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ON THE EXISTENCE OF WEAK SOLUTIONS OF A NON LINEAR MIXED PROBLEM FOR NON HOMOGENEOUS FLUIDS IN A TIME DEPENDENT DOMAIN

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Abstract: We consider the flow of a viscous incompressible non-homogeneous fluid in a tube with time dependent boundary where we give physically expressive conditions. We prove the existence of a weak solution of the problem via the Rothe method and an elliptic approximation.

Key words: Non-homogeneous fluid, weak solution, time dependent domain, Rothe method, elliptic approximation, compact set, boundary strip.

Classification: 35Q10

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1. Introduction. In this paper we study a non linear mixed problem for non homogeneous fluids in a non cylindrical domain in $R^3 \times (0,T)$. We deal with the flow of a fluid in a tube with time dependent initial and final sections where we give physically expressive boundary conditions.

In [9] we considered this problem for the Navier-Stokes equations.

We shall prove the existence of a weak solution via the Rothe method and an elliptic approximation. It has to be pointed out that many authors considered problems of this type (see [1],[2],[4],[5],[8]) but our approach is new and works well to

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study more general non linear mixed problems.

The essential point of our proof is the estimate of a time difference quotient. The paper is composed of two sections. In § 2 we describe the problem introduce particular functional spaces and give the definition of weak solutions.

2. Statement of the problem and notations. Let $\Omega(t)$ be an open set of R^3 depending on $t \in (0,T)$, T is a finite positive number. As t increases over $(0,T)$, $\Omega(t)$ generates an (x,t)-domain $\hat{\Omega}$ and the boundary $\Gamma(t)$ of $\Omega(t)$ generates an (x,t)-hypersurface $\hat{\Gamma}$. We assume that $\hat{\Gamma}$ is a C^3-hypersurface and $\Gamma(t) = \Gamma_1(t) \cup \Gamma_2(t) \cup \Gamma_3$ (Γ_3 is independent of t) with mes. $\Gamma_3 \neq 0$. Then we can represent $\hat{\Gamma}$ by $X_1 = \psi(x_1, x_2, t)$ in terms of C^3-functions ψ (in each path of a finite covering of $\hat{\Gamma}$).

The motion of an inhomogeneous incompressible fluid of viscosity 1 subject to the external force $f = f(x,t) = \{f_1(x,t), f_2(x,t), f_3(x,t)\}$ is governed by the equations

$$
(2.1) \qquad \begin{aligned}
&\rho \frac{\partial u}{\partial t} - \Delta u + \rho u \cdot \nabla u + \nabla p = \rho f \\
&\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0 \\
&\nabla \cdot u = 0
\end{aligned}
$$

where $u = u(t) = u(x,t) = \{u_1(x,t), u_2(x,t), u_3(x,t)\}$ denotes the velocity, $\rho = \rho(t) = \rho(x,t)$ the density and $p = p(x,t)$ the pressure. In the first relation of (2.1) $\rho u \cdot \nabla u = \Sigma \rho u_i \partial u / \partial x_i$ and the second is the equation of continuity.

Denoting by ν_t the outside normal to $\Gamma(t)$, we shall consider the boundary and initial conditions defined by the relations

$$\frac{1}{2}\rho|u|^2\nu_t - \frac{1}{2}\rho u\frac{\partial v}{\partial t} + p\nu_t - \frac{\partial u}{\partial \nu_t} = -\alpha \text{ on } ((\Gamma_1(t)\cup\Gamma_2(t)),t)$$

(2.2) $$u = 0 \qquad \text{on } \Gamma_3\times(0,T)$$

$$\rho(x,0) = \rho_0, \quad u(x,0) = u_0 \text{ in } \Omega(0).$$

The first relation in (2.2) determines the value of the density of the energy flux of the fluid on $\Gamma_1(t) \cup \Gamma_2(t)$. The other conditions are standard.

We shall give the weak formulation of the problem (2.1), (2.2). Let us begin by giving some definitions and basic notations.

