any function f with $dom(f) \in \mathfrak{M}$. It is easy to see that $dom(f) \cap Z \cap X$ is infinite for every $Z \in \mathfrak{M}$. If $f = f_\beta$, then, by the clause (v), there is some $u \in \mathfrak{M}_\beta$ such that either $f_\beta \upharpoonright u$ is one-to-one or $rng(f_\beta \upharpoonright u)$ is finite. In the latter case there is an $x \in rng(f_\beta \upharpoonright u)$ such that $f_\beta^{-1}\text{"}\{x\} \in \mathfrak{M}$ and f_β is constant on $f_\beta^{-1}\text{"}\{x\}$.

<u>Construction of</u> $(\mathfrak{M}_\alpha)_{\alpha \in \Omega}$. Assume that $\mathfrak{M}_\beta$, for $\beta \in \alpha \cap \Omega$ have been defined. Let $\mathfrak{M}'_\alpha$ be the closure of $\bigcup\{\mathfrak{M}_\beta ; \beta \in \alpha \cap \Omega\} \cup \{u_\alpha\}$ under finite intersections. Note that

$(\exists Z \in \bigcup\{\mathfrak{M}_\beta ; \beta \in \alpha \cap \Omega\})(dom(f_\alpha) \cap Z \cap X$ is finite $\longleftrightarrow$

$(\exists Z \in \mathfrak{M}'_\alpha)(dom(f_\alpha) \cap Z \cap X$ is finite),

thus we distinguish the following cases:

1) For some $Z \in \mathfrak{M}'_\alpha$, $dom(f_\alpha) \cap Z \cap X$ is finite. Put, then, $\mathfrak{M}_\alpha =$ $= \mathfrak{M}'_\alpha$.

2) For all $Z \in \mathfrak{M}'_\alpha$, $dom(f_\alpha) \cap Z \cap X$ is infinite but for some $Z_0 \in \mathfrak{M}'_\alpha$, $rng(f_\alpha) \upharpoonright Z_0 \cap X)$ is finite.

Since $\mathrm{dom}(f_\alpha) \cap Z_0 \cap X$ is countable we can find a set $v \supseteq \mathrm{dom}(f_\alpha) \cap Z_0 \cap X$ such that $\mathrm{rng}(f_\alpha \upharpoonright v)$ is finite. For every $Z \in \mathfrak{M}'_\alpha$

$$v \cap Z \cap X \supseteq \mathrm{dom}(f_\alpha) \cap Z \cap Z_0 \cap X,$$

hence, if we put $\mathfrak{M}_\alpha$ = closure of $\mathfrak{M}'_\alpha \cup \{v\}$ under finite intersections, then $\mathfrak{M}_\alpha$ has all properties (i) - (v).

3) Suppose, finally, that for all $Z \in \mathfrak{M}'_\alpha$, both $\mathrm{dom}(f_\alpha) \cap Z \cap X$ and $\mathrm{rng}(f_\alpha \upharpoonright Z \cap X)$ are infinite. Take an enumeration $(Z_n)_{n \in FN}$ of $\mathfrak{M}'_\alpha$ and chhose a countable class $Y = \{y_0, y_1, \dots\}$ such that $y_n \in X \cap Z_0 \cap \dots \cap Z_n$ for every $n \in FN$, and $f_\alpha \upharpoonright Y$ is one-to-one. The choice is always possible. Indeed, assume $\{y_0, \dots, y_n\}$ have been chosen. Then, as $Z_0 \cap \dots \cap Z_{n+1} \in \mathfrak{M}'_\alpha$ and f_α is infinite on $X \cap Z_0 \cap \dots \cap Z_{n+1}$, we can find a $y_{n+1} \in X \cap Z_0 \cap \dots \cap Z_{n+1}$ such that $f_\alpha(y_{n+1}) \neq f_\alpha(y_i)$ for every $i \leq n$. Now, let v be a prolongation of Y such that $f_\alpha \upharpoonright v$ is one-to-one. It is clear that $Z \cap Y$ is infinite for every $Z \in \mathfrak{M}'_\alpha$ and

$$v \cap Z \cap X \supseteq v \cap Z \cap Y = Z \cap Y.$$

Thus, putting again $\mathfrak{M}_\alpha$ = closure of $\mathfrak{M}'_\alpha \cup \{v\}$ under finite intersections, $\mathfrak{M}_\alpha$ satisfies (i) - (v) and the proof is complete.

1.4. <u>Lemma.</u> Let $\mathfrak{M}$ be a non trivial ultrafilter containing a set, let F be an endomorphism and let $F, \mathfrak{M}, d$ be coherent. Then, $\mathfrak{M}$ is minimal iff for every $a \in A[d] - A$, $A[d] = A[a]$, where $A = F''V$.

<u>Proof.</u> Assume that $\mathfrak{M}$ is minimal and $a \in A[d] - A$. There is some $f \in A$ such that $d \in \mathrm{dom}(f)$ and $f(d) = a$. Let $f = F(g)$. Clearly, g is a function and $F(\mathrm{dom}(g)) = \mathrm{dom}(f)$. As $d \in \mathrm{dom}(f)$ and $F, \mathfrak{M}, d$ are coherent, it follows that $\mathrm{dom}(g) \in \mathfrak{M}$. By minimality, there is a $u \in \mathfrak{M}$ such that $g \upharpoonright u$ is either constant or

one-to-one. Clearly, $g \upharpoonright u$ is constant iff $f \upharpoonright F(u)$ is and $g \upharpoonright u$ is one-to-one iff $f \upharpoonright F(u)$ is. The former case is impossible because $a \notin A$. Hence $f \upharpoonright F(u)$ is one-to-one and it follows from [S-V], 5th theorem of § 1 that $A[d] = A[a]$.

