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A CONSTRUCTIVE PROOF OF THE TYCHONOFF'S THEOREM FOR LOCALES

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Abstract: A choice- and replacement-free proof of the Tychonoff's theorem is given for compact locales.

Key words: Locales, compact locales, Tychonoff's theorem.

Classification: 54D30, 54H99

The Tychonoff's theorem ([12]) stating that a product of compact spaces is compact is well known to be equivalent to the axiom of choice (see [10]). A surprising result was obtained by P.T. Johnstone in [8]: if we consider compact locales (i.e., spaces represented as lattices of "open sets" - with points disregarded and, indeed, often not present in any form), the analogon of the Tychonoff's theorem can be proved without the axiom of choice. This is particularly interesting in connection with the fact that compact locales are always spatial, i.e. open-sets lattices of classical topological spaces ([2]; thus, the use of AC is localized in the formation of points, not in the preservation of the compactness property).

The proof in [8] contains a non-constructive element, namely the axiom of replacement. P.T. Johnstone formulated the problem whether one can get rid of this, too (for the special case of the locally compact locales he presented a positive answer himself). In this article, this problem is solved in the affirmative in full generality. The procedure is based on a new description of the product of locales, considerably more constructive as compared with the usually used ones ([5],[8]).

1. Locales. The basic theory of locales has been developed by Bénabou [1], Dowker and Strauss [3, 4, 5], Isbell [6] and Simmons [11]. There are considerable differences in the terminology; we follow that of Johnstone [8]. A frame is a complete lattice A in which the infinite distribution law

$$a \wedge (\bigvee S) = \bigvee \{ a \wedge s \mid s \in S \}$$

holds for all $a \in A$, $S \subseteq A$. We shall denote the maximal resp. minimal element of A by 1 resp. 0. A frame homomorphism $A \longrightarrow B$ is a map preserving finite meets and arbitrary joins (i.e., in particular, the elements 0, 1). Thus, we have a category Frm of frames. If X is a topological space, the lattice $\Omega(X)$ of its open sets is a frame. If $f: X \longrightarrow Y$ is a continuous map, then $f^{-1}: \Omega(Y) \longrightarrow \Omega(X)$ is a frame homomorphism. Thus Ω is a contravariant functor from the category Top of topological spaces to Frm.

Following Isbell [6] and Johnstone [8] we shall write Loc for the opposite category $\mathrm{Frm}^{\mathrm{Op}}$, and call its objects locales. This dual terminology enables us to make $\Omega: \mathrm{Top} \longrightarrow \mathrm{Loc}$ a covariant functor and, in consequence, to generalize familiar concepts from topology to Loc (see [7],[8]).

2. Products of locales. Products in the category Loc (sums in Frm) were defined by Dowker-Strauss [5] and Johnstone [8]. Their description is elegant, but rather non-constructive. It does not give any explicit formula for the join operation in the sum $\coprod_{j \in J} X_j$ of frames X_j. Johnstone [7] suggests to

construct the sum of X_j ($j \in J$) as a free frame over the cartesian product $\prod_{j \in J} X_j$ of the sets X_j, factorized through a congruence generated by certain relations. (In the case of an infinite J, it is of an advantage to exclude from $\prod_{j \in J} X_j$ those $(a_j)_{j \in J}$ in which we have $a_j < 1$ for infinitely many j.) This shows an analogy between frames and commutative rings (see [9]). However, frames, being ∞-ary algebras, turn out to be in this respect much more complex. In fact, the congruence generated by the obvious relations is rather obscure.

In this section we give a quite explicit description of the congruence generated by the relations [7], which enables us to describe the structure of $\bigvee_{j \in J} X_j$, explicit formulas for finite meets and arbitrary joins included.

