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UTILITY THEORY IN THE ALTERNATIVE SET THEORY

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Abstract. Theory of utility in the alternative set theory enables us to comprehend more delicately a preference relation, especially its infinitesimal differences. We prove that there is a valuation for any class with a preference relation .New, non-traditional, but natural questions arise and we solve some of them.

Key words: Alternative set theory, utility theory, preference relation, valuation.

Classification: 03E70, 90A06, 90A12.

Introduction. Utility theory was formulated by John von Neumann and Oscar Morgenstern in 1943.In the present paper we develop it from the point of view of the alternative set theory. The both theories are compared in § 4. We modify the approach of von Neumann and Morgenstern in the following way.

Let S be a class of objects. Let us imagine a man before whom we put various elements of this class and he chooses among them. When we put before him two elements of S he is able to choose one of the two.

Let us extend this picture. Let him choose not only between objects, but also between their combinations with stated probabilities. A combination of n elements $u_1,\dots,u_n$ ($n \in FN$) of S with probabilities $\alpha_1,\dots,\alpha_n$ ($\alpha_i \in FRN$, $\alpha_i \geq 0$, $\sum \alpha_i = 1$) represents a game in which α_1 is a probability of gaining u_1, α_2 is a probability of gaining u_2, etc. We denote this game by

$\alpha_1 u_1 + \ldots + \alpha_n u_n$ or $\sum_{i=1}^{n} \alpha_i u_i$. Thus when we put before the man two combinations $\sum_{i=1}^{n} \alpha_i u_i$, $\sum_{j=1}^{m} \beta_j v_j$, he either prefers $\sum \alpha_i u_i$ to $\sum \beta_j v_j$ ($\sum \alpha_i u_i \succ \sum \beta_j v_j$) or vice versa ($\sum \beta_j v_j \succ \sum \alpha_i u_i$) or he considers them to be indifferent ($\sum \alpha_i u_i \sim \sum \beta_j v_j$).

We use notions defined in [V]. N denotes the class of natural numbers, ZN of integers, RN of rational numbers. FN is the class of finite natural numbers, FRN of finite rational numbers.

§ 1. Class S and a Preference Relation. In what follows we shall denote the class of objects by S.

Definition. Let α be an element of FRN such that $0 \leq \alpha \leq 1$. Then we call α a probability coefficient.

Definition. Let S_{FRN} be a linear space with the basis S over FRN. Let us denote $S[1] \subseteq S_{FRN}$ the class of all convex combinations of elements of S, i.e.

$$S[1] = \{\sum_{i=1}^{n} \alpha_i u_i ;\ (\forall i)(1 \leq i \leq n)(u_i \in S \ \&\ \alpha_i \text{ is a probability coefficient } \& \sum_{i=1}^{n} \alpha_i = 1\}.$$

S[1] is a class of games described in the introduction. We write $\sum \alpha_i u_i$ instead of $\sum_{i=1}^{n} \alpha_i u_i$ for some $n \in FN$ and letters $u, v, w, u_i, \ldots$ for elements of S and $a, b, c, a_i, \ldots$ for elements of S[1] in the short-hand notation.

Definition. $\succ$ is a preference relation on S, provided $\succ \subseteq S[1] \times S[1]$ and it holds: if a,b,c are elements of S[1], α being a probability coefficient, then

(P1) $\neg (a \succ a)$,

(P2) $(a \succ b \,\&\, b \succ c) \Rightarrow a \succ c$,

(P3) $(\neg(a \succ b) \,\&\, \neg(b \succ c)) \Rightarrow \neg(a \succ c)$,

(P4) $a \succ b \equiv \alpha a + (1 - \alpha)c \succ \alpha b + (1 - \alpha)c$.

In what follows let S always be a class of objects with a preference relation $\succ$.

