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A FIXED POINT RESULT OF SEGHAL-SMITHSON TYPE

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Abstract: A maximality principle due to the author is used to obtain a metrical generalization of a Sehgal-Smithson fixed point result involving multivalued mappings.

Key words: Fixed point, multivalued mapping, maximal element, contractivity condition, cluster point, inward set.

Classification: 54H25

Let (X,d) be a complete metric space. Indicating by 2^X the family of all nonempty subsets of X and by $C(X)$ the class of all compact Y in 2^X, let $d(\cdot,Y)$ stand for the usual distance function from the points of X to the element Y of 2^X and $p(Y)(\cdot)$ the associated projection function from X to $Y\in C(X)$ given by

$$p(Y)(x) = \{y\in Y; d(x,Y) = d(x,y)\},\ x\in X.$$

Suppose Y is a nonempty subset of X and let the (multivalued) mapping T from Y to $C(X)$ be given. After a terminology of Sehgal and Smithson [8] we shall say T is a weak directional contraction if a number k in $(0,1)$ may be found such that, for each $z\in Y$ there exists u in $p(Tz)(z)$ with $DT(z,u)<k$, where

$$DT(z,u) = 0 \text{ if } z = u;\ \infty \text{ if } Y(z,u] = \emptyset$$
$$= \inf\ H(Tz,Tw)/d(z,w); w\in Y(z,u)\ ,\ \text{if } Y(z,u)\neq\emptyset$$

(H being the usual(generalized) Hausdorff metric on 2^X and $Y(z,u]$ the subset of all $w\in Y$ distinct from z with the property $d(z,w) + d(w,u) = d(z,u)$) and a directional contraction if $k = 1$

in this definition. The following result established by Sehgal and Smithson in the above quoted paper may be considered as the start point of our developments.

<u>Theorem 1</u>. Suppose Y is closed and T is such that

(i) the function $\varphi = \varphi_T$ from Y to $[0,\infty)$ defined by $\varphi(x) = d(x,Tx)$, $x \in Y$, is lower semicontinuous.

If, in addition, either of the following conditions holds

(ii) T is a weak directional contraction

(iii) T is a directional contraction and each sequence (z_n) in Y with $DT(z_n,w_n) \longrightarrow 1$ for some (w_n) in X fulfilling $w_n \in p(Tz_n)(z_n)$, $n \in N$, has a cluster point

the considered mapping T has at least a fixed point (in Y).

Concerning the first part of this result (which extends a similar one due to Husain and Sehgal [5]) it immediately follows by definition that (ii) may be written as

(ii)′ for each z in Y not belonging to Tz there exists u in $p(Tz)(z)$ with $H(Tz,Tw) < k\cdot d(z,w)$ for some w in $Y(z,u]$.

In this context, let us note that, z,u,w being as above

(1) $d(z,w) + d(w,Tz) = d(z,Tz)$

because $d(z,w) + d(w,u) = d(z,u) = d(z,Tz)$ gives at once

$d(w,u) = d(z,Tz) - d(z,w) \leq d(z,u') - d(z,w) \leq d(w,u')$, $u' \in Tz$

that is, $d(w,u) \leq d(w,Tz)$ and since the reverse inequality $(d(w,u) \geq d(w,Tz))$ also holds, our claim is proved; observing that the reciprocal of (ii)′ (given z and w satisfying (1) an element u in $p(Tz)(z)$ may be determined with $w \in Y(z,u]$) is not valid - as simple examples show - when the range of T is not in C(X), it is clear that the following condition strictly includes, in general, the above one (ii)'' for each z in Y not belonging to Tz there exists w in $Y(z,Tz]$ with $H(Tz,Tw) < k \cdot d(z,w)$

(where, given A in 2^X we denoted, for each $z \in X$

(2) $Y(z,A] = \{w \in Y;\ w \neq z,\ d(z,w) + d(w,A) = d(z,A)\}$)

so that, a natural problem is whether a replacement of (ii) by (ii)'' would produce the same conclusion in the statement of Theorem 1 (the first part). At the same time, noting that, under the acceptance of (ii)'' with $k = 1$, a more general formulation of (iii) is

(iii)' any sequence (z_n) in Y for which $z_n \notin Tz_n$, $n \in N$, $H(Tz_n,Tw_n) < d(z_n,w_n)$, $n \in N$, and $H(Tz_n,Tw_n)/d(z_n,w_n) \to 1$ for some (w_n) in X fulfilling $w_n \in Y(z_n,Tz_n]$, $n \in N$, has a cluster point

is again of interest to ask whether a substitution of (iii) by (iii)' leads us to the same conclusion about the fixed points of T. The answer to both questions is positive (as we shall see below) and is based on a maximality principle stated by the author in [12]. Some further aspects of the problem will be discussed in a future paper.