Let Ω be a bounded open set in R^3 with boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We will need the following function spaces.

$$D(\Omega) = \{\varphi \mid \varphi \in (C^\infty(\bar{\Omega}))^3,\ \varphi = 0 \text{ on } \Gamma_3,\ \nabla\cdot\varphi = 0\}$$
$$H(\Omega) = \{\text{the completion of } D(\Omega) \text{ under the } (L^2(\Omega))^3\text{-norm}\}$$
$$V(\Omega) = \{\text{the completion of } D(\Omega) \text{ under the } (H^1(\Omega))^3\text{-norm}\}$$

We let

$$(u,v)_\Omega = \Sigma\int_\Omega u_i v_i\, dx; \quad |u|^2_\Omega = (u,u)_\Omega$$
$$((u,v))_\Omega = \Sigma\int_\Omega \nabla u_i\cdot\nabla v_i\, dx;\ \|u\|^2_\Omega = ((u,u))_\Omega$$
$$(u,v)_\Gamma = \Sigma\int_\Gamma u_i v_i\, dx;\ \|u\|_{\mathcal{L}} = \text{norm in } \mathcal{L}.$$

Let $\hat{\Omega}$ be the (x,t)-domain $(\Omega(t),t)$. For functions u defined in $\hat{\Omega}$ we define $\beta(u)$ by

$$\beta(u) = \int_0^T \|u\|^2_{\Omega(t)}\, dt$$

whenever the integral makes sense. Then we introduce

$$D(\hat{\Omega}) = \{\varphi \mid \varphi \in C^\infty(\bar{\hat{\Omega}}))^3,\ \varphi = 0 \text{ on } \Gamma_3,\ \nabla\cdot\varphi = 0\}$$
$$V(\hat{\Omega}) = \{\text{the completion of } D(\hat{\Omega}) \text{ under the norm } \beta(u)\}$$

By $U(\hat{\Omega})$ we mean the set of all $u \in V(\hat{\Omega})$ such that $\sup_t |u|_{\Omega(t)}$ over $(0,T)$ is finite. We set the definition of weak solutions.

(u,ρ) will be a weak solution of the problem (2.1),(2.2) if one has ,

i) $u \in U(\hat{\Omega})$, $\rho \in L^{\infty}(\hat{\Omega})$

ii) $\forall \varphi \in D(\hat{\Omega})$ with $\varphi(T) = 0$

$$(2.3) \quad \int_0^T \{ (\rho u, \frac{\partial \varphi}{\partial t})_{\Omega(t)} + (\rho u, u \cdot \nabla \varphi)_{\Omega(t)} + \\ + \frac{1}{2} (\rho u, \varphi \frac{\partial \psi}{\partial t})_{\Gamma(t)} + \frac{1}{2} (\rho |u|^2 \nu_t, \varphi)_{\Gamma(t)} + (\alpha, \varphi)_{\Gamma(t)} + \\ + (\rho f, \varphi)_{\Omega(t)} - ((u, \varphi))_{\Omega(t)} - (\rho u \nu_t, u \cdot \varphi)_{\Gamma(t)} \} \, dt = \\ = (\rho_0 u_0, \varphi(0))_{\Omega(0)}$$

iii)

$$(2.4) \quad \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0 \quad \text{(in the distribution sense).}$$

(2.3) is obtained from the first part of (2.1) multiplying it by a test function φ integrating over $\hat{\Omega}$ and bearing in mind the equation of continuity and the three conditions of (2.2).

The following theorem holds:

<u>Theorem 1</u>. <u>We assume</u>

$u_0 \in H(\Omega(0))$; $f \in L^2(0,T;H(\Omega(t))$; $0 < R_1 \leq \rho_0 \leq R_2$

(R_1, R_2 <u>are positive constants</u>)

<u>Then there exists a weak solution</u> (u,ρ) <u>of the problem</u> (2.1), (2.2) <u>such that</u> $0 < R_1 \leq \rho \leq R_2$.

3. <u>Proof of the theorem 1</u>. Let us begin by considering the following approximating problem.

3.1 Auxiliary problem. We look for u^m, ρ^m such that $\forall \varphi \in (H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$

$$(3.11)\quad \int_0^T \Big\{ \frac{1}{m} \left(\frac{\partial u^m}{\partial t}, \frac{\partial \varphi}{\partial t}\right)_{\Omega(t)} + ((u^m, \varphi))_{\Omega(t)} +$$

$$+ (\rho^m u^m \cdot \nabla u^m, \varphi)_{\Omega(t)} + (u^m \cdot \nabla \rho^m, u^m \varphi)_{\Omega(t)} -$$

$$- \left(\rho^m, u^m, \frac{\partial \varphi}{\partial t}\right)_{\Omega(t)} - \frac{1}{2m} (\Delta \rho^m u^m, \varphi)_{\Omega(t)} -$$

$$- \frac{1}{2}\left(\rho^m u^m, \varphi \frac{\partial \psi}{\partial t}\right)_{\Gamma(t)} - \frac{1}{2} (|u^m|^2 \cdot \nu_t, \rho^m \varphi)_{\Gamma(t)} -$$

