Marc De Wilde; Pierre B. A. Lecomte; Dominique Mélotte

Invariant cohomology of the Poisson Lie algebra of a symplectic manifold

Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 2, 337--352

Persistent URL: http://dml.cz/dmlcz/106375

COMMENTATIONES MATHEMATICAE UNVERSITATIS CAROLINAE

26,2 (1985)

INVARIANT COHOMOLOGY OF THE POISSON LIE ALGEBRA OF A SYMPLECTIC MANIFOLD

M. De WILDE, P. B. A. LECOMTE, D. MELOTTE

Abstract. Let (M,F) be a symplectic manifold and let $\mathfrak{G}$ be a subalgebra of its Lie algebra of symplectic vector fields. It is shown that if (M,F) has a $\mathfrak{G}$-invariant connection, the subcomplex of the Chevalley complex of differential cochains of the Poisson algebra of (M,F) generated by the $\mathfrak{G}$-invariant cochains and the 1-differentiable cochains has the same cohomology as the total complex. Moreover, the second and third cohomology spaces of the complex of invariant cochains are computed.

Key words: Symplectic manifolds. Chevalley cohomology. Poisson algebra. Invariance.

Classification: 17B65, 17B56, 53C15

---

1. Introduction. Let M be a connected, Hausdorff, second countable smooth manifold of dimension $2n>2$. Let F be a symplectic form on M; Λ will denote its contravariant version, i.e. the contravariant 2-tensor obtained by lifting the indices of F by the duality defined by F. The Poisson Lie algebra of M is (N,P), N being the space of all smooth real functions on M and P the Poisson bracket.

We denote by ∂ the coboundary operator of the Chevalley cohomology of the adjoint representation of (N,P). A cochain C (i.e. an alternating multilinear map from N^q into N) is called differential if it is a differential operator of some fixed order allower M. It is vanishing on the constants (in short nc) if $i(1)C = 0$.

The space $\Lambda_{diff}(N)$ of all differential cochains is stable by ∂ . Its cohomology $H^q(\Lambda_{diff}(N), \partial)$ is not known in general, due to the lack of a reasonable model for the cohomology of formal symplectic vector fields. The spaces H^q ($q = 1,2,3$) play an important role in various problems of symplectic geometry, namely in the study of formal deformations of (N,P), and they have been computed in [1, 5, 8].

Suppose now that $\mathcal{G}$ is a Lie algebra of symplectic vector fields on M. A cochain C is $\mathcal{G}$-invariant if $L_X C = 0$ for all $X \in \mathcal{G}$.

Denote by $\Lambda^{\mathcal{G}}_{diff}(N)$ the space of all $\mathcal{G}$-invariant differential cochains. It is still stable by ∂ and the knowledge of $H^q(\Lambda^{\mathcal{G}}_{diff}(N), \partial)$ $(q \leq 3)$ is essential in the study of $\mathcal{G}$-invariant formal deformations of (N,P).

It is known that the study of $H(\Lambda_{diff}(N), \partial)$ reduces to that of $H(\Lambda_{diff,nc}(N), \partial)$ [3] and the same holds true for the invariant cohomology. An important subspace of $\Lambda_{diff,nc}(N)$ is the space of 1-differentiable cochains (i.e. of order 1 in each argument) isomorphic to the space $\Lambda(M)$ of smooth forms on M by $\mu^*: \Lambda(M) \to \Lambda_{1-diff,nc}(N)$, where

$$\mu^*\omega(u_0, \dots, u_{q-1}) = \omega(X_{u_0}, \dots, X_{u_{q-1}}),$$

X_u being the Hamiltonian vector field of u. The space $\Lambda_{1-diff,nc}(N)$ is stable by ∂ and μ^* intertwins d (the exterior differential) with ∂ .

We will assume that M admits a $\mathcal{G}$-invariant linear connection. From results of [6, 7], it seems to be a reasonable conjecture that the difference between the cohomology and the invariant cohomology only comes from 1-differentiable cochains.

We prove in this paper that $\Lambda^{\mathcal{G}}_{diff,nc}(N) \wedge \Lambda_{1-diff,nc}(N)$ and $\Lambda_{diff,nc}(N)$ have the same cohomology and we compute $H^q(\Lambda^{\mathcal{G}}_{diff,nc}(N))$ for q = 2,3 (the case q = 1 is trivial).