Let (V,d) be a metric space. Given the ordering $\leq$ on V, let us call the sequence (v_n) in V, ascending provided that $v_i \leq v_j$ whenever $i \leq j$, and the element v in V, maximal when $v \leq w \in V$ implies $v = w$. The following Zorn principle obtained by the author in the above quoted reference will be in effect in the sequel.

<u>Theorem 2</u>. Suppose that

(iv) any ascending sequence in V is a Cauchy sequence bounded from above.

Then, to every v in V there corresponds a maximal element w in V such that $v \leq w$.

An interesting particular form of this theorem (which, under its quasi-metric variant [10] appears as a generalization of the well-known Brésis-Browder ordering principle [2]) can be stated along the following lines. Let (X,d) be a metric space. Given the function $\varphi: X \to [0,\infty)$ call the subset Y of X, φ-closed when any convergent sequence (y_n) in Y with $(\varphi(y_n))$ decreasing has its limit in Y too, and φ-complete provided each Cauchy sequence (y_n) in Y with $(\varphi(y_n))$ decreasing converges (in X); at the same time, let us call the ambient function φ, Y-self-lsc when for each sequence (y_n) in Y with $(\varphi(y_n))$ decreasing, $y_n \to$ $\to y \in Y$ and $\varphi(y_n) \leq t$ for all $n \in N$, we have $\varphi(y) \leq t$. Observe that if these properties involving Y and φ are verified, the condition (iv) holds in the structure $(Y,d,\leq)$ where $\leq$ is the ordering on X defined by the convention

$$x \leq y \text{ if and only if } d(x,y) \leq \varphi(x) - \varphi(y)$$

and therefore, we have (see also the above quoted author's paper).

<u>Theorem 3</u>. Let the couple Y in 2^X and $\varphi: X \to [0,\infty)$ be such that Y is φ-closed and φ-complete while φ is Y-self-lsc. Then, to every y in Y there corresponds z in Y with the properties (a) $d(y,z) \leq \varphi(y) - \varphi(z)$, (b) $d(z,w) > \varphi(z) - \varphi(w)$ for all $w \in Y$, $w \neq z$.

Under these preliminaries, let (X,d) be a metric space. It will be sufficient in the sequel to work with the (generalized) Hausdorff pseudo-metric D on 2^X defined as

$$D(Y,Z) = \sup\{d(y,Z); y \in Y\}, \quad Y,Z \in 2^X$$

rather than the (generalized) Hausdorff metric H (observe at this moment that, $B(X)$ indicating the class of all bounded Y in 2^X, we have

$$H(Y,Z) = \max\{D(Y,Z),D(Z,Y)\}, \quad Y,Z \in B(X)$$

as well as (by standard computations)

(3) $\quad d(x,Z) \leq d(x,Y) + D(Y,Z),\ x \in X,\ Y,Z \in B(X))$.

Let Y be a nonempty subset of X and $T: Y \longrightarrow B(X)$ a (multivalued) map. As basic assumptions about the couple (Y,T) we shall admit that

(v) Y is T-convex (for each z in Y not belonging to Tz the subset Y(z,Tz] defined as in (2) is not empty)

and moreover (letting $\varphi = \varphi_T$ from X to $[0,\infty)$ be introduced as in (i) with $\varphi = 0$ outside Y)

(vi) Y is both φ-closed and φ-complete

(vii) φ is Y-self-lsc.

Concerning the problem of extending Theorem 1 is the sense we already precised, the first main result of the present note is

Theorem 4. Assume that, in addition to the above hypotheses, an lsc strictly increasing function $f: [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0$, $f(t) > t$, $t > 0$ (so, $f(\infty) = \infty$) may be found with the property

(viii) for each z in Y not belonging to Tz there exists w in Y(z,Tz] with

$$f(D(Tz,Tw) + t) \leq f(d(z,w) + t) - d(z,w),\ 0 \leq t < \varphi(z).$$

Then, T has at least a fixed point (in Y).