$$- (\alpha, \varphi)_{\Gamma(t)} - (\rho^m f, \varphi)_{\Omega(t)} \Big\} dt = (\rho_o^m u_o^m, \varphi(0))_{\Omega(0)} -$$

$$- (\rho^m(T) u^m(T), \varphi(T))_{\Omega(T)},$$

$$(3.12)\quad \frac{\partial \rho^m}{\partial t} + u^m \cdot \nabla \rho^m = \frac{1}{m} \Delta \rho^m \quad \text{in } \hat{\Omega}$$

$$(3.13)\quad u^m \in (H^1(\hat{\Omega}))^3 \cap H(\Omega);\ \rho^m \in L^\infty(\hat{\Omega}) \cap H^1(\hat{\Omega}) \cap L^2(0,T); H^2(\Omega(t))$$

with

$$\rho_o^m \in C^1(\Omega(0));\ \rho_o^m \longrightarrow \rho_o \text{ in } L^2(\Omega(0));\ \frac{\partial \rho^m}{\partial \nu} = 0 \text{ on } (\Gamma(t), t)$$

$$0 < R_1 \le \rho_o^m \le R_2;\ |\nabla \rho_o^m|^2_{\Omega(0)} \le m;\ u_o^m \longrightarrow u_o \text{ in } L^2(\hat{\Omega}).$$

Assuming u^m to be known and applying the Rothe method one has a solution ρ^m of (3.12) in $\hat{\Omega}$ (as to details see [3]). The following uniform estimates for ρ^m hold.

By the maximum principle we have

$$(3.14)\qquad 0 < R_1 \le \rho^m \le R_2$$

Then multiplying (3.12) by ρ^m and integrating over $\Omega(t)$ one obtains

$$\frac{1}{2} \int_{\Omega(t)} \frac{\partial}{\partial t}(\rho^m)^2\, dx + \frac{1}{m} \|\rho^m\|^2_{\Omega(t)} = \frac{1}{2} \int_{\Omega(t)} u^m \cdot \nabla (\rho^m)^2 dx$$

or

$$\frac{1}{2} |\rho^m|^2_{\Omega(t)} + \frac{1}{m} \int_0^t \|\rho^m\|^2_{\Omega(t)} dt = \frac{1}{2} |\rho_o^m|^2 +$$

$$+ \int_0^t \Big\{ (u^m \cdot \nu_t + \frac{\partial \psi}{\partial t}, (\rho^m)^2)_{\Gamma(t)}\, dt$$

hence

$$\frac{1}{m}\int_0^T \|\varphi^m\|^2_{\Omega(t)}dt < c.$$

Now multiplying (3.12) by $\Delta\varphi^m$ and integrating over $\Omega(t)$, and after the integration by parts one gets

$$\left(\frac{\partial\nabla\varphi^m}{\partial t}, \nabla\varphi^m\right)_{\Omega(t)} + \frac{1}{m}|\Delta\varphi^m|^2_{\Omega(t)} = -(\nabla\varphi^m\cdot\nabla u^m, \nabla\varphi^m)_{\Omega(t)}$$

Using the estimates (3.14),(3.15) and the interpolation inequality (see [7])

$$\|\nabla\varphi^m\|^2_{L^4(\Omega(t))} \le c\,|\Delta\varphi^m|_{\Omega(t)}\|\varphi^m\|_{L^\infty(\Omega(t))}$$

one gets

$$\frac{1}{2}|\nabla\varphi^m|^2_{\Omega(t)} + \frac{1}{m}\int_0^T|\Delta\varphi^m|^2_{\Omega(t)}dt \le \frac{1}{2}|\nabla\varphi^m_o|_{\Omega(0)} -$$

$$-\int_0^T\left(\nabla\varphi^m, \nabla\varphi^m\frac{\partial\psi}{\partial t}\right)_{\Gamma(t)} dt + c$$