The converse is shown similarly.

1.5. Corollary. There is an endomorphic universe with standard extension, A, such that $A[a] = V$ for every $a \notin A$.

Proof. Take $\mathfrak{M}$ minimal, non trivial, containing the supersets of some countable class and d such that o, $\mathfrak{M}$, d are coherent. By the last but two theorem of [S-V], there is an endomorphism F such that $F''V[d] = V$. Then $A = F''V$ has a standard extension and, by 1.4, $A[a] = A[d] = V$ for all $a \notin A$.

§ 2. The equivalence $\overset{=}{\underset{A}{}}$. We come, now, to examine the status of the relation $\underset{A}{\equiv}$, when A is an endomorphic universe with standard extension.

Let us recall that $x \underset{A}{\equiv} y$ iff for every set-formula $\varphi(x,x_1,\dots,x_n)$ of FL and any parameters $y_1,\dots,y_n \in A$, $\varphi(x,y_1,\dots$ $\dots,y_n) \leftrightarrow \varphi(y,y_1,\dots,y_n)$.

Let $\mathrm{Mon}_A(x)$ denote the equivalence class of x w.r.t. $\underset{A}{\equiv}$.

Given a (proper) endomorphic universe A, let us put $D_a =$ $= \{u \in A; a \in u\}$ for every $a \notin A$.

D_a is a non trivial filter on A and it is easily seen that

$$\bigcap D_a = \mathrm{Mon}_A(a) \text{ and } D_a = D_b \leftrightarrow a \underset{A}{\equiv} b.$$

In fact, D_a is a kind of "ultrafilter" since the following holds:

2.1. Lemma. For every $u \in A$,

$$u \in D_a \leftrightarrow (\forall v \in D_a)(u \cap v \neq \emptyset).$$

Proof. " $\rightarrow$ " is trivial. Conversely, let $u \notin D_a$. Take $v \in D_a$. Then $a \in v - u$, $v - u \in A$ and $(v-u) \cap u = \emptyset$.

Let f be a function in A. We say that D_b is the image of D_a by f, if $a \in dom(f)$ and $f''u; a \in u \subseteq dom(f))$ generates D_b. We write, then, $f''D_a = D_b$.

The next lemma is a classical combinatorial result (see, e.g. [B] Th.3.3, or [C-N], p. 207) so we omit the proof.

2.2. Lemma. Let X be a countable class and let $F: X \rightarrow X$ be a function such that $F(x) \neq x$ for every $x \in X$. Then there is a partition of X into two classes X_0, X_1 such that

$$F''X_i \cap X_i = \emptyset, \ i = 0,1.$$

2.3. Lemma. Let A be an endomorphic universe (with standard extension) and let $a \notin A$. Then for every function $f \in A$,

$$f''D_a = D_a \leftrightarrow f(a) = a.$$

Proof. If $f(a) = a$ it is clear, by 2.1, that $f''D_a = D_a$.

Conversely, suppose $f''D_a = D_a$. It suffices to show that $\{x \in dom(f);\ f(x) = x\} \in D_a$, or, equivalently, that the set

$$u = \{x \in dom(f);\ f(x) \neq x\} \notin D_a.$$

Assume the contrary. There is a countable X such that $a \in Ex(X)$. Since $a \in Ex(X) \cap u = Ex(X \cap u)$ we can suppose that $X \subseteq u$. Moreover, we can suppose that $f''X \subseteq X$.

(Otherwise, put $u_1 = u \cup f''u$ and extend f to f_1 on u_1 by defining for $x \in u_1 - u: f_1(x) = y$ iff $f(y) = x$ and y is the least in the usual linear ordering of V. Then, $f_1 \in A$, $f_1(x) \neq x \ \ \forall x \in u_1$, $u_1 \in D_a$ and $f_1''D_a = D_a$. Put

$$X_1 = X \cup f_1''X \cup f_1^{2}{}''X \cup \ldots$$

X_1 is countable, $X_1 \subseteq u_1$, $f''X_1 \subseteq X_1$ and $a \in Ex(X_1)$.)

By the preceding lemma, there is a partition of X into X_0, X_1 such that $f"X_i \cap X_i = \emptyset$, $i = 0,1$. But if $a \in Ex(X_i)$, it follows from $f"D_a = D_a$ that $a \in Ex(f"X_i)$. Indeed, let $f"X \subseteq v$. Then, $X_i \subseteq f^{-1}"f"X_i \subseteq f^{-1}"(v)$, hence $f^{-1}"(v) \in D_a$ or equivalently, $v \in f"D_a = D_a$, whence $a \in v$. But this is a contradiction.

It is well known that if X is at most countable, then $\overset{\pmb{=}}{X}$ is an indiscernibility equivalence and $Mon_X(x) = \{x\}$ iff $x \in Def_X$, while for every class C,

$$x \in Def_C \longrightarrow Mon_C(x) = \{x\}.$$

That the arrow in the last implication cannot be reversed, is seen by the next counterexample.

2.4. <u>Corollary.</u> If A is an endomorphic universe with standard extension and $A[d] = V$, for some $d \notin A$, then $Mon_A(d) = \{d\}$.

<u>Proof.</u> Let $d' \in Mon_A(d)$ and $d' \neq d$. Since $A[d] = V$, there is an $f \in A$ such that $d \in dom(f)$ and $f(d) = d'$. Obviously, $f"D_d = D_{d'} = D_d$ and this contradicts the preceding lemma.

2.5. <u>Corollary.</u> If A is a maximal endomorphic universe with standard extension then

$$x \underset{A}{\overset{\pmb{=}}{}} y \longleftrightarrow x = y.$$

<u>Proof.</u> This is an immediate consequence of 1.5 and 2.4.

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