Proof. Let the function $\psi: X \longrightarrow [0,\infty)$ be defined as

$$\psi(x) = f(\varphi(x)),\ x \in X.$$

As $(\psi(y_n))$ is decreasing if and only if $(\varphi(y_n))$ has such a property, it is clear that (vi) holds with φ replaced by ψ. Furthermore, let $g: [0,\infty) \longrightarrow [0,\infty)$ be defined by the convention

$$g(t) = \sup \{s \geq 0;\ f(s) \leq t\},\ t \geq 0.$$

The fact that g is well defined (and increasing on $[0,\infty)$)

results in essence from $f(\infty) = \infty$. We also note the equivalence

(4) $f(s) \leq t$ if and only if $s \leq g(t)$

obtained from the property

$$f(g(t)) \leq t, \quad t \geq 0,$$

which is immediate in view of the fact that f is lsc on its existence domain. Now, if (y_n) is a sequence in Y with $(\psi(y_n))$ decreasing, $y_n \to y \in Y$ and $\psi(y_n) \leq t$, $n \in N$, we have by (4), $\varphi(y_n) \leq g(t)$, $n \in N$, so that $\varphi(y) \leq g(t)$ which, again by (4), gives $\psi(y) \leq t$; in other words, (vii) also holds with φ replaced by ψ . In this case, Theorem 3 being applicable, given y in Y there exists z in Y with

(a) $d(y,z) \leq \psi(y) - \psi(z)$

(b) $d(z,w) > \psi(z) - \psi(w)$ for all $w \in Y$, $w \neq z$.

Suppose z is not belonging to Tz and let $w \in Y(z,Tz]$ (which is not empty, by (v)). We have by (b) plus the inequality (3)

$$d(z,w) > f(d(z,Tz)) - f(d(w,Tw)) \geq f(d(z,w) + d(w,Tz)) - f(d(w,Tz) + D(Tz,Tw))$$

that is

$$f(D(Tz,Tw) + d(w,Tz)) > f(d(z,w) + d(w,Tz)) - f(d(z,w))$$

while

$$0 \leq d(w,Tz) < d(z,Tz) = \varphi(z)$$

which contradicts (viii). So, z belongs to Tz and the result follows. Q.E.D.

As a basic particular case, let $f(t) = ht$, $t \geq 0$, for some $h > 1$; then, clearly (viii) reduced to (ii)'' with $k = (h-1)/h$ and Theorem 4 becomes Theorem 1 (the first part). Another particular case of practical interest is $f(t) = e^t - 1$, $t \geq 0$; that it does not reduce to the preceding one is a consequence of the fact that if $a,b>0$ satisfy (for some $t(a,b)>0$)

$$f(b+t)\leq f(a+t)-a,\ 0\leq t<t(a,b)$$

then, a relation like

$$b\leq k\cdot a \text{ for some } k \text{ in } (0,1) \text{ independent of } a,\ b$$

does not hold since, otherwise, its immediate consequence

$$f(ka)\leq f(a)-a \text{(with } a,\ k \text{ sufficiently close to } 1)$$

would produce a contradiction as it can be readily verified. Returning to the first choice, it is clear that the case $h\to 1$, when (viii) becomes

(ix) for each z in Y not belonging to Tz there exists w in $Y(z,Tz]$ with $D(Tz,Tw)<d(z,w)$

cannot be handled by these procedures so, it would be of interest to find out under what supplementary assumptions (constructed after the model of (iii)$'$) conclusion of Theorem 4 continues to hold. An appropriate answer to this question is contained in the following second main result of this note.

Theorem 5. With the same general assumptions like before, let us admit that (ix) replaces (viii) and, in addition

(x) each sequence (z_n) in Y with $z_n\notin Tz_n$, $n\in N$, $(\varphi(z_n))$ decreasing, $D(Tz_n,Tw_n)<d(z_n,w_n)$, $n\in N$, and $D(Tz_n,Tw_n)/d(z_n,w_n)\to 1$ for some (w_n) in X fulfilling $w_n\in Y(z_n,Tz_n]$, $n\in N$, has a cluster point.