2. The main result

Theorem 2.1. *Let (M,F) be a symplectic manifold of dimension $2n>2$ and let $\mathcal{G}$ be a Lie algebra of symplectic vector fields over M.*

If M admits a $\mathcal{G}$-invariant linear connection, then the inclusion

$$i: (\Lambda^{\mathcal{G}}_{diff,nc}(N) \wedge \Lambda_{1-diff,nc}(N), \partial) \rightarrow (\Lambda_{diff,nc}(N), \partial)$$

induces an isomorphism in cohomology.

We will set

$$I^{\mathcal{G}}(N) = \Lambda^{\mathcal{G}}_{diff,nc}(N) \wedge \Lambda_{1-diff,nc}(N).$$

The proof goes in two steps. First, M is supposed to be a contractible open subset of $\mathbb{R}^{2n}$ with its canonical symplectic structure. Next the result has to be extended to an arbitrary M.

The proof of this second step is entirely similar to [2], p. 211,B and will be omitted here.

For the first step, a proof based on a study of the symbols in lexicographical order and on an induction with respect to this order would be possible. Since this type of proof seems to hide an argument based on spectral sequences, we have preferred the latter approach. We thus introduce appropriate spectral sequences on $(\Lambda_{diff,nc}(N), \partial)$ and $(I^{\mathcal{G}}(N), \partial)$ and show that their terms E_1 are isomorphic and that the sequences converge. Surprisingly, the latter point does not seem to follow from classical convergence arguments.

The assumption that M admits a $\mathbb{G}$-invariant connection can be slightly relaxed, as shown in § 3, but the improvement is not of obvious interest.

3. <u>The case of the symplectic manifold</u> $\mathbb{R}^{2n}$. In this section, M denotes some fixed contractible open subset of $\mathbb{R}^{2n}$, equipped with its canonical symplectic form F.

By substituting the i-th component of $\xi_j \in \mathbb{R}^{2n}$ to the i-th partial derivative of u_j in a cochain $C(u_o,\dots,u_{q-1})$, we define a linear map a which transforms the nc cochains into alternating polynomials on $\mathbb{R}^{2n*}$, of order ≥ 1 in each argument and smoothly depending on $x \in M$.

Let Q be the space of all such polynomials and $\mathcal{P}$ be the space of all alternating polynomials on $\mathbb{R}^{2n*}$, of order >1 in each argument. Define b: $\mathcal{P} \otimes \Lambda(M) \rightarrow Q: P \otimes \omega \rightarrow \alpha(P \cdot \mu^*\omega)$, where α is the antisymmetrization projector and

$$(P\cdot Q)(\xi_o,\dots,\xi_{q-1}) = P(\xi_o,\dots,\xi_{\ell-1})Q(\xi_\ell,\dots,\xi_{q-1}).$$

The map b is a linear bijection and $b^{-1}\circ a$ identifies $\Lambda_{diff,nc}(N)$ to $\mathcal{P}\otimes \Lambda(M)$. An easy computation shows that, by this identification, ∂ transforms into

$$\partial(P\otimes\omega) = d'P\otimes\omega + P\otimes d''\omega,$$

where d' is given by

$$(d'P)(\xi_o,\dots,\xi_q) =$$

$$\sum_{i<j}(-1)^j\Lambda(\xi_i,\xi_j)[P(\dots,\underset{(i)}{\xi_i+\xi_j},\dots\hat{j}\dots) - P(\dots,\underset{(i)}{\xi_i},\dots\hat{j}\dots) - P(\dots,\xi_j,\dots\hat{j}\dots)]$$

(recall that Λ is the contravariant 2-tensor obtained by lifting the indices in F) and if ω is a q-form.

The coboundary d' can be interpreted as follows.

A homogeneous polynomial $P(\xi_0,\dots,\xi_{q-1})$ of degree r_i in ξ_i identifies to a q-linear form on $\mathcal{S}_{r_0}\times\dots\times\mathcal{S}_{r_{q-1}}$, where $\mathcal{S}_i$ is the symmetric i-th power of $\mathbb{R}^{2n*}$. Thus $\mathcal{P}$ may be regarded as the space of all cochains on $\mathcal{S}=\prod_{i>1}\mathcal{S}_i$, continuous with respect to the product topology.

By lowering indices by means of F, the Lie algebra of formal symplectic vector fields without constant term on $\mathbb{R}^{2n}$ identifies to $\mathcal{S}$, its subalgebra $sp(n,\mathbb{R})$ corresponding to $\mathcal{S}_2$.

