Commentationes Mathematicae Universitatis Carolinae

Le Van Hot

A fixed point theorem

Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 2, 299--308

Persistent URL: http://dml.cz/dmlcz/106369

COMMENTATIONES MATHEMATICAE UNVERSITATIS CAROLINAE

26,2 (1985)

A FIXED POINT THEOREM

LE VAN HOT

Abstract: Using the maximal principle we prove a new fixed point theorem.

Key words: Banach space, fixed point theorem, uniformly convex function.

Classification: Primary: 47H10
Secondary: 47H15, 47H17

Since recent years many authors have used the maximal principle to prove fixed point theorems, for example [1],[2],[3].

In this paper, using that idea we prove a nex fixed point theorem and show some applications.

Let X be a Banach space, D a subset of X. By conv D we denote the convex hull of D. Let P be a binary relation on D. We say that P is reflexive if P(x,x) for all $x \in D$, P is closed if the set $\{(x,y) \in D \times D: P(x,y)\}$ is closed on $D \times D$. The function $h: \mathrm{conv}\, D \longrightarrow R$ is said to be uniformly convex if it is convex and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$h\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\,(h(x) + h(y)) - \delta$$

for all $x,y \in \mathrm{conv}\, D$, $\|x-y\| > \varepsilon$. If S is a subset of D, (h/S) denotes the restriction of h on S, R(h/S) denotes the range of (h/S).

Theorem: Let D be a closed subset of a Banach space X, P a reflexive closed relation on D, $h: \mathrm{conv}\, D \longrightarrow R_+$ a uniformly convex continuous bounded function attaining its minimum $x_0 \in D$. Let $f: D \longrightarrow D$ be a map such that:

1) if $x \in D$ and $P(x_0, x)$, then $P(x_0, f(x))$,

2) if $x, y \in D$, $P(x,y)$ and $h(\frac{1}{2}(x+y)) \geq h(x)$,

then $P(f(x), f(y))$ and $h(\frac{1}{2}(f(x)+f(y)) \geq h(f(x)))$.

Then f has a fixed point.

Proof: Let $\mathfrak{M}$ be the family of all nonempty subsets S of D containing x_0 and satisfying the following conditions:

a) if $x, y \in S$, $h(x) < h(y)$, then $P(x,y)$ and $h(x) < h(f(x)) \leq$ $\leq h(y)$ and $h(f(x)) = h(y)$ if and only if $f(x) = y$;

b) if $x, y \in S$, $h(x) \leqq h(y)$, then $h(x) \leq h(\frac{1}{2}(x+y))$,

c) if $a \in R_+$, $h(x_0) < a < \sup \{h(x) \mid x \in S\}$ and $a \notin R(h|S)$, then there exists an $x \in S$ such that $h(x) < a < h(f(x))$.

Obviously, $\{x_0\} \in \mathfrak{M}$, thus $\mathfrak{M} \neq \emptyset$.

Lemma 1: If $S \in \mathfrak{M}$, then $h(x_1) \neq h(x_2)$ for all $x_1, x_2 \in S$ and $x_1 \neq x_2$.

Proof: Suppose that there are $x_1, x_2 \in S$, $x_1 \neq x_2$ and $h(x_1) =$ $= h(x_2)$, then by b) and by uniform convexity of h we have:
$h(x_1) < h(\frac{1}{2}(x_1 + x_2)) < \frac{1}{2}(h(x_1) + h(x_2)) \Longrightarrow h(x_2) > h(x_1)$,
a contradiction. This finishes the proof of Lemma 1.

Lemma 2: If $S \in \mathfrak{M}$, $(x_n) \subseteq S$, $h(x_n) \uparrow a$, then (x_n) is a Cauchy sequence and moreover, if $x \in S$, $h(x) = a$, then $x = \lim x_n$.

Proof: Suppose that (x_n) is not a Cauchy sequence, then there exists an $\epsilon > 0$ and a subsequence (x_{n_i}) such that:
$\|x_{n_i} - x_{n_j}\| \geq \varepsilon$ for $i \neq j$. By the uniform convexity of h there exists a $\delta > 0$ such that:

$$h(x_{n_i}) \leq h(\tfrac{1}{2}(x_{n_i} + x_{n_{i+1}})) \leq \tfrac{1}{2}(h(x_{n_i}) + h(x_{n_{i+1}})) - \delta .$$

Thus $h(x_{n_{i+1}}) \geq h(x_{n_i}) + 2\delta \geq h(x_1) + 2i\delta$ for all i.