Let J be a set. We call a J-connector a system $(M, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}$ $(j \in J)$, $M_1, M_2)$, where $M_1, M_2 \subseteq M$, $\mathcal{R}_j^{\uparrow} \subset 2^M \times M$, $\mathcal{R}_j^{\downarrow} \subset M \times 2^M$ for $j \in J$ such that the following condition holds:

Let $K \subseteq M$. Whenever

$$
(C) \begin{cases} (1) & (M_1 \subseteq K) \,\&\, [(N \mathcal{R}_j^{\uparrow} x) \,\&\, (N \subseteq K) \Longrightarrow \\ & \qquad \Longrightarrow x \in K] \,\&\, [(x \mathcal{R}_j^{\uparrow} N) \,\&\, (x \in K) \Longrightarrow N \subseteq K] \\ \text{or} & \\ (2) & (M_2 \subseteq K) \,\&\, [(N \mathcal{R}_j^{\uparrow} x) \,\&\, (x \in K) \Longrightarrow \\ & \qquad \Longrightarrow N \subseteq K] \,\&\, [(x \mathcal{R}_j^{\downarrow} N) \,\&\, (N \subseteq K) \Longrightarrow x \in K] \end{cases}
$$

holds, it is $K = M$.

Now let X_j $(j \in J)$ be a system of frames. Denote by B the cartesian product $\prod_{j \in J} X_j$. There is a natural ordering " $\precsim$ " of B, making $\prod_{j \in J} X_j$ a Frm-product of X_j (see [5]). Let $B' \subseteq B$ be the subset of all $x = \prod_{j \in J} a_x^j \in B$ such that we have $a_x^j < 1$ for at most finitely many $j \in J$. It is easy to see that B' is a sublattice of B, preserving finite meets and non-empty joins, but it is not a locale: There is no minimal element in B'. Denote by Z the lattice of all subsets of B' ordered by inclusion.

We call $m_1, m_2 \in Z$ strongly equivalent ($m_1 \sim_s m_2$), if there exists an $m \in Z$ and a J-connector $(m, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}, m_1, m_2)$ (in the sequel called simply the connector) such that it holds

$$(3) \quad (x\,\mathcal{R}_j^{\downarrow}\,m' \text{ or } m'\,\mathcal{R}_j^{\uparrow}\,x) \Rightarrow (a_x^j = \bigvee_{y \in m'} a_y^j) \,\& \\ \&(a_x^k = a_y^k \text{ for } k \neq j,\ y \in m') \,\&\, (m' \neq \emptyset).$$

We will call a <u>kernel</u> of $m \in Z$ the set

$$s(m) = \{x \in m \mid (\forall j \in J)\ a_x^j > 0\}.$$

We set $u \sim v \equiv_{df} s(u) \sim_s s(v)$. The element u is called <u>standard</u>, if $u = s(u)$.

2.1. <u>Observation</u>: " $\sim$ " is an equivalence relation, containing " $\sim_s$ ".

<u>Proof</u>: It suffices to show that $u \sim_s v \Rightarrow s(u) \sim_s s(v)$. Let $(m, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}, u, v)$ be a connector. Denoting by $\bar{\mathcal{R}}_j^{\uparrow}$, $\bar{\mathcal{R}}_j^{\downarrow}$ the restrictions of $\mathcal{R}_j^{\uparrow}$ ($\mathcal{R}_j^{\downarrow}$) to $s(m) \times 2^{s(m)}$, $2^{s(m)} \times s(m)$, respectively, we obtain a connector $(s(m), \bar{\mathcal{R}}_j^{\uparrow}, \bar{\mathcal{R}}_j^{\downarrow}, s(u), s(v))$.

Denote by $[m]$ the class of $m \in Z$ in $(Z/\sim)$.

2.2. <u>Further observations</u>: 1. Assume $x, y, z \in Z$, $x \subseteq y$, $x \sim z$. Then there exists a $t \in Z$ such that $z \subseteq t$, $y \sim t$. Thus, we can define a canonical ordering on $(Z/\sim)$ by the formula $[x] \leq [y] \equiv_{df} (\exists z \in Z)(z \sim y \,\&\, x \subseteq z)$.