Theorem 1.1. Let a,b,c, d be elements of S[1], α be a probability coefficient. Then

$(a \succ b \,\&\, c \succ d) \Rightarrow \alpha a + (1 - \alpha)c \succ \alpha b + (1 - \alpha)d$ and

$(\neg(a \succ b) \,\&\, \neg(c \succ d)) \Rightarrow \neg(\alpha a + (1 - \alpha)c \succ \alpha b + (1 - \alpha)d)$.

Remark. Assuming (P4), these two assertions are equivalent to (P2) and (P3).

Definition. Let a,b be elements of S[1]. $a \sim b$ (a *is indifferent to* b) if $\neg(a \succ b) \,\&\, \neg(b \succ a)$.

Theorem 1.2. An indifference relation is an equivalence.

Proof. Reflexivity from (P1), symmetry from the Definition, transitivity from (P3).

Theorem 1.3. Let us define a relation $\dot{\succ}$ on $S[1]/\sim$ as follows: $[a] \dot{\succ} [b]$ iff $a \succ b$. $\dot{\succ}$ is a preference relation on $S/\sim$ and it is a strict linear ordering of $S/\sim$.

Proof. The definition of $\dot{\succ}$ is correct. This follows from (P2) and (P3). The assertion $\sum \alpha_i [u_i] = [\sum \alpha_i u_i]$ follows from 1.1. Hence $(S/\sim)[1] = S[1]/\sim \subseteq S_{FRN}/\sim$. All axioms (P1) - (P4) hold. When u,v are elements of S such that $[u] \neq [v]$ then either $[u] \dot{\succ} [v]$ or $[v] \dot{\succ} [u]$.

By 1.3, from now on, we shall w.l.o.g. consider any preference relation to be also a strict linear ordering of S.

§ 2. The Definition and the Existence of a Valuation. One of our main goals is to find a valuation of S, i.e. an embedding of S into RN, which preserves a preference relation. In this way we find prices of objects of S.

The following definition proves to be useful.

Definition. $S[0] = \{\Sigma \alpha_i u_i;\ (\forall i)(u_i \in S\ \&\ \alpha_i \in FRN)\ \&$
$\&\ \Sigma \alpha_i = 0\ \&\ \Sigma |\alpha_i| > 0\}$.

Each element of S[0] can be evidently written in this form: $\Sigma \alpha_i u_i + \Sigma - \beta_i v_i$, where the relations $\Sigma \alpha_i = \Sigma \beta_i =$
$= \gamma > 0$ and $\Sigma \frac{\alpha_i}{\gamma} u_i \in S[1]$ and $\Sigma \frac{\beta_i}{\gamma} v_i \in S[1]$ hold.

Definition. Let $\Sigma \alpha_i u_i + \Sigma - \beta_i v_i$ be an element of S[0], $\Sigma \alpha_i = \Sigma \beta_i = \gamma > 0$. We define

$\Sigma \alpha_i u_i + \Sigma - \beta_i v_i > 0$ if $\Sigma \frac{\alpha_i}{\gamma} u_i \succ \Sigma \frac{\beta_i}{\gamma} v_i$ and

$\Sigma \alpha_i u_i + \Sigma - \beta_i v_i < 0$ if $\Sigma \frac{\alpha_i}{\gamma} u_i \prec \Sigma \frac{\beta_i}{\gamma} v_i$.

According to this definition we divided the class S[0] into two parts. In the first part, there are elements greater than or equal to zero, and in the second one, there are elements less than or equal to zero.

Theorem 2.1. Let a,b be elements of S[0], ϑ being an element of FRN. Then

(1) $(a > 0\ \&\ b > 0) \Longrightarrow (a + b > 0)$,

$(a < 0\ \&\ b < 0) \Longrightarrow (a + b < 0)$.

(2) $\vartheta > 0 \Rightarrow (a>0 \equiv \vartheta \cdot a>0)$,

$\vartheta < 0 \Rightarrow (a>0 \equiv \vartheta \cdot a<0)$.