It is a matter of computation to check that, by this isomorphism, d' corresponds to the differential of the Chevalley cohomology of the trivial representation of $\mathcal{S}$ on $\mathbb{R}$.

4. <u>The spectral sequences</u>. In the sequel, S denotes one of the spaces $\mathcal{P}\otimes\Lambda(M)$ or $\hat{I}^{G}(N) = b^{-1}\circ a\, I^{G}(N)$.

The space S is graded by $S=\bigoplus_{q\geq 0} S^q$, S^q being the space of all q-cochains belonging to S. It admits the decreasing filtration F^p $(p\in\mathbb{Z})$, where $F^p = \bigoplus F^{p,q}$ and $F^{p,q}$ is the space of elements of S^q of total order at most 2q-p. The total order of $P\otimes\omega$ is the sum of the total order of P and of the number of arguments of ω. Thus $F^{p,q}=0$ if $p>q$. Moreover, $d'F^p\subset F^p$ and $d''F^p\subset F^{p+1}$. Thus (S,∂) is a graded filtered differential space (in the sense of [4]). The corresponding spectral sequence will be denoted $(E_i(S),d_i)$.

5. <u>The terms</u> E_0. It is clear that, for $S=\mathcal{P}\otimes\Lambda(M)$,

$$(E_0(S),d_0)\cong(S,d'\otimes 1).$$

The case $S=\hat{I}^{G}(N)$ reouires some attention. If C is a cochain, the homogeneous part of aC of highest order is the symbol σ_C of C. If C is invariant, so is σ_C because, for every $X \in \mathcal{H}(M)$, $\sigma_{L_X C} = L_{X^*}\sigma_C$, where X^* denotes the natural lifting of X to T^*M. Conversely, since M admits an invariant connection, every smooth homogeneous polynomial on T^*M, invariant by $\mathfrak{G}$, is the symbol of an invariant cochain [7, § 11]. It follows that $E_0^{p,q}(S) = F^{p,p+q}/F^{p+1,p+q}$ corresponds by b to the space of symbols of the (p+q)-cochains of order 2q+p of $I^{\mathfrak{G}}(N)$. Thus

$$E_0(\hat{I}^{\mathfrak{G}}(N)) \cong [b^{-1}\{p \in Q : L_{X^*}P = 0,\ \forall X \in \mathfrak{G}\}]\cdot\Lambda(M)$$

where $\cdot$ is defined by

$$(P \otimes \omega)\cdot(P' \otimes \omega') = (-1)^{kk'}(P \wedge P') \otimes \omega \wedge \omega'$$

k (resp. k') being the number of arguments of ω (resp. P'). Moreover, d_0 identifies again to $d' \otimes \mathbf{1}$.

A more precise description of $E_0(\hat{I}^{\mathfrak{G}}(N))$ will be useful. If $p \in \mathcal{P}$,

$$L_{X^*}P = \rho(DX)P,$$

where DX is the Jacobian matrix of X and ρ the natural representation of $gl(n,\mathbb{R})$ on $\mathcal{P}$. If X is symplectic, it follows that

$$L_{X^*} \circ b = b \circ \Theta(X)$$

where

$$\Theta(X)(P \otimes \omega) = \rho(DX)\,P \otimes \omega + P \otimes L_X\omega .$$

Hence

$$E_0(\hat{I}^{\mathfrak{G}}(N)) = \ker\Theta \cdot \Lambda(M)$$

where

$$\ker\Theta = \{C \in \mathcal{P} \otimes \Lambda(M) : \Theta(X)C = 0,\ \forall X \in \mathfrak{G}\}.$$

For the sake of simplicity, we will denote $\ker\Theta \cdot \Lambda(M)$ by

$I^{\Theta}(N)$.

<u>Remark</u>. The only point where the existence of a $\mathfrak{G}$-invariant connection is used is the assertion that every homogeneous polynomial on $T^* M$ invariant by $\mathfrak{G}$ is the symbol of an invariant cochain.

6. Isomorphism of the terms E_1

<u>Lemma 6.1</u>. <u>The inclusion</u> $i: I^{\Theta}(N) \longrightarrow \mathcal{P} \otimes \Lambda(M)$ <u>induces an isomorphism</u> $i_{\#}: E_1(I^{\Theta}(N)) \longrightarrow E_1(\mathcal{P} \otimes \Lambda(M))$.