<u>Proof</u>: Let $(m, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}, s(x), s(z))$ be a connector. Putting $t = (y \setminus x) \cup z$, we obtain an obvious connector $(m \cup s(t), \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}, s(y), s(t))$. $\square$

2. If $u \subseteq v \subseteq w$ and $u \sim w$, then $u \sim v$. Hence, " $\leq$ " is a partial ordering.

<u>Proof</u>: It suffices to show that $v \sim w$. But if $(m, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow},$

$s(u), s(w))$ is a connector, then $(m, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}, s(v), s(w))$ is a connector, as well. □

3. Let $u, v \in Z$, $u \subseteq v$. Then $(\forall y \in v)(\exists x \in u)(x \prec y) \Rightarrow u \sim v$.

Proof: For $u \in Z$ put $d(u) = \{x \in B' \mid (\exists y \in u)\ x \prec y\}$. Since evidently $(\forall y \in v)(\exists x \in u)(x \prec y) \,\&\, u \subseteq v \Rightarrow d(u) = d(v)$, it suffices to show that $u \sim_s d(u)$ for $u \in Z$. Let $\mathcal{R}_j^{\uparrow}$, $\mathcal{R}_j^{\downarrow}$ be maximal relations on $2^{d(u)} \times d(u)$, $d(u) \times 2^{d(u)}$, satisfying (3). (The condition (3) is obviously preserved by the union of relations.) From the fact that for $x \in B'$ there are only finitely many j with $a_x^j < 1$, we easily obtain that $(d(u), \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}, u, d(u))$ is a connector. □

4. For any $u_i \in Z$ we have $[\bigvee_{i \in I} u_i] = \bigvee_{i \in I} [u_i]$.

Proof: The union $t(\alpha)$ of all elements of a given class $\alpha \in (Z/\sim)$ belongs to α, since a union of connectors (in the obvious meaning) is a connector. Moreover, the mapping $t: (Z/\sim) \rightarrow Z$ preserves ordering and for arbitrary $z \in Z$, $\alpha \in (Z/\sim)$ it holds $z \subseteq t(\alpha) \equiv [z] \leq \alpha$. Thus, " [ ] " is a left adjoint to t so that it preserves joins. □

5. Denote by $\wedge_{B'}$ the meet operation in B'. For $u, v \in Z$ let $u \barwedge v = \{x \wedge_{B'} y \mid x \in u, y \in v\}$. Then $[u \barwedge v]$ depends only on $[u]$, $[v]$.

Proof: Assume that $(m^{(i)}, \mathcal{R}_j^{\uparrow\,(i)}, \mathcal{R}_j^{\downarrow\,(i)}, u^{(i)}, v^{(i)})$ are connectors, $i = 1,2$. Put $\mathcal{R}_j^{\downarrow} = \{(x \wedge y, m \barwedge \{y\}) \in B' \times 2^{B'} \mid (x \mathcal{R}_j^{\downarrow\,(1)}\, m \,\&\, y \in m^{(2)})$ or $(x \mathcal{R}_j^{\downarrow\,(2)}\, m \,\&\, y \in m^{(1)})\}$. $\mathcal{R}_j^{\uparrow} = \{(m \barwedge \{y\}, x \wedge y) \in 2^{B'} \times B' \mid (m\, \mathcal{R}_j^{\uparrow(1)} \,\&\, y \in m^{(2)})$ or $(m\, \mathcal{R}^{\uparrow(2)} x \,\&\, y \in m^{(1)})\}$. It is easy to see that $(m^{(1)} \barwedge m^{(2)}, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}, u^{(1)} \barwedge u^{(2)}, v^{(1)} \barwedge v^{(2)})$ is a connector. □

6. The operation " $\wedge$ " in $(Z/\sim)$ defined by $[u] \wedge [v] =$

$= [u \bar{\wedge} v]$ is the ordinary meet ($\equiv$ infimum in $\leq$) in $(Z/\sim)$.

Proof: By 2, 4, $(Z/\sim)$ is a complete lattice. Denote by "$\wedge_{(Z/\sim)}$" the true meet in $(Z/\sim)$. By 3, we have

$$(+) \qquad (\forall x \in u)(\exists y \in v)(x \prec y) \Rightarrow [u] \leq [v],$$

and hence trivially $[u] \wedge_{(Z/\sim)} [v] \geq [u] \wedge [v]$. Moreover, $[u] \wedge_{(Z/\sim)} [v] \leq [u],[v]$, by definition. Thus, by 1, there exist $s \sim u$, $t \sim v$ such that for some representative uv of the class $[u] \wedge_{(Z/\sim)} [v]$ it holds $uv \subseteq s$, $uv \subseteq t$. By 5, (+), we have now $[u] \wedge_{(Z/\sim)} [v] \leq [s \bar{\wedge} t] = [u] \wedge [v]$.