Definition. Function F is a valuation of S if $dom(F) \subseteq S$, $rng(F) \subseteq RN$ and if $\sum \alpha_i u_i$ is an element of S[0] then

$$\sum \alpha_i u_i > 0 \equiv \sum \alpha_i F(u_i) > 0 \text{ and}$$

$$\sum \alpha_i u_i < 0 \equiv \sum \alpha_i F(u_i) < 0.$$

A valuation F of S is a total valuation if dom(F) = S.

Theorem 2.2. Let $\{F_n, n \in FN\}$ be a sequence of valuations of S such that $F_n \subseteq F_{n+1}$ for all n. Then $F = \bigcup \{F_n, n \in FN\}$ is a function and it is a valuation.

Theorem 2.3. Let F be a valuation of S and let F be at most countable. Let w be an element of S which is not an element of dom(F). Then there is a $z \in RN$ such that $F \cup \{\langle w,z\rangle\}$ is a valuation.

Proof. Put $Y = \{\sum \alpha_i u_i;\ (\forall i)(u_i \in dom(F)\ \&\ \alpha_i \in RN)\ \&\ \&\ \sum \alpha_i = -1\}$. If $\sum \alpha_i u_i$ is an element of Y then $w + \sum \alpha_i u_i$ is an element of S[0]. Put

$X_{\sum \alpha_i u_i} = \{x \in RN;\ x \leq \sum - \alpha_i F(u_i)\}$, for $w + \sum \alpha_i u_i \leq 0$,

$X_{\sum \alpha_i u_i} = \{x \in RN;\ x \geq \sum - \alpha_i F(u_i)\}$, for $w + \sum \alpha_i u_i \geq 0$.

Evidently $X_{\sum \alpha_i u_i}$ are intervals on one side unbounded. There are at most countably many of them.

We claim that $Z = \bigcap \{X_{\sum \alpha_i u_i},\ \sum \alpha_i u_i \in Y\}$ is not empty. Classes $X_{\sum \alpha_i u_i}$ are set-theoretically definable. Thus if Z is empty, the intersection of finitely many of these intervals is empty. And by the definition of $X_{\sum\alpha_i u_i}$ we see that already the intersection of two of these intervals is empty. Say $\{x;\ x \leq \sum - \alpha_i F(u_i)\} \cap \{x;\ x \geq \sum - \beta_i F(v_i)\} = \emptyset$. Hence $w + \sum \alpha_i u_i \leq 0$ and $w + \sum \beta_i v_i \geq 0$. By the theorem 2.1, $-w - \sum \beta_i v_i \leq 0$ and $\sum \alpha_i u_i - \sum \beta_i v_i \leq 0$. Consequently

$$\sum \alpha_i F(u_i) - \sum \beta_i F(v_i) \leq 0, \quad \sum - \beta_i F(v_i) \leq \sum - \alpha_i F(u_i)$$

and it is a contradiction.

Now, we take $z \in Z$. The function $F \cup \{\langle w,z \rangle\}$ is the desired function.

Theorem 2.4. For any S, there is a total valuation of S.

Proof. If S is finite, we use several times the last theorem. Otherwise either $\{u_n, n \in FN\}$ or $\{u_\alpha, \alpha \in \Omega\}$ is an enumeration of S depending on S being countable or uncountable.

We choose $F(u_1) \in RN$ arbitrarily. By the induction, we prolongate the function by 2.2 and 2.3,

Theorem 2.5. Let k, q be elements of RN, k be a positive number, F is a valuation of S iff $k \cdot F + q$ is a valuation of S.

Theorem 2.6. For any class S there is a valuation G such that

(1) $rng(G) \subseteq N$,

(2) for $\sum \alpha_i u_i \in S[1]$ holds $\sum \alpha_i G(u_i) \in N$.

Remark. The theorem has the following economic interpretation. For any class S there exists a monetary unit so small that both values of elements of S and all values of their combinations can be expressed in this unit.