It is well-known that

$$E_1(S) \cong H(E_0(S), d_0).$$

We have already seen that $\mathcal{P} \cong \Lambda_c(\mathcal{S})$, the space of continuous cochains on $\mathcal{S}$. Moreover $sp(n,\mathbb{R}) \cong \mathcal{S}_2$ is a subalgebra of $\mathcal{S}$.

Denote by $\mathcal{R}$ the space of skew-symmetric polynomials of order ≥ 3 in each argument. Then $\mathcal{R} \cong \Lambda_c(\mathcal{S}/\mathcal{S}_2)$.

Consider the Hochschild-Serre filtration of $\mathcal{P} \otimes \Lambda(M)$ $(\cong \Lambda_c(\mathcal{S}) \otimes \Lambda(M))$ related to the subalgebra $\mathcal{S}_2 \cong sp(n,\mathbb{R})$. The first term of the corresponding spectral sequence of the differential space $(\mathcal{P} \otimes \Lambda(M), d' \otimes \mathbb{1})$ is

$$E_0' \cong \Lambda_c(\mathcal{S}) \otimes \Lambda(M) \cong \Lambda(sp(n,\mathbb{R}), \mathcal{R}) \otimes \Lambda(M)$$

with the differential $d_0' = \partial_{\varphi} \otimes \mathbb{1}$, where ∂_{φ} is the differential of the Chevalley cohomology of the representation φ restricted to $sp(n,\mathbb{R})$ and $\mathcal{R}$.

Similarly, for the corresponding filtration of $I^{\Theta}(N)$, $E_0'' = I^{\Theta}(N)$ and $d_0'' = \partial_{\varphi} \otimes \mathbb{1}$ when identifying E_0'' to a subspace of $\Lambda(sp(n,\mathbb{R}),\mathcal{R}) \otimes \Lambda(M)$ by the isomorphism above.

Since $sp(n,\mathbb{R})$ is simple, it is well-known that

$$H(E_o',d_o') \cong \Lambda^{ad} \otimes \mathcal{R}^{\rho} \otimes \Lambda(M)$$

and more precisely,

$$\ker \partial_{\rho} = \Lambda^{ad} \otimes \mathcal{R}^{\rho} \otimes \Lambda(M) \oplus \operatorname{im} \partial_{\rho}$$

where Λ^{ad} denotes the space of all ad-invariant elements of $\Lambda(sp(n,\mathbb{R}),\mathbb{R})$ and $\mathcal{R}^{\rho}$ the space of ρ-invariant elements of $\mathcal{R}$. Moreover, there exists a linear map $k: \Lambda(sp(n,\mathbb{R}),\mathcal{R}) \longrightarrow$ $\longrightarrow \Lambda(sp(n,\mathbb{R}),\mathcal{R})$ which intertwins the representation

$$\mathcal{L}_{\rho} : A \longrightarrow i(A) \circ \partial_{\rho} + \partial_{\rho} \circ i(A)$$

of $sp(n,\mathbb{R})$ and which is a right inverse of ∂_{ρ} on $\operatorname{im} \partial_{\rho}$.

Indeed, $\ker \partial_{\rho}$ is stable by $\mathcal{L}_{\rho}$ thus it has an $\mathcal{L}_{\rho}$-stable algebraic supplement $\mathbb{E}$. Thus

$$\Lambda(sp(n,\mathbb{R}),\mathcal{R}) = (\Lambda^{ad} \otimes \mathcal{R}^{\rho}) \oplus \operatorname{im} \partial_{\rho} \oplus \mathbb{E} .$$

Observe that $\partial_{\rho} : \mathbb{E} \longrightarrow \operatorname{im} \partial_{\rho}$ has a unique inverse. If α, β, γ are the projectors on $\Lambda^{ad} \otimes \mathcal{R}^{\rho}$, $\operatorname{im} \partial_{\rho}$ and $\mathbb{E}$ associated to this decomposition $k = (\partial_{\rho} |_{\mathbb{E}})^{-1} \circ \beta$ has the required properties.

Observe that, by the isomorphism $\mathcal{P} \cong \Lambda(sp(n,R),\mathcal{R})$, ρ becomes $\mathcal{L}_{\rho}$. Thus k commutes with $\Theta(X)$ $(X \in \mathcal{G})$ and $k \otimes 1$ stabilizes $I^{\Theta}(N)$.