7. Given a system $f_j : X_j \to C$ of join-preserving mappings, there exists a unique join-preserving mapping $f: (Z/\sim) \to C$ such that it holds that

$$(4) \qquad f([\{x\}]) = \bigwedge_{j \in J} f_j(a_x^j) \text{ for any } x \in B'.$$

Proof: By 4, the mapping f is uniquely determined by the formula $f([m]) = \bigvee_{x \in m} f([\{x\}])$, and it obviously preserves joins. Our only task is to show that f is correctly defined. Let $(m, \mathfrak{R}_j^{\uparrow}, \mathfrak{R}_j^{\downarrow}, u, v)$ be a connector. We will show that, by our definition, $f([u]) = f([v])$. (This will be enough, since the definition obviously gives $f([u]) = f([s(u)])$.) In fact, since the set $K = \{x \in M \mid f([\{x\}]) \leq f([u])\}$ trivially satisfies the condition (1), it is $K = m$. Thus, $f([v]) \leq f([m]) \leq f([u])$. Analogously, $f([u]) \leq f([v])$. □

2.3. Theorem: The set $(Z/\sim)$ ordered by "$\leq$" is a frame with joins and meets given by the formulas

$$(5) \qquad \bigvee_{i \in I} [u_i] = [\bigcup_{i \in I} u_i]$$
$$[u] \wedge [v] = [\{x \wedge_{B'} y \mid x \in u, y \in v\}]$$

If we define $\iota_j : X_j \to (Z/\sim)$ by $\iota_j(a) = [\{\tau_j(a)\}]$, where

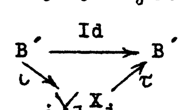
$\tau_j(a) \in B'$ and $a^j_{\tau_j(a)} = a$, $a^k_{\tau_j(a)} = 1$ for $k \neq j$, then ι_j are frame homomorphisms and $(Z/\sim)$ is the sum of X_j with injections ι_j.

Proof: By 2.2.2, 2.2.4, 2.2.6, $(Z/\sim)$ is a complete lattice with joins and meets given by (5). However, (5) trivially implies the distributive law so that $(Z/\sim)$ is a frame. The mappings ι_j are frame homomorphisms by (5). (Note that namely the behaviour of the zero element forces us to set $u \sim v \equiv s(u) \sim_s s(v)$.) Given homomorphisms $f_j : X_j \to C$, there exists (by 2.2.7) a unique join-preserving mapping $f : (Z/\sim) \to C$ satisfying (4). This mapping obviously preserves finite meets. □

2.4. Observation: For arbitrary standard $x, y \in B'$ we have

$$[\{x\}] \le [\{y\}] \equiv x \prec y.$$

Proof: Consider the mapping $\tau_j : X_j \to B'$ defined by Theorem 2.3. Obviously τ_j preserve joins, and thus, by 2.2.7, there exists a unique join-preserving $\tau : \bigvee_{j \in J} X_j \to B'$ satisfying (4). Since B is the product of X_j and B' is a sublattice of B, we have a canonical join- and finite meet-preserving map $\iota : B' \to$ $\to \bigvee_{j \in J} X_j$ induced by $\iota_j : X_j \to \bigvee_{j \in J} X_j$. By (4), the diagram



commutes. Thus, ι is injective and hence $\{x\} \sim_s \{y\} \equiv x = y$ (for standard x, y). Now $[\{x\}] \le [\{y\}] \equiv [\{x\}] \wedge [\{y\}] =$ $= [\{x\}] \equiv [\{x \wedge y\}] = [\{x\}] \equiv x \wedge y = x \equiv x \prec y$. □

2.5. Remark: This result is proved in [5] and it can be reformulated to say that ι_j preserve arbitrary (even infinite) meets. This property could be called the openness of ι_j. This is motivated by the following

Fact: Let X, Y be topological T_1-spaces. Then a continuous $f: X \to Y$ is open iff $f^{-1}: \Omega(Y) \to \Omega(X)$ preserves arbitrary (even infinite) meets.

