Charles W. Swartz

The Farkas lemma of Glover

Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 4, 651--654

Persistent URL: http://dml.cz/dmlcz/106403

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

26,4 (1985)

THE FARKAS LEMMA OF GLOVER

Charles SWARTZ

Abstract: We use standard functional analysis techniques to establish a result of Glover which he employs to obtain a nonlinear version of the classical Farkas Lemma.

Key words: Farkas Lemma, convex functions, subgradients, Krein-Smulian Theorem.

Classification: 90C25

In this brief note we present a proof of a theorem which has been used in optimization to establish a nonlinear version of the Farkas Lemma ([1], Lemma 3) and to establish Kuhn-Tucker Theorems ([3] 2.1,[4] 2.3, 2.4). The proof given by Glover in [1] uses machinery from set-valued mappings; we present a proof below which only employs standard topics from functional analysis, namely, the Krein-Smulian Theorem.

Let X and Y be locally convex spaces with S a closed convex cone in Y. Let $g:X \longrightarrow Y$ be positive homogeneous, S-convex and such that $s' \circ g = s'g$ is lower semi-continuous for each $s' \in S^*$, where $S^* = \{s' \in Y' : (s', s) \geq 0 \ \forall s \in S\}$ is the dual cone of S. As usual we write $\partial f(0)$ for the subgradient of a convex function $f:X \longrightarrow \mathbb{R}$ at 0 ([7]). Glover shows that if $A = \bigcup_{s' \in S^*} \partial(s'g)(0)$, then $-(g^{-1}(-S))^* = \bar{A}$, where the closure is in the weak* topology of Y' ([1] Lemma 1), and then uses this result to establish a general nonlinear Farkas Lemma ([1] Theorem 1). Glover's Farkas Lemma contains the linear Farkas Lemmas of Zalinescu ([6]) and Schirotzek ([5]). In order to obtain a sharper form of the Farkas Lemma, Glover gives sufficient conditions which guarantee that the set A above is weak* closed ([1] Lemma 3). We state and prove a version of this result which uses only standard functional analysis techniques whereas Glover's proof uses results of Robinson on set-valued mappings. Our proof also covers the case when X is only metrizable and not necessarily a normed space, but we must assume that the range space is barrelled although not necessarily normed.

Theorem 1. Let X be complete, metrizable and let Y be barrelled and suppose that $g(X) + S = Y$. Then $-(g^{-1}(-S))^* = A$ so in particular A is weak* closed.

Proof: By Lemma 1 of [1] it suffices to show A is weak* closed and by the Krein-Smulian Theorem ([2], 3.10.2) it suffices to show that $A \cap U^o$ is weak* closed for each closed, absolutely convex neighborhood of 0, U, in X. Let (x'_ν) be a net in $A \cap U^o$ such that (x'_ν) is weak* convergent to x'. It suffices to show that $x' \in A \cap U^o$. Let p be the Minkowski functional of U. Choose $s'_\nu \in S^*$ such that $x'_\nu \in \partial(s'_\nu g)(0)$ and let $y \in Y$. By hypothesis, $y = g(x) + s$ for some $x \in X$, $s \in S$. Then

(1) $\langle s'_\nu, y\rangle = \langle s'_\nu, g(x)\rangle + \langle s'_\nu, s\rangle \geq \langle x'_\nu, x\rangle \geq -p(x)$.

Also $-y = g(\bar{x}) + \bar{s}$ for some $\bar{x} \in X$, $\bar{s} \in S$ so $\langle s'_\nu, -y\rangle = \langle s'_\nu, g(\bar{x})\rangle + \langle s'_\nu, \bar{s}\rangle \geq \langle x'_\nu, \bar{x}\rangle \geq -p(\bar{x})$ and

(2) $\langle s'_\nu, y\rangle \leq p(\bar{x})$.

Thus, if $r = \max\{p(x), p(\bar{x})\}$, (1) and (2) imply that $|\langle s'_\nu, y\rangle| \leq r$. Hence, $\{s'_\nu\}$ is weak* bounded and, therefore, relatively weak* compact by the barrelledness ([2], 3.6.2).

Thus, $\{s'_\nu\}$ has a subnet, which we continue to denote by $\{s'_\nu\}$, which is weak* convergent to some $y' \in Y'$. Since $\langle s'_\nu, s\rangle \geq 0$ for $s \in S$, $\langle y', s\rangle \geq 0$ so that $y' \in S^*$. For $x \in X$, we have $\langle y', g(x)\rangle = \lim \langle s'_\nu, g(x)\rangle \geq \lim \langle x'_\nu, x\rangle = \langle x', x\rangle$ so $x' \in \partial(y'g)(0)$ and $x' \in A \cap U^0$ since U^0 is weak* closed.

In Glover's version he assumes that X is a Banach space and Y is a normed space, but there is no completeness assumption on Y.

If $f: X \longrightarrow \mathbb{R}$ is lower semicontinuous and sublinear and $x' \in X'$, then under the assumptions of Theorem 1 Glover's Farkas Lemma ([1], Theorem 1) is: $x' \in \partial f(0) \dot{-} A$ iff $-g(x) \in S$ implies $f(x) \geq \langle x', x\rangle$. For the case when f and g are linear, this yields the Farkas Lemmas of Zalinescu ([6]) and Schirotzek ([5]).

References

[1] B. GLOVER: A generalized Farkas Lemma with applications to quasi-differentiable programming, Zeit. Oper. Res. 26(1982), 125-141.

[2] J. HORVATH: Topological Vector Spaces and Distributions, Addison-Wesley, Reading, Mass., 1966.

[3] S. KURCYUSZ: On the existence and nonexistence of Lagrange multipliers in Banach spaces, J. Opt. Theory Appl. 20(1976), 81-110.

[4] J-P. PENOT: On the existence of Lagrange multipliers in nonlinear programming in Banach spaces, Lecture Notes in Control and Information Sciences, 30,1980, 89-104.

[5] W. SCHIROTZEK: On Farkas type theorems, Comment. Math.Univ. Carolinae 22(1981), 1-14.

[6] C. ZALINESCU: A generalization of the Farkas Lemma and applications to convex programming, J. Math. Anal. Appl. 66(1978), 651-678.

[7] C. ZALINESCU: On an abstract control problem, Numer.Func. Anal. Opt. 2(1980), 531-542.

Department of Mathematical Sciences, New Mexico State University
Las Cruces, NM 88003, U.S.A.

(Oblatum 15.4. 1985)