Since $\Lambda^{ad} \otimes \mathcal{R}^{\rho} \otimes \Lambda(M) \subset I^{\Theta}(N)$, we have

$$\ker \partial_{\rho} \cap I^{\Theta}(N) = (\Lambda^{ad} \otimes \mathcal{R}^{\rho} \otimes \Lambda(M)) \oplus (I^{\Theta}(N) \cap \operatorname{im} \partial_{\rho})$$

and

$$I^{\Theta}(N) \cap \operatorname{im} \partial_{\rho} = \partial_{\rho} \circ k(I^{\Theta}(N) \cap \operatorname{im} \partial_{\rho}) \subset \partial_{\rho} I^{\Theta}(N).$$

Thus

$$H(E_o'',d_o'') \cong H(E_o',d_o')$$

and the lemma follows.

7. Convergence of the spectral sequences $E_i(S)$. It does not seem that the convergence of the spectral sequences $E_i(S)$, $S = \mathcal{P} \otimes \Lambda(M)$ or $I^{\theta}(N)$ follows from standard arguments.

We need the following

Lemma 7.1. Let $(X = \bigoplus_q X^q, \delta)$ be a graded differential space. Suppose that $\delta = \delta_0 + \delta_1$, where δ_i $(i = 0,1)$ is homogeneous of degree i and that $\operatorname{im} \delta_0 \cap \operatorname{im} \delta_1 \subset \operatorname{im} \delta_0 \circ \delta_1$. If (E_j, d_j) is the spectral sequence defined by the filtration $F^p = \oplus_{q \geq p} X^q$, then $E_2^p \cong E_\infty^p$.

Recall that

$$E_i^p = Z_i^p/(Z_{i-1}^{p+1} + D_{i-1}^p)$$

where

$$Z_i^p = F^p \cap \delta^{-1} F^{p+i} \text{ and } D_i^p = F^p \cap \delta F^{p+i}.$$

It is clear that $Z_2^p \supset Z_\infty^p + Z_1^{p+1}$. These spaces are equal. Indeed, if $x \in Z_2^p$, $x = x_p + x_{p+1} \bmod F^{p+2}$, with $x_p \in X^p$ and $x_{p+1} \in X^{p+1}$. Then $\delta_0 x_p = 0$ and $\delta_1 x_p + \delta_0 x_{p+1} = 0$, thus there exists $z \in X^p$ such that $\delta_1 x_p = -\delta_0 x_{p+1} = \delta_0 \delta_1 z$. Note that $x_{p+1} + \delta_1 z \in X^{p+1} \cap \ker \delta_0 = Z_1^{p+1}$ and $x_p - \delta_1 z = x_p + \delta_0 z - \delta z \in F^p \cap \ker \delta = Z_\infty^p$. Since $F^{p+2} \subset Z_1^{p+1}$, the equality follows.

Moreover, $D_1^p = D_\infty^p$. Indeed, if $x \in D_\infty^p$, choose the largest $q \leq p-1$ such that $x = \delta y$ with $y \in F^q \setminus F^{q+1}$. Then $q = p-1$. Otherwise, $y \in Z_2^q = Z_\infty^q + Z_1^{q+1}$ and $x = \delta y \in \delta F^{q+1}$, contradicting the choice of q.

Suppose now that $Z_1^{p+1} = Z_\infty^{p+1} \oplus V$. Since $Z_1^{p+1} \cap D_\infty^p = Z_\infty^{p+1} \cap D_\infty^p$, $Z_1^{p+1} + D_\infty^p = (D_\infty^p + Z_\infty^{p+1}) \oplus V$. Moreover $Z_\infty^p \cap Z_1^{p+1} = Z_\infty^{p+1}$, thus $Z_\infty^p + Z_1^{p+1} = Z_\infty^p \oplus V$. Then

$$E_2^p = (Z_\infty^p + Z_1^{p+1})/(Z_1^{p+1}+D_\infty^p) = (Z_\infty^p \oplus V)/[(D_\infty^p+Z_\infty^{p+1})\oplus V] \cong E_\infty^p .$$

Hence the lemma.

Let us now prove the convergence of the spectral sequences $E_1(S)$ of § 3.