Proof: If $f: X \to Y$ is open, then the image mapping $f_1:$ $: \Omega(X) \to \Omega(Y)$ is evidently left adjoint to f^{-1}. Thus, f^{-1} preserves meets. On the other hand, if f^{-1} preserves meets, it has a left adjoint f_*. For $U \in \Omega(X)$, $V \in \Omega(Y)$ we have $f_*(U) = \bigwedge_{f^{-1}(V) \geq U} V \leq \bigwedge_{f^{-1}(V) \geq U} V = f_1(U)$ (for, since Y is T_1, we have $x \in \bigcap_{f^{-1}(V) \geq V} V \equiv f^{-1}(Y \setminus \{x\}) \not\supseteq U \equiv x \in f_1(U)$). On the other hand, $f^{-1} f_*(U) = f^{-1}(\bigwedge_{f^{-1}(V) \geq U} V) = \bigwedge_{f^{-1}(V) \geq U} f^{-1}(V) \supseteq U$, and hence $f_*(U) \supseteq f_1(U)$. Thus, $f_* = f_1$. □

3. <u>The Tychonoff's theorem.</u> A frame (locale) is said to be <u>compact</u>, if for any $S \subseteq A$ with $\bigvee S = 1$ there exists a finite $F \subseteq S$ with $\bigvee F = 1$. In this section we give a choice- and replacement-free proof of the theorem that the product of compact locales is compact.

Let A be a frame. A set $S \subseteq A$ is called a <u>covering</u> of A, if it holds $\bigvee S = 1$. For coverings s, t of a frame A we set $s \leq t$, if it holds $(\forall x \in s)(\exists y \in t)\ x \leq y$. (This is the ordinary concept of refinement.) Let now s be a covering of A and let $t \subseteq A$ such that $\bigvee t \geq a$. We will use the notation $s \wedge_a t = \{x \in s \mid x \leq a\} \cup \{x \wedge y \mid x \in s \,\&\, y \in t\}$. Obviously, $s \wedge_a t$ is a covering of A and $s \wedge_a t \leq s$. Analogously, $(\forall x \in s \wedge_a t)((x \leq a) \Rightarrow (\exists y \in t)(x \leq y))$.

Now let X_j $(j \in J)$ be a system of frames. Consider a system s_j of coverings such that $s_j = 1$ except for, at most, finitely many j. Then the system

$$\prod_{j\in J} s_j = \{[\{x\}] \in \bigvee_{j\in J} X_j \mid (\forall j \in J)\ a_x^j \in s_j\}$$

is a covering of $\bigvee_{j\in J} X_j$.

In the last section we remarked that $\iota_j : X_j \to \bigvee_{j\in J} X_j$ preserve arbitrary meets. Thus, they have left adjoints $p_j : \bigvee_{j\in J} X_j \to X_j$ (which, of course, are __not__ frame homomorphisms). We can easily check that

$$p_j([u]) = \bigvee\{a_x^j \mid x \in u\}. \tag{6}$$

3.1. __Lemma:__ Let $\{u,v\}$ be a covering of $\bigvee_{j\in\{0,1\}} A_j$. Then $p_0(u) = 1$ or $p_1(v) = 1$.

__Proof:__ There should exist a connector $(m, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow} (j \in \{0,1\}), \bar{u} \cup \bar{v}, \{1\})$ for some standard representatives $\bar{u}$, $\bar{v}$ of the classes u, v. Consider a system $x_i \in B'$, $i \in I$ such that x_i differ at most at one coordinate. Then the statement $(\forall i \in I)[(a^0_{x_i} \leq p_0(u))$ or $(a^1_{x_i} \leq p_1(v))]$ implies the statement $(a^0_{\bigvee x_i} \leq p_0(u))$ or $(a^1_{\bigvee x_i} \leq p_1(v))$. Thus, by (3), the set $K = \{x \in m \mid (a^0_x \leq p_0(u))$ or $(a^1_x \leq p_1(v))\}$ satisfies (1), and hence $K = m$. In particular, $1 \leq p_0(u)$ or $1 \leq p_1(v)$. □

3.2. __Observation:__ Any element of a finite lattice is a join of join-irreducible elements.

__Proof:__ An obvious induction. □

3.3. __Lemma:__ Consider a finite covering $t = \{[\{x_i\}] \mid i \leq n, x_i \in B'\}$ of the frame $\bigvee_{j\in J} X_j$. Then there exist finite coverings s_j of the frames X_j such that $\prod_{j\in J} s_j \leq t$.