Proof. Let F be any valuation of S. By [V], it holds: there is a Y and a set d such that F"S is similar to Y and $Y \subseteq d$. As $x \subseteq RN$ is the set-formula and as $(\forall x)(x \subseteq F"S \,\&\, Fin(x) \rightarrow x \subseteq RN)$

we see that $d \subseteq RN$. As $F''S$ is similar to Y, there is an endomorphism H such that $H''(F''S) = Y$.

Each element of d can be written as $\frac{x}{y}$ where x and y are elements of N which are prime to each other. The set $\{y; (\exists z \in d)(z = \frac{x}{y})\}$ is linearly ordered by $<$. Thus it has the greatest element m. If $m \in N \setminus FN$, put $k = m!$, if $m \in FN$, put $k = n!$, where n is any element of $N \setminus FN$.

Each element $H \circ F(u)$ of d can now be written as $\frac{d_u}{k}$, where $d_u \in ZN$. If d contains also negative numbers, put $q = -\min\{d_u, \frac{d_u}{k} \in d\}$, otherwise put $q = 0$.

We define the function G as follows. Let $u \in S$. Supposing $H \circ F(u) = \frac{d_u}{k}$ put $G(u) = d_u + q$.

$H \circ F$ is a valuation, as H is an endomorphism and as the property "to be a valuation" is set-theoretically definable with parameter S. G is also a valuation, as $G = k \cdot (H \circ F) + q$.

It is easy to prove that $\sum \alpha_i G(u_i) \in N$ for $\sum \alpha_i u_i \in S[1]$.

§ 3. Partial Preference Relation

Definition. Let $\succ$ be a preference relation on S. Let $\succsim \subseteq \succ$. Then $\succsim$ is called a partial preference relation on S.

Let $n \in FN$. A partial preference relation of degree n is defined by $\succ_n = \succ \restriction \{\langle \sum_{i=1}^{k} \alpha_i u_i ; \sum_{i=1}^{m} \beta_i v_i \rangle \in S[1] \times S[1];\ 1 \leq m, 1 \leq k,\ m+k \leq n\}$.

The structure of S with a preference relation can be rather rich and complicated. A question offers: does not a partial preference relation, say of a certain degree n, suffice for finding a total valuation? And if not, what is determined by a given partial preference relation?

Definition. $\Sigma(S) = \{\succ, \succ$ is a preference relation on $S\}$.

Definition. Let $\succsim$ be a partial preference relation on S.

(1) Let a, b be elements of S[1]. A relation of a, b is determined by $\succsim$ if $(\forall \succ \in \Sigma(S))(\forall \succ' \in \Sigma(S))$ $((\succsim \subseteq \succ \,\&\, \succsim \subseteq \succ') \Rightarrow \succ \restriction \{\langle a,b\rangle\} = \succ' \restriction \langle a,b\rangle\})$.

(2) Let $a = \Sigma \alpha_i u_i + \Sigma - \beta_i v_i \in S[0]$, $\Sigma \alpha_i = \Sigma \beta_i = \gamma > 0$. The a is determined by $\succsim$ if the relation of $\Sigma \frac{\alpha_i}{\gamma} u_i, \Sigma \frac{\beta_i}{\gamma} v_i$ is determined by $\succsim$.

(3) $\succsim$ determines a preference relation if every $a \in S[0]$ is determined by $\succsim$.

Example 1. Let α be a probability coefficient. Let $S = \{u,w,v\}$. Define $\succ_2$ by: $u \succ_2 w \succ_2 v$. Obviously $\succ_2$ does not determine a relation of w and $\alpha u + (1 - \alpha)v$.

Example 2. Let α be a probability coefficient. Let $S = \{u,v,w,z\}$. Define $\succ_3$ by $u \succ_3 v \succ_3 w \succ_3 z$; $v \succ_3 (1 - \frac{1}{n})u + \frac{1}{n} w$ and $v \succ_3 (1 - \frac{1}{n})u + \frac{1}{n} z$ and $\frac{1}{n} u + (1 - \frac{1}{n})z \succ_3 w$ and $\frac{1}{n} v + (1 - \frac{1}{n}) z \succ_3 w$ for every $n \in FN$.