This contradicts the boundedness of h. Now, let $x \in S$ and $h(x) = a$. If $x \neq \lim x_n$, then there is an $\varepsilon > 0$ and n_0 such that: $\| x_n - x \| \geq \varepsilon$ for all $n > n_0$. Then there is a $\delta > 0$ such that:

$$h(x_n) \leq h(\tfrac{1}{2}(x_n + x)) \leq \tfrac{1}{2}(h(x_n) + h(x)) - \delta,$$

$$h(x) \geq h(x_n) + 2\delta \quad \text{for all } n \geq n_0.$$

This contradicts the assumption $h(x) = \lim h(x_n)$ and the proof of Lemma 2 is complete.

Lemma 3. Let $S \in \mathcal{M}$ and $x \in S$ be such that $h(x_0) < h(x) < \sup \{ h(x) | x \in S \}$, then $f(x) \in S$.

Proof: Suppose that $f(x) \notin S$. We claim that $h(f(x)) \notin R(h|S)$. In fact, if $h(f(x)) = h(y)$ for some $y \in S$, then $h(x) < h(y)$ and $y \neq f(x)$; then by a) $h(f(x)) < h(y)$, a contradiction. This shows that $h(f(x)) \notin R(h|S))$ and $h(f(x)) < \sup \{ h(x) | x \in S \}$. Now by c) there exists a $z \in S$ such that $h(z) < h(f(x)) < h(f(z))$ but by Lemma 1) and by a) it is impossible. That proves that $f(x) \in S$ and ends the proof of Lemma 3.

Lemma 4. Let $S \in \mathcal{M}$, $x \in D$, $h(x_0) < h(x) \leq h(u)$ for some $u \in S$. Suppose that there exists a sequence $(x_n) \subseteq S$ such that $\lim x_n = x; h(x_n) \uparrow h(x)$, then $x \in S$.

Proof: If $h(x) \notin R(h|S)$, then $h(x) < h(u)$. In fact if $h(x) = h(u)$ then by Lemma 2, $u = \lim x_n = x \in S$, a contradiction. By the condition c) there is a $z \in S$ such that $h(z) < h(x) < h(f(z))$. Then there is an integer n_0 such that $h(z) < h(x_{n_0}) < h(f(z))$. This contradicts the condition a). This shows $h(x) \in R(h|S)$ and $h(x) = h(y)$ for some y, $y \in S$. By Lemma 2) $y = \lim x_n = x \in S$. This ends the proof of Lemma 4.

Lemma 5. Let $S \in \mathfrak{M}$, $y \in S$, $y \neq x_0$; then either $y = f(z)$ for a $z \in S$, $z \neq y$ or $y = \lim_{n\to\infty} f(z_n)$; $h(f(z_n)) \uparrow h(y)$ for a sequence $(z_n) \subseteq S$.

Proof. Put $M_y = \sup \{h(x) \mid x \in S; h(x) < h(y)\}$.

1) If $M_y = h(y)$, then there is a $(z_n) \subseteq S$ such that $h(z_n) \uparrow h(y)$. By the condition a) we have $h(z_n) < h(f(z_n)) \leq$ $\leq h(z_{n+1}) < h(y)$. Thus $h(f(z_n)) \uparrow h(y)$. By Lemma 3) $f(z_n) \in S$ for all n and by Lemma 2) $y = \lim f(z_n)$.

2) If $M_y < h(y)$, then by c) there is a $z \in S$ such that $h(z) <$ $< \frac{1}{2}(M_y + h(y)) < h(f(z)) \leq h(y)$. By Lemma 3) $f(z) \in S$ and by Lemma 1 $f(z) = y$. Of course $y \neq z$. This completes the proof of Lemma 5,

Lemma 6. Let $S_1, S_2 \in \mathfrak{M}$ and suppose that for each $x \in S_1$ there is a $u \in S_2$ such that $h(x) \leq h(u)$. Then $S_1 \subseteq S_2$.

