Eva Butkovičová

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SHORT BRANCHES IN THE RUDIN-FROLÍK ORDER

Eva BUTKOVIČOVÁ

__Abstract__: We construct in the Rudin-Frolík order an unbounded chain order-isomorphic to ω_1 .

__Key words__: type of ultrafilters, Rudin-Frolík order

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0. Introduction

The Rudin-Frolík order of types of ultrafilters in $\beta\omega$ has the following properties:

(1) each type of ultrafilters has at most 2^{ω} predecessors - [F1] ,

(2) the cardinality of each branch is at least 2^{ω} .

Hence, the cardinality of a branch in the Rudin-Frolík order can only be 2^{ω} or $(2^{\omega})^{+}$. It is shown in [B1] that there exists a chain order-isomorphic to $(2^{\omega})^{+}$.

The aim of this paper is to prove the following result which solves the problem of the existence of a branch having smaller cardinality.

Theorem: In the Rudin-Frolík order there exists an unbounded chain order-isomorphic to ω_1.

By (1) and (2) the branch containing this chain has cardinality 2^{ω}.

This result was announced in [B2].

1. Preliminaries

We shall need the standard set theoretical notation and terminology. By ultrafilter we mean an ultrafilter on ω.

The type of an ultrafilter p is
$\tau(p) = \{q ; \exists h$ homeomorphism from $\beta\omega$ onto $\beta\omega$ such that $h(p) = q\}$.

Let p, q be ultrafilters on ω. Then $\tau(p) \leq \tau(q)$ in the Rudin-Frolík order iff there exists a countable discrete set $X = \{x_n ; n \in \omega\}$ of ultrafilters such that $q = \sum(X, p)$ where
$\sum(X, p) = \{A ; \{n; A \in x_n\} \in p\}$. (Defined in [F1].)

Proposition 1.1: [F2]. For each ultrafilter p the set $\{\tau(q) : \tau(q) < \tau(p)\}$ is linearly ordered.

Proposition 1.2: Let $X = \{x_n; n \in \omega\}$, $Y = \{y_n; n \in \omega\}$ be discrete sets of ultrafilters and $p \in \beta\omega$. Then
$\sum(X, p) < \sum(Y, p)$ iff $\{n; x_n < y_n\} \in p$.

2. Proof of the Theorem

We want to construct an unbounded chain of types of ultrafilters $\{\tau(p_\alpha); \alpha \in \omega_1\}$ such that $\tau(p_\alpha) < \tau(p_\beta)$ whenever $\alpha < \beta$.

To do this we shall construct sets $\{X_\alpha ; \alpha \in \omega_1\}$

satisfying the following conditions:

(i) $X_\alpha = \{ x_n^\alpha \, ; \, n \in \omega \}$ is a discrete set of ultrafilters of mutually incomparable types,

(ii) $\{n \, ; \, \tau(x_n^\beta) < \tau(x_n^\gamma)\}$ is cofinite for each $\beta < \gamma$,

(iii) $|\{ \beta \, ; \, \tau(x_n^\beta) < \tau(x_n^\alpha)\}| < \omega$ for each $\alpha \in \omega_1$, $n \in \omega$,

(iv) if $\beta < \alpha$ and $\tau(x_n^\beta) \not< \tau(x_n^\alpha)$ then $\tau(x_n^\beta) \not> \tau(x_n^\alpha)$.

Let $X_o = \{x_n^o \, ; \, n \in \omega\}$ be an arbitrary discrete set of minimal incomparable ultrafilters. Suppose that X_β is defined for all $\beta < \alpha$.

Let $\alpha = \gamma + 1$. Define a discrete set X_α in such a way that for each $n \in \omega$ $\tau(x_n^\alpha)$ is a successor of $\tau(x_n^\gamma)$. It is trivial that all four conditions are fulfilled.

Let α be a limit ordinal. Then there exists a sequence $\{\alpha_k \, ; \, k \in \omega\}$ of ordinals converging to α . Let us define $A_o = \omega$ and

$A_k = \{\ell \in A_{k-1} \, ; \, \tau(x_\ell^{\alpha_k}) > \tau(x_\ell^{\alpha_{k-1}})\} - [0, k]$ for each $k > 0$.

It is evident that $\bigcap_{k \in \omega} A_k = \emptyset$.

Put $Z_k = A_k - A_{k+1}$. The set Z_k is finite by inductive assumption. For $n \in Z_k$ define x_n^α in such a way that X_α is a discrete set and $\tau(x_n^\alpha)$ is a successor of $\tau(x_n^{\alpha_k})$ incomparable with all successors of $\tau(x_n^{\alpha_k})$ which were choosen already.

Again, all four conditions are trivially fulfilled.

Let p be an arbitrary nontrivial ultrafilter. Define $p_\alpha = \Sigma(X_\alpha , p)$. We prove that $\{\tau(p_\alpha) ; \alpha \in \omega_1\}$ is the required chain.

Condition (ii) and Proposition 1.2 yield that $\tau(p_\beta) < \tau(p_\gamma)$

if $\beta < \gamma$.

Suppose now that there exists an ultrafilter y such that $\tau(y) > \tau(p_\alpha)$ for each $\alpha \in \omega_1$. Hence, there exists a countable discrete set $Y = \{ y_n ; n \in \omega \}$ such that $y = \sum(Y, p)$.

By Proposition 1.2 the set $\{ k ; \tau(y_k) > \tau(x_k^\alpha) \}$ belongs to p for each $\alpha \in \omega_1$. The set Y is countable therefore there exists an $\ell \in \omega$ such that $|\{ \beta ; \tau(y_\ell) > \tau(x_\ell^\beta) \}| = \omega_1$.

By conditions (i) and (iv) all types of the ultrafilters from the sets $X_\alpha ; \alpha \in \omega_1$ are distinct. Hence the set $\{ \tau(x_\ell^\beta) ; \tau(x_\ell^\beta) < \tau(y_\ell) \}$ has cardinality ω_1 and by Proposition 1.1 it is linearly ordered.

Now we have a linearly ordered set of cardinality ω_1 and by the condition (iii) each point from this set has finitely many predecessors. This is a contradiction.

We can take in the construction immediate successor instead of successor and p a minimal ultrafilter. Then we get a chain such that each point (except p) in the branch containing this chain has an immediate predecessor.

R e f e r e n c e s

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Matematický ústav SAV
Jesenná 5, 041 54 Košice
Czechoslovakia

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