Take first $S = \mathcal{P}\otimes \Lambda(M)$. It is graded by

$$S = \oplus X^q,\ X^q = \bigoplus_{2k-r=q} (\mathcal{P}\otimes \Lambda(M))^{r,k},$$

where $(\mathcal{P}\otimes \Lambda(M))^{r,k}$ denotes the space of all k-cochains, homogeneous of order r. The filtration of § 3 is $F^p = \bigoplus_{r\geq q} X^q$. The differential $d' + d''$ verifies the assumption of Lemma 7.1, hence

$$E_2^{p,q}(S) \cong E_\infty^{p,q}(S).$$

The space $E_2(S)$ is easily computed:

$$E_1(S) \cong H(\mathcal{P},d') \otimes \Lambda(M),$$

$$d_1([P]_{d'} \otimes \omega) = [P]_{d'} \otimes d''\omega$$

and

$$E_2(S) \cong H(\mathcal{P},d')$$

because M is contractible, hence $H(\Lambda(M),d) \cong \mathbb{R}$.

In particular, each $E_2^{p,q}$ is finite dimensional.

It is known [4] that, for $i > q+1$, there is a canonical surjective map

$$\theta_i^{p,q}: E_i^{p,q} \longrightarrow E_\infty^{p,q}.$$

We must show that it is surjective. In fact, if $d_j \neq 0$ for some $j > 2$, for $i > \sup(q+1,j)$, we get

$$\dim E_\infty^{p,q} = \dim E_2^{p,q} \geq \dim E_j^{p,q} > \dim E_i^{p,q} > \dim E_\infty^{p,q}.$$

It follows that the spectral sequence collapses at the second term and that $\theta_2^{p,q}$ is bijective.

We consider now the case of $I^\theta(N)$. We have the commutative diagram

$$\begin{array}{ccc} E_i^{p,q}(I^{\Theta}(N)) & \longrightarrow & E_i^{p,q}(\mathcal{P}\otimes\Lambda(M)) \\ \tilde{\Theta}_i^{p,q}\downarrow & & \downarrow\Theta_i^{p,q} \\ E_\infty^{p,q}(I^{\Theta}(N)) & \longrightarrow & E_\infty^{p,q}(\mathcal{P}\otimes\Lambda(M)) \end{array}$$

where the horizontal arrows are induced by the inclusion map $I^{\Theta}(N) \to \mathcal{P} \otimes \Lambda(M)$. For $i > q + 1$, $\tilde{\Theta}_i^{p,q}$ is onto, while the upper horizontal arrow and $\Theta_i^{p,q}$ are isomorphisms. Since $E_i^{p,q}(I^{\Theta}(N))$ is finite-dimensional, it follows that all the arrows are isomorphisms, hence the result.

8. The second and third invariant cohomology spaces. We conclude by describing the space $H^i(\Lambda^{\mathbb{G}}_{diff,nc}(N), \partial)$, for $i = 2,3$. Given a connection Γ on M, there exist a 2-cocycle S^3_Γ with symbol Λ^3 and a 3-cochain T_Γ with symbol

$$(\xi_o, \xi_1, \xi_2) \to \Lambda(\xi_o, \xi_1)\,\Lambda(\xi_1, \xi_2)\,\Lambda(\xi_2, \xi_o),$$

which allow an easy description of $H^i(\Lambda_{diff,nc}(N), \partial)$ $(i=2,3)$ [1]. Moreover, if Γ is invariant, S^3_Γ and T_Γ are invariant.

Proposition 8.1. Under the assumption of thm. 2.1,

a) any $\mathbb{G}$-invariant 2-cocycle C reads

$$C = r\, S^3_\Gamma + \mu^*\omega + \partial D,$$

where $r \in \mathbb{R}$, $\omega \in \Lambda^{\mathbb{G}}(M)$ and $D \in \Lambda^{\mathbb{G}}_{diff,nc}(N)$. Moreover

$$[C] \to (r, [\omega])$$

is bijective, [ ] denoting the cohomology classes and

$$H^2(\Lambda^{\mathbb{G}}_{diff,nc}(N), \partial) \cong \mathbb{R} \oplus H^2(\Lambda^{\mathbb{G}}(M), d);$$

b) any $\mathbb{G}$-invariant 3-cocycle reads

$$C' = S^3_\Gamma \wedge L_X + s\, T_\Gamma + \mu^*\omega + \partial E$$

where $s \in \mathbb{R}$, $\omega' \in \Lambda^{\mathbb{G}}(M)$, $E \in \Lambda^{\mathbb{G}}_{diff,nc}(N)$, X *is symplectic and* $\mathbb{G}$ *-invariant*, s *vanishing if* ∂T_{Γ} *is not exact in* $\Lambda^4(M)$. *Moreover*,

$$[C'] \longrightarrow (X \bmod L^{\mathbb{G}}, s, [\omega'])$$

is bijective, $L^{\mathbb{G}}$ *denoting the space of invariant Hamiltonian vector fields, and*

$$H^3(\Lambda^{\mathbb{G}}_{diff,nc}(N), \partial) \cong H^1(\Lambda^{\mathbb{G}}(M), d) \oplus H^3(\Lambda^{\mathbb{G}}(M), d) \oplus \mathcal{X},$$

with $\mathcal{X} = \mathbb{R}$ *or* $\{0\}$ *according as* ∂T_{Γ} *is exact in* $\Lambda^{\mathbb{G}}(M)$ *or not.*