Proof: Suppose that $S_1 \nsubseteq S_2$, then $S_1 \setminus S_1 \cap S_2 \neq \emptyset$. Let $\bar{x} \in S_1 \setminus$ $\setminus S_1 \cap S_2$. By assumption there is a $u \in S_2$ such that $h(u) \geq h(\bar{x})$. Put $A = \{x \in S_1 \cap S_2 : \forall y \in S_1;\ h(y) < h(x) \Rightarrow y \in S_2\}$. Of course $A \neq \emptyset$ since $x_0 \in A$. It is clear that $h(x) < h(\bar{x})$ for all $x \in A$. Put $M_A =$ $= \sup \{h(x) \mid x \in A\} \leq h(\bar{x})$.

1) If $M_A \in R(h \mid A)$, then $M_A = h(y) < h(\bar{x})$ for some $y \in A$. By Lemma 3 $f(y) \in S_1 \cap S_2$; $h(y) < h(f(y))$ and if $z \in S_1$, $h(z) < h(f(y))$, then $h(z) \leq h(y)$. Thus $z \in A$. Therefore $f(y) \in A$, a contradiction.

2) If $M_A \notin R(h|A)$, then there is an $(x_n) \subseteq A$, $h(x_n) \uparrow M_A$. By Lemma 4) $\lim x_n = x \in S_1 \cap S_2$. It is clear that $x \in A$. It contradicts the fact $h(x) = M_A \notin R(h|A)$. This shows that $S_1 \setminus S_1 \cap S_2 = \emptyset$ and $S_1 \subseteq S_2$.

Lemma 7. $\bar{S} = \bigcup \{S \mid S \in \mathfrak{M}\} \in \mathfrak{M}$.

Proof: It is easy to verify that $\bar{S}$ satisfies all conditions a), b), c).

Now we return to the proof of the theorem. Put

$M = \sup \{h(x) | x \in \overline{S}\}$. If $M \notin R(h|\overline{S})$, then there is a sequence $(x_n) \subseteq \overline{S}$, $h(x_n) \uparrow M$. By Lemma 2 there is an $\overline{x} = \lim x_n$ and $h(\overline{x}) = M$. Put $\widetilde{S} = \overline{S} \cup \{\overline{x}\}$. It is obvious that $\widetilde{S}$ satisfies the condition c).

Now we verify that $\widetilde{S}$ also satisfies the conditions a),b), too. Let $x \in \widetilde{S}$, $h(x) < h(\overline{x})$, then $x \in \overline{S}$ and there exists an n_0 such that $h(x) < h(x_n)$ for all $n > n_0$. Since $\overline{S} \in \mathfrak{M}$, we have $P(x,x_n)$ and $h(x) \leq h(f(x)) \leq h(x_n); h(x) \leq h(\frac{1}{2}(x+x_n))$ for all $n > n_0$. Since P is closed and h is continuous, it follows that $P(x,\overline{x}), h(x) < h(f(x)) < \lim h(x_n) = h(\overline{x})$ and $h(x) \leq \lim h(\frac{1}{2}(x+x_n)) = h(\frac{1}{2}(x+\overline{x}))$. This shows that $\widetilde{S} \in \mathfrak{M} \Rightarrow \widetilde{S} \subseteq \overline{S}$ and $\overline{x} \in \overline{S}$. This contradicts the fact $M = h(\overline{x}) \notin R(h|\overline{S})$. Then there is a $u \in \overline{S}$ such that $h(u) = M$. Put $\widetilde{\widetilde{S}} = \overline{S} \cup \{f(u)\}$. Of course $\widetilde{\widetilde{S}}$ satisfies the condition c). Let $x \in \widetilde{\widetilde{S}}$, $h(x) < h(f(u))$, then $x \in \overline{S}$. If $x = x_0$, then of course $P(x_0,u)$ and $h(x_0) \leq h(\frac{1}{2}(x_0+f(u))) \leq \frac{1}{2}(h(x_0)+h(f(u))) \Rightarrow h(f(u)) \geq h(x_0)$ and by assumption 1) we have $P(x_0,f(u))$.

If $x \neq x_0$, then either $x = f(z)$ for a $z \in \overline{S}$ or $x = \lim f(z_n)$, $h(f(z_n)) \uparrow h(x)$ for a sequence $(z_n) \subseteq \overline{S}$.