Then, T has at least a fixed point (in Y).

Proof. Suppose by contradiction T has no fixed points in Y. Given $z_o\in Y$ arbitrary fixed, let us apply Theorem 3 with φ introduced by the procedure we already indicated; there exists then a point z_1 in Y with

$$(a)_1\quad d(z_o,z_1)\leq\varphi(z_o)-\varphi(z_1)$$

(b)$_1$ $d(z_1,w) > \varphi(z_1) - \varphi(w)$ for all $w \in Y$, $w \neq z_1$.

Furthermore, given z_1 in Y let us again apply Theorem 3 with φ replaced by 2φ ; then, a $z_2 \in Y$ may be found with

(a)$_2$ $d(z_1,z_2) \leq 2(\varphi(z_1) - \varphi(z_2))$

(b)$_2$ $d(z_2,w) > 2(\varphi(z_2) - \varphi(w))$ for all $w \in Y$, $w \neq z_2$,

and so on. By induction, we get a sequence (z_n) in Y with $z_n \notin Tz_n$, $n \in N$, $(\varphi(z_n))$ decreasing and (for each $n \in N$)

(a)$_n$ $d(z_{n-1},z_n) \leq n(\varphi(z_{n-1}) - \varphi(z_n))$

(b)$_n$ $d(z_n,w) > n(\varphi(z_n) - \varphi(w))$ for all $w \in Y$, $w \neq z_n$.

Let (w_n) in Y be such that $w_n \in Y(z_n,Tz_n]$, $n \in N$. We have by (b)$_n$ plus (3)

$$d(z_n,w_n) > n(d(z_n,Tz_n) - d(w_n,Tw_n)) \geq n(d(z_n,w_n) + d(w_n,Tz_n) - d(w_n,Tz_n) - D(Tz_n,Tw_n)) = n(d(z_n,w_n) - D(Tz_n,Tw_n)),\ n \in N,$$

that is

$$D(Tz_n,Tw_n) > (1-1/n)\ d(z_n,w_n),\ n \in N.$$

In particular, letting (w_n) above be taken as in (ix) it follows from this relation that

$$1 - 1/n < D(Tz_n,Tw_n)/d(z_n,w_n) < 1,\ n \in N$$

and therefore $D(Tz_n,Tw_n)/d(z_n,w_n) \longrightarrow 1$, which, in view of (x) gives us (eventually on a subsequence)

$$z_n \longrightarrow z \text{ for some } z \text{ in } Y$$

(this last property being an immediate consequence of (vi)).

Again by the evaluations (b)$_n$,

$$d(z_n,Tz_n) \leq d(w,Tw) + (1/n)d(z_n,w),\ n \in N,\ w \in Y$$

so that, passing to the limit as $n \longrightarrow \infty$, one gets (taking (vii) into account)

(4) $d(z,Tz) \leq d(w,Tw)$, for all w in Y.

Combining this with (3) we have

$$d(z,w) + d(w,Tz) \leq d(w,Tz) + D(Tz,Tw),\ w \in Y(z,Tz]$$

that is

$d(z,w) \leq D(Tz,Tw)$, for each w in $Y(z,Tz]$

contradicting (ix). So, the supposition T has no fixed points is false and this ends our argument. Q.E.D.

Now, clearly, (ix) extends (iii)′ so that, correspondingly, Theorem 5 may be deemed as a generalization of Theorem 1 (the second part). This fact, combined with the discussion following the preceding statement allows us to conclude that the initial program of extending Theorem 1 modulo the couple (ii)″/(iii)′ has been accomplished.