Dropping the conditions of invariance in Prop. 7.1 exactly gives back the description of $H^i(\Lambda_{diff,nc}(N), \partial)\,(i=2,3)$ mentioned above.

The proof requires some preparation. We keep the notations of the end of the proof of Lemma 6.1. Let $\mathbb{P}$ be the bundle associated to the bundle $L_S(M)$ of symplectic linear frames of M and the natural representation $\wp$ of its structure group $Sp(n, \mathbb{R})$ on $\mathcal{P}$ (observe that the differential of $\hat{\wp}$ is $\wp$). Since the projectors α, β, γ of $\mathcal{P}$ commute with $\wp$ and thus with $\hat{\wp}$, they induce projectors, also denoted α, β, γ on the fibres of $\mathbb{P}$. For the same reason, the maps $\partial_{\wp}$ and k induce linear endomorphisms on $\mathbb{P}$, which we shall again denote by $\partial_{\wp}$ and k. It is clear that $\partial_{\wp}(\mathbb{P}) = \beta(\mathbb{P})$ and that $k \circ \partial_{\wp} = \gamma$.

If C is a cochain of lexicographical order $(r_0,\dots,r_{q-1})$ with $r_{q-1} > r_{\ell} = \dots = r_{q-1} = 1$, its lexicographical symbol $\bar{\sigma}_C$ identifies to a $(q-\ell)$-form on M with values in $\mathbb{P}$ and, by this identification, $L_{\underline{X}^*} \bar{\sigma}_C$ corresponds to the natural Lie derivative L_X with respect to X on the space $\Lambda(TM, \mathbb{P})$ of $\mathbb{P}$-valued forms on M. In particular, $L_{\underline{X}^*}$ commutes with $\alpha, \beta, \gamma, \partial_{\wp}$ and k.

Indeed, in the trivialisation associated to a canonical chart, the latter have the (constant) local forms $\alpha, \beta, \gamma, \partial_\rho$ or k of Lemma 6.1, while the local form of L_{X^*} is

$$L_X P = \sum X^i D_{x^i} P - \rho(DX)P.$$

Let now $C = \partial E$. It is known [1] that $\partial_\rho \bar{\sigma}_E = \bar{\sigma}_{\partial E}$ or 0. Suppose that i is invariant. One has

$$\bar{\sigma}_E = \alpha \bar{\sigma}_E + \beta \bar{\sigma}_E + \gamma \bar{\sigma}_E$$

and $\beta \bar{\sigma}_E = \partial_\rho \circ k \circ \beta \bar{\sigma}_E$. It is thus possible to correct E by a coboundary in order to cancel $\beta \bar{\sigma}_E$ without changing C. On the other hand, since ∂_ρ is injective on im γ and commutes with L_X, $\gamma \bar{\sigma}_E$ is invariant and it is the symbol of an invariant operator.

Let us now prove a). Let

$$C = rS^3_\Gamma + \mu^*\omega + \partial D$$

be an invariant 2-cocycle. Unless D is of order 1, $\partial_\rho \bar{\sigma}_D$ is invariant. Since there is no 1-cochain in $\Lambda^{ad} \otimes \mathcal{R}^\rho$, the above argument shows, by an induction on the order, that $D = D' + \mu^*\eta$, where D' is invariant. Thus $C = rS^3_\Gamma + \mu^*(\omega + d\eta) + \partial D'$ and $\omega + d\eta$ is invariant.

If now C is the coboundary of an invariant 1-cochain r = 0 and

$$\mu^*\omega + \partial D = \partial D''$$

where ω, D and D" are invariant. By the same kind of argument, D" - D is of order 1. Hence the result.

For b), let

$$C' = S^3_\Gamma \wedge L_X + \mu^*\omega + \partial E$$

be invariant (anyway, T_Γ is invariant).