1) Let $x = f(z)$ for a $z \in \overline{S}$, $x \neq z$, then $h(z) < h(x) \leq h(u)$. By the conditions a),b) we have $P(z,u)$ and $h(\frac{1}{2}(z+u)) \geq h(z)$. By assumption 2) it follows that $P(x,f(u))$ and $\frac{1}{2}(h(f(u)) + h(x)) \geq h(\frac{1}{2}(x+f(u))) \geq h(x) \Rightarrow h(f(u)) \geq h(x)$.

2) If $x = \lim f(z_n): h(f(z_n)) \uparrow h(x)$ for a sequence $(z_n) \subseteq \overline{S}$, then $P(z_n,u)$ and $h((\frac{1}{2}(u+z_n)) \geq h(z_n)$. By assumption 2) we have $P(f(z_n),f(u))$ and $\frac{1}{2}(h(f(u)) + h(f(z_n))) \geq h(\frac{1}{2}(f(u) + f(z_n)) \geq h(f(z_n))$. Since P is closed and h is continuous, it follows that: $P(x,f(u))$ and $\frac{1}{2}(h(f(u)) + h(x)) \geq h(\frac{1}{2}(f(u) + x)) \geq h(x) \Rightarrow h(f(u)) \geq h(x)$.

This proves that $P(x,f(u))$, $h(\frac{1}{2}(f(u) + x)) \geq h(x)$, $h(f(u)) \geq h(x)$ for all $x \in \overline{S}$, especially for $x = u$.

Now let $x \in \tilde{\bar{S}}$, $h(x) < h(f(u))$. Then $x \in \bar{S}$. If $x \neq u$, then $h(x) < h(u)$. Since $\bar{S} \in \mathfrak{M}$, we have $h(f(x)) > h(x)$, $h(f(x)) \leq h(u) \leq h(f(u))$. This proves that $\tilde{\bar{S}}$ satisfies the conditions a), b), too, and $\tilde{\bar{S}} \in \mathfrak{M}$. Therefore $\tilde{\bar{S}} \subseteq \bar{S} \Rightarrow f(u) \in \bar{S}$ and $h(f(u)) = h(u)$. By Lemma 1) $f(u) = u$. This completes the proof of the theorem. For the sake of completeness we include the following

Lemma 8. Let X be a uniformly convex Banach space, D a convex bounded subset of X, then the function $h(x) = \|x\|^2$ is uniformly convex, continuous and bounded on D.

Proof: The boundedness and the continuity of h are obvious.

Now without loss of generality we can suppose that D is contained in the unit ball $B_1(0)$ of X. Suppose that h is not uniformly convex, then there exist an $\varepsilon > 0$ and subsequences (x_n), $(y_n) \subseteq D$ such that: $\|\frac{1}{2}(x_n+y_n)\|^2 \geq \frac{1}{2}(\|x_n\|^2 + \|y_n\|^2) - \frac{1}{n}$ for all $n = 1, 2, \ldots$. We can suppose that $a = \lim \|x_n\| \geq \lim \|y_n\| = b$. Put $\lambda_n = \|y_n\|(\|x_n\|)^{-1}$, then $\lim \lambda_n = \lambda = ba^{-1}$.

1) Let $\lambda < 1$, then $\|\frac{1}{2}(x_n+y_n)\| \leq \frac{1}{2}(\|x_n\| + \|y_n\|) = \frac{1}{2}(1+\lambda_n)\|x_n\|$. By assumption it follows that:

$$-\frac{1}{n} + \frac{1}{2}(1+\lambda_n)^2 \|x_n\|^2 \leq \|\frac{1}{2}(x_n+y_n)\|^2 \leq \frac{1}{4}(1+\lambda_n)^2 \|x_n\|^2.$$

Taking limit we have a contradiction: $\frac{1}{4}(1-\lambda)^2 \leq 0$.

2) Let $\lambda = 1$. We can suppose that $\|x_n - \lambda_n y_n\| \geq \frac{1}{2}\varepsilon$ for all n. Then $\|x_n\| = \|\lambda_n y_n\| > \frac{1}{4}\varepsilon$. Of course

$\|(\|x_n\|)^{-1}x_n - (\|y_n\|)^{-1}y_n\| = (\|x_n\|)^{-1}\|x_n - \lambda_n y_n\| > (2\|x_n\|)^{-1}\cdot\varepsilon > \frac{1}{2}\varepsilon$. By the uniform convexity of X there exists a $\delta > 0$ such that $\|(2\|x_n\|)^{-1}(x_n - \lambda_n y_n)\| < 1 - \delta$.