$\succ_3$ does not determine the relation of $\alpha u + (1 - \alpha)z$, $\alpha v + (1 - \alpha)w$, elements of S[1], i.e. $\succ_3$ does not determine $\alpha u + (1 - \alpha)z - \alpha v - (1 - \alpha)w \in S[0]$.

We have proved that for each S with a preference relation $\succ$ there is its valuation , i.e. its embedding F into RN, such that $\langle S[1], \succ\rangle$ is isomorphic to $\langle (F''S)[1], >\rangle$.
Thus w.l.o.g. we consider S to be a subclass of RN.

Definition. Let x, y be elements of RN .
$x \lesssim y$ (x is less in order than y) if

(1) $0 \leq x < y \;\&\; \frac{x}{y} \doteq 0$, or

(2) $x < y \leq 0 \;\&\; \frac{y}{x} \doteq 0$, or

(3) $x < 0 < y$.

$x \overset{\sigma}{=} y$ (x <u>and</u> y <u>are equal in order)</u> if $\neg(x \overset{\sigma}{<} y) \;\&\; \neg(y \overset{\sigma}{<} x)$.

<u>Remark.</u> $x \overset{\sigma}{\leq} y$ denotes $x \overset{\sigma}{<} y \vee y \overset{\sigma}{=} x$

<u>Example 3.</u> In the example 2, there is:

(1) $u-v \overset{\sigma}{<} u-w$, $u-v \overset{\sigma}{<} u-z$, $w-z \overset{\sigma}{<} u-z$, $w-z \overset{\sigma}{<} u-w$;

(2) $\alpha u + (1-\alpha) z = \alpha v + (1-\alpha) w$ iff $u-v = \frac{1-\alpha}{\alpha}(w-z)$,

$\alpha u + (1-\alpha) z > \alpha v + (1-\alpha) w$ iff $u-v > \frac{1-\alpha}{\alpha}(w-z)$.

<u>Lemma 3.1.</u> Let x, y be elemen ts of RN.

(1) $x \overset{\sigma}{<} y$ implies $x < y$.

(2) $x \overset{\sigma}{=} 1$ iff $x \in BRN \setminus Mon(0)$.

(3) If $x \overset{\sigma}{=} 1$ then $x \cdot y \overset{\sigma}{=} y$.

(4) If the both x and y are positive then $x+y \overset{\sigma}{=} \max\{x,y\}$.

<u>Theorem 3.2.</u> Let $u_1, \ldots, u_n \in S$. Then $|\Sigma \alpha_i u_i| \overset{\sigma}{\leq}$ $\overset{\sigma}{\leq} \max\{|u_1|, \ldots, |u_n|\}$ for all $\Sigma \alpha_i u_i \in S[0]$.

<u>Proof.</u> By 3.1, $|\Sigma \alpha_i u_i| \leq \Sigma |\alpha_i u_i| \overset{\sigma}{\leq} \max\{|\alpha_1 u_1|, \ldots$ $\ldots, |\alpha_n u_n|\} \overset{\sigma}{=} \max\{|u_1|, \ldots, |u_n|\}$.

<u>Lemma 3.3.</u> Let X be a countable subclass of positive rational numbers. Then there is a positive rational number d such that $(\forall x \in X)(d \leq x)$.

<u>Proof.</u> If X has a minimal element m, put d=m. Otherwise, put $H = \{\alpha \in N;\ (\exists x \in X)\ \frac{1}{\alpha+1} \leq x < \frac{1}{\alpha}\}$. Evidently H is countable. Thus there is a $\beta \in N$ such that $(\forall \alpha \in H)(\alpha < \beta)$. Put $d = \frac{1}{\beta}$.

<u>Theorem 3.4.</u> Let $u_1, \ldots, u_n$ be elements of S. Then there is

a $d \in RN$ such that, for all $\sum \alpha_i u_i \in S[0]$, if $\sum \alpha_i u_i \neq 0$ then $|\sum \alpha_i u_i| \geq d$.