Concerning the basic T-convexity hypothesis (v) about Y, the following remarks are in order. Let $J_Y(z)$ denote (for each element z of Y) the (eventual empty) subset of all u in X with the property $Y(z,u]$ is not empty; of course, in the linear normed case, this is nothing but the inward set of z with respect to Y in the sense of Caristi [4], whose closed $(-z)$-translate contains the tangent cone $K_Y(z)$ of z with respect to Y (see, e.g., Penot [7]) or, equivalently, the asymptotic direction set $A_Y(z)$ of z with respect to Y in Browder′s sense [3]. It is now clear that, if the range of T is in $C(X)$, (v) may be clearly deduced from the stronger hypothesis

(v)′ $p(Tz)(z) \cap J_Y(z)$ is not empty for each z in Y, not belonging to Tz

(this fact being a consequence of the reasoning we used in the proof of (1)) and consequently, our main results could be also viewed as a straightforward (multivalued) extension of Caristi′s fixed point theorems (cf. the above reference). It is interesting to note at this moment that, still assuming (v)′ is to be satisfied, conclusion of Theorem 4 remains valid in case (viii) would be replaced by the following hypothesis (viii)$'$ for each z in Y, not belonging to Tz there exists u in $p(Tz)(z)$ and w in $Y(z,u]$ such that

$$f(d(u,Tw) + t) \leq f(d(z,w) + t) - d(z,w),\ 0 \leq t < \varphi(z);$$

indeed, it suffices to observe that relation (b) of the above quoted theorem

$$d(z,w) > f(d(z,Tz)) - f(d(w,Tw)),\ w \in Y,\ w \neq z$$

gives at once (if we take $u \in p(Tz)(z)$ and $w \in Y(z,u]$ as in (viii)$'$)

$$d(z,w) > f(d(z,w) + d(w,u)) - f(d(w,u) + d(u,Tw))$$

that is,

$$f(d(u,Tw) + t) > f(d(z,w) + t) - d(z,w),$$

where $0 \leq t = d(w,u) < \varphi(z)$, contradicting (viii)$'$ and proving our claim. In the same context, condition (ix) being substituted by the following one

(ix)$'$ for each z in Y, not belonging to Tz there exists u in $p(Tz)(z)$ and w in $Y(z,u]$ with $d(u,Tw) < d(z,w)$

and (x) by

(x)$'$ each sequence (z_n) in Y with $z_n \notin Tz_n$, $n \in N$, $(\varphi(z_n))$ decreasing, $d(u_n,Tw_n) < d(z_n,w_n)$, $n \in N$, and $d(u_n,Tw_n)/d(z_n,w_n) \to 1$ for some (u_n) in X and (w_n) in Y with $u_n \in p(Tz_n)(z_n)$, $w_n \in Y(z_n,u_n]$, $n \in N$, has a cluster point

conclusion of Theorem 5 will also remain valid. In fact, $(b)_n$ of that result would imply then (taking (u_n) in X and (w_n) in Y in such a way that $u_n \in p(Tz_n)(z_n)$, $w_n \in Y(z_n,u_n]$, $n \in N$)

$$d(z_n,w_n) > n\,(d(z_n,Tz_n) - d(w_n,Tw_n)) \geq n\cdot(d(z_n,w_n) + d(w_n,u_n) -$$
$$d(w_n,u_n) - d(u_n,Tw_n)) = n(d(z_n,w_n) - d(u_n,Tw_n)),\ n \in N$$

and this immediately gives (taking (u_n) and (w_n) above as in (ix)$'$)

$$1-1/n < d(u_n,Tw_n)/d(z_n,w_n) < 1,\ n \in N$$

that is, $d(u_n,Tw_n)/d(z_n,w_n) \to 1$. By (x)$'$, $z_n \to z$ (eventually on a subsequence) for some $z \in Y$, in which case, relation (4) (obtained by the same procedure as before) gives us for any couple $u \in p(Tz)(z)$, $w \in Y(z,u]$,

$$d(z,w) + d(w,u) \leq d(w,u) + d(u,Tw)$$

that is, $d(z,w) \leq d(u,Tw)$, a contradiction with respect to (ix)$'$, completing the argument. Of course, when T is univalent, conditions (viii)$'$, (ix)$'$, (x)$'$ will respectively coincide with (viii),(ix),(x) but, in general, they are distinct even in the linear normed case. Returning to the key condition (v)$'$, note that an interpretation of it in terms of variable drops was indicated by Turinici [9] (see also the fixed point approach used in Kirk and Caristi [6]) which allows us to connect our statements with those of Altman [1], based on a contractor directions viewpoint. It seems to be not without interest to formulate a corresponding variant of the above theorems for metrizable uniform structures founded, e.g., on the appropriate variant of Theorem 3 in these structures due to the author [11]; some aspects of this problem will be discussed elsewhere.

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