By assumption it follows that:

$-\frac{1}{n} + \frac{1}{2}(1+\lambda_n^2)\|x_n\|^2 \leq (\|(2\|x_n\|)^{-1}(x_n+\lambda_n y_n)\| \ \|x_n\| +$

$+ (1-\lambda_n)\frac{1}{2}\|y_n\|)^2 \leq [(1-\delta)\|x_n\| + (1-\lambda_n)\frac{1}{2}\|y_n\|]^2.$

Then $0 < a^2 < (1-\delta)^2 a^2$, a contradiction. This proves that h is uniformly convex.

Corollary 1. Let $0 \in D$ be a bounded closed subset of a uniformly convex Banach space, P a reflexive closed relation on D. Let $f: D \to D$ be a map such that:

1) if $x \in D$, $P(0,x)$, then $P(0,f(x))$

2) if $x,y \in D$; $P(x,y)$ and $\|\frac{1}{2}(x+y)\| \geq \|x\|$,

then $P(f(x),f(y))$ and $\|\frac{1}{2}(f(x)+f(y))\| \geq \|f(x)\|$.

Then f has a fixed point.

Now if the relation P is defined by $P(x,y)$ for all $x,y \in D$, then we have:

Corollary 2. Let D be a closed subset of a Banach space, $h: \mathrm{conv}\, D \to R_+$ a uniformly convex continuous bounded function attaining its minimum at $x_0 \in D$. Suppose that $f: D \to D$ is a map such that if $x,y \in D$, $h(\frac{1}{2}(x+y)) \geq h(x)$, then $h(\frac{1}{2}(f(x) + f(y))) \geq h(f(x))$. Then f has a fixed point.

If the relation P on D is defined by: $P(x,y)$ if and only i $h(\lambda x + (1-\lambda)y) \geq h(x)$ for all $\lambda \in [0,1]$ then we have:

Corollary 3. Let D, h be as in Corollary 2 and $f: D \to D$ map such that: if $x,y \in D$, $h((1-\lambda)x + \lambda y) \geq h(x)$, then $h((1-\lambda)f(x) + \lambda f(y)) \geq h(f(x))$ for all $\lambda \in [0,1]$. Then f has a fixed point.

All notions concerning Banach lattices used here are standard, we refer the reader for instance to [6].

Corollary 4. Let X be a uniformly convex Banach lattice, $0 \in D$ a closed, bounded subset of the positive cone C^+ of X. L $f: D \to D$ be a map such that: if $x,y \in D$, $x \leq y$, then $f(x) \leq f(y)$.

Then f has a fixed point.

Proof: It is sufficient to note that if $x, y \in D$ and $x \leq y$, then $\|x\| \leq \|y\|$.

Let X be a Banach space. $L_2^X([0,1])$ denotes the Lebesgue space of all strongly measurable functions $x:[0,1] \to X$ such that:

$$\|x\|_{L_2} = \left(\int_0^1 \|x(t)\|^2 dt \right)^{\frac{1}{2}} < \infty .$$

Lemma 9. Let X be a uniformly convex Banach lattice, $D = \{x \in L_2^X([0,1]) : \|x(t)\|_X \leq K$ for all $t \in [0,1]\}$ for some positive number K, then the function $h(x) = \|x\|_{L_2}^2$ is uniformly convex on D.

Proof: Let ε be a given positive number, $x, y \in D$ such that $\|x-y\|_{L_2} > \varepsilon$. Put $I = [0,1]$; $A = \{t \in I, \|x(t)-y(t)\|_X \geq \frac{1}{2} \varepsilon\}$.

Then $\int_0^1 \|x(t)-y(t)\|^2 dt \leq \int_A \|x(t)-y(t)\|^2 dt + \int_{I \setminus A} (4)^{-1} \cdot \varepsilon^2 dt <$

$< 4K^2 \mu(A) + \frac{1}{4} \varepsilon^2 \Longrightarrow \mu(A) \geq \frac{3}{16} \varepsilon^2$.

By Lemma 8, there exists a $\delta > 0$ such that:

$\|\frac{1}{2}(x(t)+y(t))\|^2 \leq \frac{1}{2}(\|x(t)\|^2 + \|y(t)\|^2) - \delta$ for all $t \in A$.