__Proof:__ will be done for $J = \{0,1\}$. This, by induction, obviously implies the case of J finite; the case of J infinite is executed by the finiteness of t. Let, hence, $J = \{0,1\}$. Let A_j ($j = 0,1$) be sets of all possible elements of X_j obtained from a_y^j ($i \leq n$) by join and meet-operations in X_j. Obviously A_j are finite lattices. Write s_j for the set of all join-irreducible elements in A_j. By 3.2, s_j is a covering of X_j. We will show that $s_0 \times s_1 \leq t$. Suppose the contrary. Then there exists a $y \in B'$ such that $a_y^j \in s_j$ for $j = 0,1$ and $x_i \not\geq y$ for any $i \leq n$. From the join-irreducibility of a_y^j it follows that

$$p_j([\{x_i \mid a_{x_i}^j \not\geq a_y^j\}]) \geq a_y^j \text{ for } j = 0,1.$$

By 2.4 and by the properties of y, however,

$$\bigvee_{j \in \{0,1\}} [\{x_i \mid a_x^j \not\geq a_y^j\}] = 1,$$

contradicting 3.1. □

3.4. <u>Lemma</u>: Consider compact frames X_j ($j \in J$). Let $(m, \mathcal{R}_j^{\uparrow}, \mathcal{R}_j^{\downarrow}\ (j \in J), k, \{1\})$ be a connector. Then for any finite $m' \subseteq m$ such that $m' \sim_s 1$ and for any $x \in m'$ there exists a finite $m'' \subseteq (m' \setminus \{x\}) \cup k$ such that $m'' \sim_s 1$.

<u>Proof</u>: Let $\bar{k}$ be the set of all $x \in m$, satisfying the statement of Lemma 3.4. We will show that k satisfies the condition (1), and hence $\bar{k} = m$.

The inclusion $k \subseteq \bar{k}$ is obvious.

α) Let $y \mathcal{R}_{\varkappa}^{\downarrow} u$ & $x \in u$, $y \in \bar{k}$. Then, of course, $x \prec y$ so that if $1 \sim_s m' \ni x$, it is $1 \sim_s (m' \setminus \{x\}) \cup \{y\}$. Thus, $x \in \bar{k}$.

β) Let $u \mathcal{R}_{\varkappa}^{\uparrow} x$ & $u \subseteq \bar{k}$. Assume $1 \sim_s m' \ni x$. Put $M' = \{[\{y\}] \mid y \in m'\}$. Then M' is a covering of $\bigvee_{j \in J} X_j$. By 3.3, there exists a covering $\prod_{j \in J} s_j \leq M'$. We take the covering $s_{\varkappa} \wedge_{a_x^{\varkappa}} \{a_y^{\varkappa} \mid y \in u\}$ of the frame $X_{\varkappa}$. By compactness, it possesses a finite subcovering $\bar{s}_{\varkappa}$. Putting $\bar{s}_j = s_j$ for $j \neq \varkappa$, we obviously obtain $\prod_{j \in J} \bar{s}_j \leq (M' \setminus [\{x\}]) \cup \{[\{y\}] \mid y \in u\}$. The left hand set is finite. Thus, there exists a finite subset $t \subseteq u$ with $(m' \setminus x) \cup \cup t \sim_s 1$. From $t \subseteq \bar{k}$ we easily obtain $x \in \bar{k}$ (by induction on card t). □

3.5. __Theorem__: In the Zermelo set theory (without the axioms of choice and replacement) Tychonoff's theorem holds for locales; i.e., the product of compact locales is compact.

__Proof__: Let X_j ($j \in J$) be a system of compact frames and let S be a covering of $\bigvee_{j \in J} X_j$. Put $k = s(\bigcup_{x \in S} t(x))$, where s is the kernel and t is defined in 2.2.4. It will be $k \sim_s 1$. By Lemma 3.4 (with $m' = \{1\}$, $x = 1$), there exists a finite subset $k' \subseteq k$ with $k' \sim_s 1$. Since k' is finite, however, there exists a finite $F \subseteq S$ such that $(\forall x \in k')(\exists \alpha \in F)(x \in s(t(\alpha)))$. Thus, of course, $\bigvee F = 1$. □

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