Proof. Put $X = \{|\sum \alpha_i u_i|;\ \alpha_1, \ldots, \alpha_n \in FRN,\ \sum \alpha_i = 0\}$ and use 3.3.

Definition. We denote the d from 3.4 by $d(u_1, \ldots, u_n)$.

Theorem 3.5. Let $u_1, \ldots u_n, v_1, \ldots, v_n$ be elements of S such that $(\forall i)(1 \leq i \leq n)\ |u_i - v_i| \lessapprox d(u_1, \ldots, u_n)$. Let $\sum \alpha_i u_i$ be an element of S[0]. Then

$$\sum \alpha_i u_i > 0 \Rightarrow \sum \alpha_i v_i > 0 \text{ and}$$
$$\sum \alpha_i u_i < 0 \Rightarrow \sum \alpha_i v_i < 0.$$

Proof. $\sum \alpha_i v_i = \sum \alpha_i u_i + \sum \alpha_i (v_i - u_i)$ and $|\sum \alpha_i (v_i - u_i)| \lessapprox d(u_1, \ldots, u_n) \lessapprox |\sum \alpha_i u_i|$. By 3.1 we have the desired properties.

Theorem 3. 6. Let $\succ_n$ be a partial preference relation on S. Let $\sum_{i=1}^{n+1} \alpha_i u_i$ be an element of S[0] which is not determined by $\succ_n$. Then no element of S[0] which is a combination of n elements of $\{u_1, \ldots, u_{n+1}\}$ equals zero.

Proof. Let $\{1, \ldots, n+1\} = \{a_1, \ldots, a_n\} \cup \{b\}$ and suppose $0 = \sum_{j=1}^{n} \beta_j u_{a_j}$ is an element of S[0]. Hence $u_{a_1} = -\frac{1}{\beta_1} \sum_{j=2}^{n} \beta_j u_{a_j}$. Since $\sum_{i=1}^{n+1} \alpha_i u_i = \alpha_{a_1} u_{a_1} + \sum_{j=2}^{n} \alpha_{a_j} u_{a_j} + \alpha_b u_b = \sum_{j=2}^{n} (\alpha_{a_j} - \alpha_{a_1} \cdot \frac{\beta_j}{\beta_1}) u_{a_j} + \alpha_b u_b$, we have $\sum_{i=1}^{n+1} \alpha_i u_i$ can be expressed as a combination of n elements. Thus it is determined by $\succ_n$, a contradiction.

Theorem 3.7. For each n there is an $S = \{u_1, \ldots, u_{n+1}\}$, a partial preference relation of a degree n $\succ_n$ on S and an element $\sum_{i=1}^{n+1} \alpha_i u_i$ of S[0] which is not determined by $\succ_n$.

Proof. By induction we construct $\{u_1,\dots,u_{n+1}\} = S$ and $\alpha_1,\dots,\alpha_{n+1}$ elements of FRN, $\sum_{i=1}^{n+1}\alpha_i = 0$, such that $\sum_{i=1}^{n+1}\alpha_i u_i =$ is not determined by $\succ_n$ and in addition $\sum_{i=1}^{n+1}\alpha_i u_i = 0$.

For n=3 see the example 2.

Suppose the assertion holds for n-1, we prove for n. Let v,w be such that $v \succ u_1 \succ w$ and $v-u_1 \lessapprox d(u_1,\dots,u_n)$, $u_1-w \lessapprox d(u_1,\dots,u_n)$.