It follows that:

$$\|\tfrac{1}{2}(x+y)\|_{L_2}^2 \leq \int_A \|\tfrac{1}{2}(x(t)+y(t))\|^2 dt + \int_{I \setminus A} \|\tfrac{1}{2}(x(t)+y(t))\|^2 dt \leq$$

$$\leq \tfrac{1}{2} \int_A (\|x(t)\|^2 + \|y(t)\|^2 - \delta) dt + \int_{I \setminus A} \tfrac{1}{2}(\|x(t)\|^2 + \|y(t)\|^2) dt \leq$$

$$\leq \tfrac{1}{2}(\|x\|_{L_2}^2 + \|y\|_{L_2}^2) - \tfrac{3}{16} \cdot \varepsilon^2 .$$

This ends the proof of Lemma 9.

Now we consider the Cauchy problem of differential equation in Banach lattice X:

$$\text{(I)} \qquad \begin{cases} \dot{x} = f(t,x) \\ x(0) = x_0 \end{cases}$$

where $f:[0,1]\times X\to X$ satisfies the Carathéodory conditions, i.e.:

1) $f(t,\cdot)$ is continuous for a.e. $t\in[0,1]$,

2) $f(\cdot,x)$ is strong measurable for every $x\in X$.

We say that (I) has a solution, if there exists a continuous function $x:[0,1]\to X$ such that: $x(t)=x_0+\int_0^t f(s,x(s))ds$ for all $t\in[0,1]$.

Corollary 5. Let X be a uniformly convex Banach lattice, $f:[0,1]\times X\to X$ satisfies the Carathéodory conditions, and:

1) there is a function $\beta(t)\in L_1([0,1])$ such that $\|f(t,x)\|\le\beta(t)$ for all $t\in[0,1]$; $x\in X$,

2) $0\le f(t,x)\le f(t,y)$ if $0\le x\le y$; $t\in[0,1]$.

Then for each $x_0\in C^+$ the problem (I) has a solution.

Proof. Put $D=\{x\in L_2^X([0,1]):x(t)\ge 0$ and $\|x(t)\|_X\le\|x_0\|+\int_0^1\beta(t)dt$ for all $t\in[0,1]\}$, $F_f(x)(t)=x_0+\int_0^t f(s,x(s))ds$ for $x\in D$, $t\in[0,1]$. One can verify that $F_f:D\to D$ and $F_f(x)\le F_f(y)$ if $x,y\in D$; $x\le y$. Now we define a relation P on D such that $P(x,y)$ if and only if $x\le y$. Put $h(x)=\|x\|_{L_2}^2$. By Lemma 9, h is a uniformly convex continous bounded function on $D\ni 0$. If $x,y\in X$, $x\le y$, then $\frac{1}{2}(x+y)\ge x$ and $\|\frac{1}{2}(x+y)\|^2\ge\|x\|^2$. Therefore if $x,y\in D$, $x\le y$, then $F_f(x)\le F_f(y)$, $\frac{1}{2}(F_f(x)+F_f(y))\ge F_f(x)$ and $\|\frac{1}{2}(F_f(x)+F_f(y))\|^2\ge\|F_f(x)\|^2$. By the theorem F_f has a fixed point $\bar{x}\in D$. It is easy to see that $\bar{x}$ is a solution of (I).

R e f e r e n c e s

[1] H. BRÉZIS, F. BROWDER: A general ordering principle in nonlinear functional analysis, Advances in Math. 21 (1976), 355-364.

[2] F. BRØNSTED: Fixed points and partial orders, Proc. Amer. Math. Soc. 60(1978), 365-366.

[3] J. CARISTI: Fixed point theorems for mapping satisfying inwardness conditions, Trans. Amer. Math. Soc. 215 (1976), 241-251.

[4] I. EKELAND: Nonconvex minimization problems, Bull. Amer. Math. Soc. (New Series) 1(1979), 443-474.

[5] B. FUCHSTEINER: Iterations and fixed points, Pacific J. Math. 68(1977), 73-79.

[6] H. SCHAEFER: Banach lattices and positive operators, Springer-Verlag, New York (1974).

Math. Dept. of Economical Institute, Hanoi, Vietnam

(Oblatum 28.5. 1984)