If E is of order $(p,q) > (1,1)$, $\partial_{\wp} \bar{\sigma}_E$ is of order $(p,q,2)$ (in decreasing order), while $S^3_\Gamma \wedge L_X$ is of order $(3,3,1)$. These orders are never equal. If $(p,q,2)$ dominates, up to a correction of E by an invariant operator, or by a coboundary, we may assume that $\bar{\sigma}_E = \alpha\bar{\sigma}_E$. The only 2-cochains in $\Lambda^{ad} \otimes \mathcal{R}^{\wp}$ are $1 \otimes \Lambda^k$ for k odd ≥ 3. Thus if $(p,q,2) > (3,3,1)$ and $(p,q) \neq (k,k)$ (k odd $\geq$ ≥ 3), we may correct E by an invariant operator and decrease its order. If $(p,q) = (k,k) > (3,3)$, $\alpha\bar{\sigma}_E = f\Lambda^k$ ($f \in N$). The term of ∂E of order $(k,k-1,3)$ is then

$$k(k-1)\; f\Lambda(\xi_0, \xi_1)^{k-2}\Lambda(\xi_0, \xi_2)^2\Lambda(\xi_1, \xi_2).$$

It follows that f is invariant and E is the symbol of an invariant cochain. We have thus an induction process which allows to decrease the order of C' by correcting it by ∂T for some invariant T as long as E is of order $(p,q) \neq (3,3)$ with $(p,q,2) >$ $> (3,3,1)$.

Suppose that $(p,q) = (3,3)$. Then $\alpha\bar{\sigma}_E = f\Lambda^3$ and, for some invariant T',

$$C' - \partial T' = S^3_\Gamma \wedge L_{X'} + \mu^*\omega' + \partial E'$$

where E' is of order $< (3,3)$ and $X' = X - X_f$.

Suppose next that $C' - \partial T'$ is of order $(3,3,1)$. Its symbol is $\Lambda^3 \otimes L_{X'}$; since it is invariant, X' is invariant. Thus

$$C' - \partial T' - S^3_\Gamma \wedge L_{X'} = \mu^*\omega' + \partial E'$$

is of order $< (3,3,1)$.

So the induction leads to the existence of some invariant T and X such that

$$C' - \partial T - S^3_\Gamma \wedge L_X$$

is 1-differentiable. It is then of the type $\mu^*\eta$ for some invariant η . Hence the first part of b).

If C is the coboundary of an invariant 2-cochain and

$$C' = S^3 \wedge L_X + \mu^* \omega' + \partial E$$

with invariant X, ω' and E, we have

$$S^3 \wedge L_X + \mu^* \omega' = \partial T$$

where T is invariant. The same proof shows that $X = X_f$ and $\omega =$ $= d\eta$ where, this time, f and η are invariant. Hence the conclusion.

References

[1] M. De WILDE, S. GUTT, P. LECOMTE: A propos des deuxieme et troisième espaces de cohomologie de l'algèbre de Lie de Poisson d'une variété symplectique, Ann. Inst. H. Poincaré,Vol. 40,no 1(1984), 77-93.

[2] M. De WILDE, P. LECOMTE: Cohomology of the Lie algebra of smooth vector fields of a manifold, J. Math. P. et Appl. 62(1983), 197-214.

[3] M. De WILDE, P. LECOMTE: Cohomologie 3-différentiable de l'algèbre de Poisson d une variété symplectique, Ann. Inst. Fourier XXXIII,fasc. 4(1983), 83-94.

[4] W. GREUB, S. HALPERIN, R. VANSTONE: Connections, curvature and cohomology, III, Academic Press, New York-London, 1972.

[5] S. GUTT: Second et troisième espaces de cohomologie différentiable de l'algèbre de Lie de Poisson d'une variété symplectique, Ann. Inst. H. Poincaré 33, 1 (1981), 1-31.

[6] S. GUTT: Déformations formelles de l'algèbre des fonctions différentiables sur une variété symplectique, Doctor Thesis, Brussels, 1979.

[7] A. LICHNEROWICZ: Déformations d'algèbres associées à une variété symplectique (les $*_\nu$-produits), Ann.Inst.

Fourier 32,1(1982), 157-209.

[8] J. VEY: Déformation du crochet de Poisson d'une variété symplectique, Comm. Math. Helv. 50(1975), 421-454.

Université de Liège, Institut de Mathématique, Avenue des Tilleuls, 15, B-4000 Liège (Belgium)

(Oblatum 20.9. 1984)