We prove that $\succ_n \upharpoonright \langle v,w,u_2,\dots,u_n\rangle = \succ_n \upharpoonright \langle u_1,u_1,u_2,\dots,u_n\rangle$ for all such v, w. Let $a = \beta_1 v + \beta_2 w + \dots + \beta_n u_n$ be an element of $\{v,w,u_2,\dots,u_n\}[0]$ which is a combination of n elements. Denote $b = (\beta_1+\beta_2)u_1 + \dots + \beta_n u_n$, $c = \beta_1(v-u_1) + \beta_2(w-u_1)$. Thus $a = b + c$. We have $|c| = |\beta_1(v-u_1) + \beta_2(w-u_1)| \lessapprox d(u_1,\dots,u_n) \lessapprox d(u_1,u_1,\dots,u_n) \leq |b|$. By 3.5, we have $a>0 \Longrightarrow b>0$, $a<0 \Longrightarrow b<0$. Never $a = 0$. Indeed if $a = 0$ then either $0 \neq b = -c$, a contradiction with $|c| \lessapprox |b|$ or $0 = b = c$, a contradiction with the induction premise, by 3.6.

Let γ be a probability coefficient. Put $d = \alpha_1(\gamma v + (1-\gamma)w) + \sum_{i=2}^{n}\alpha_i u_i$. For example, let $\alpha_1 > 0$. Since $\sum_{i=2}^{n}\alpha_i u_i = -\alpha_1 u_1$, we have $d>0$ iff $\gamma v + (1-\gamma)w > u_1$ and $d<0$ iff $\gamma v + (1-\gamma)w < u_1$. Thus d is not determined by $\succ_n$.

We take v, w such that d=0. Then $S = \{v,w,u_2,\dots,u_n\}$, $d \in S[0]$ have all desired properties.

Corollary 3.8. There is no n such that every partial preference relation of a degree n determines a preference relation.

§ 4. A comparison of the Theory of Utility in the Cantor Set Theory and in the Alternative Set Theory

The classical theory of utility differs from ours in the following three points. See [F].

(1) A preference relation is not given on the whole S[1] but

only on its subclass on $S \cup \{\langle w, \alpha u + (1-\alpha)v\rangle ; u,w,v \in S,\ u \succ w \succ v,\ \alpha \in E_1,\ 0 \leq \alpha \leq 1\}$. I.e. only $\succ_3$ is given.

(2) So called Archimedean axiom is assumed: $(\forall u,w,v \in S)(u \succ w \succ v)(\exists \alpha \in E_1)(w = \alpha u + (1-\alpha)v)$.

(3) A total valuation exists iff S contains a countable dense.

Commentary.

ad (1). Assuming (2), $\succ_3$ determines a preference relation. Indeed, if $\sum \alpha_i u_i \in S[0]$, $u_1 \succ \ldots \succ u_n$, then $(\forall i)(\exists \beta_i \in E_1)$ $(u_i = \beta_i u_1 + (1-\beta_i)u_n)$. Thus $\sum \alpha_i u_i = \sum \alpha_i \beta_i (u_1 - u_n) > 0$ iff $\sum \alpha_i \beta_i > 0$. Hence (1) is sufficient.

ad (2). Von Neumann and Morgenstern wrote about the Archimedean axiom: It is probably desirable to require it, since its abandonment would be tantamount to introducing infinity utility differences [NM]. Infinity differences are one of the basic notion s in the alternative set theory.

In some situation the Archimedean axiom seems to be restrictive . Let us return to the Introduction to the man who chooses between elements of S[1]. Let S contain a TV(t), a similar TV with a small hash (h), a pencil (p). Let us imagine the man prefers t to h and h to p. We can also imagine he prefers h to a game in which he gains either t or p, though the probability of gaining only p would be the smallest possible. I.e. if α is a probability coefficient ($\alpha \neq 0,1$) then

$$t \succ h \succ \alpha t + (1-\alpha)p \succ p.$$

By this way we can describe incomparability of some values. Here $t - h \not\lesssim t - p$.

ad (3). In the alternative set theory there is a total valuation for any S.

There is another point very important for our conception. Probabilities which appear in our games are finite rational numbers. This means that "the smallest possible" stands for very small but perceptible, before the horizon of our discernibility. Also elements of S[1] represent games for n elements of S, where n is a finite natural number, i.e.easy to survey, before the horizon.

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