From (3.15) one has

$$\frac{1}{m}\int_0^T|\Delta\varphi^m|^2_{\Omega(t)}dt \le c\,|\nabla\varphi^m_o|^2 + c$$

hence

$$(3.16) \qquad \int_0^T|\Delta\varphi^m|^2_{\Omega(t)}dt \le c\, m^2.$$

Next we consider the existence of the solution of (3.11). We set

$$a(\varphi^m u^m, u^m, \varphi) = \int_0^T\Big\{\frac{1}{m}\left(\frac{\partial u^m}{\partial t}, \frac{\partial\varphi}{\partial t}\right)_{\Omega(t)} +$$

$$+ (\varphi^m u^m\cdot\nabla u^m, \varphi)_{\Omega(t)} + (u^m\cdot\nabla\varphi^m, u^m\varphi)_{\Omega(t)} - \frac{1}{2}(\varphi^m|u^m|^2\cdot\nu_t, \varphi) -$$

$$-\frac{1}{2}\left(\varphi^m u^m, \varphi\frac{\partial\psi}{\partial t}\right)_{\Gamma(t)} - \left(\varphi^m u^m, \frac{\partial\varphi}{\partial t}\right)_{\Omega(t)} - \frac{1}{2m}(\Delta\varphi^m u^m, \varphi)_{\Omega(t)} +$$

$$+ ((u^m, \varphi))_{\Omega(t)}\Big\}\, dt + (\varphi^m(T)u^m(T), \varphi(T))_{\Omega(T)}$$

$$\langle L, \varphi\rangle = \int_0^T\{(\varphi^m f, \varphi)_{\Omega(t)} + (\alpha, \varphi)_{\Gamma(t)}\}dt + (\varphi^m_o u^m_o, \varphi(0))_{\Omega(0)}.$$

By the following well known theorem one obtains the existence of a solution in $(H^1(\hat{\Omega}))^3$ of the equation

(3.17) $\quad a(\rho^m u^m, u^m, \varphi) = \langle L, \varphi \rangle$

(for convenience we denote different constants by the same symbol c).

<u>Theorem 2.</u> If

i) <u>there exists a constant</u> $c > 0$ <u>such that</u>

$$a(\rho^m u^m, u^m, u^m) \geq c \, \|u^m\|^2_{(H^1(\hat{\Omega}))^3}$$

ii) <u>the form</u> $u^m \longrightarrow a(\rho^m u^m, u^m, \varphi)$ <u>is weakly continuous in</u> $H^1(\hat{\Omega})$ i.e.

$u^m_n \rightarrow u^m$ <u>weakly in</u> $(H^1(\hat{\Omega}))^3$ <u>implies</u>

$$\lim_{m \to \infty} a(\rho^m u^m, u^m_n, \varphi) = a(\rho^m u^m, u^m, \varphi).$$

<u>Then</u> (3.17) <u>has a solution in</u> $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$.

The condition ii) is obvious. The condition i) can be easily proved; in fact

$$a(\rho^m u^m, u^m, u^m) = \int_0^T \left\{ \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|^2_{\Omega(t)} + \|u^m\|^2_{\Omega(t)} \right\} dt +$$

$$+ \frac{1}{2} \left| \sqrt{\rho^m(T)} u^m(T) \right|^2_{\Omega(T)} + \frac{1}{2} \left| \sqrt{\rho^m_0} u^m(0) \right|^2_{\Omega(0)} \geq c \, \|u^m\|_{H^1(\hat{\Omega})}$$

Then there exists a solution in $(H^1(\hat{\Omega}))^3 \cap H(\hat{\Omega})$ of (3.17).

To passing to the limit in (3.11) we will need a priori estimates of the approximations u^m.

3.2. <u>Standard a priori estimates.</u> We can replace in (3.11) φ by u^m, it comes

$$\int_0^T \left\{ \frac{1}{m} \left| \frac{\partial u^m}{\partial t} \right|^2_{\Omega(t)} + \|u^m\|^2_{\Omega(t)} - \left(\rho^m u^m, \frac{\partial u^m}{\partial t}\right)_{\Omega(t)} + \right.$$

$$+ (\rho^m u^m \cdot \nabla u^m, u^m)_{\Omega(t)} + (u^m \cdot \nabla \rho^m, u^m, u^m)_{\Omega(t)} -$$

$$- \frac{1}{2}(\rho^m |u^m|^2 \cdot \nu_t, u^m)_{\Gamma(t)} - \frac{1}{2}\left(u^m \rho^m, u^m \frac{\partial \psi}{\partial t}\right)_{\Gamma(t)} - (\alpha, u^m)_{\Gamma(t)} -$$

$$- \frac{1}{2m} (\Delta\rho^m u^m, u^m)_{\Omega(t)} - (\rho^m f, \varphi)_{\Omega(t)}\Big\} dt = |\sqrt{\rho_0^m} u_0^m|^2_{\Omega(0)} -$$

$$- |\sqrt{\rho^m}(T) u^m(T)|^2_{\Omega(T)}$$

Bearing in mind (3.12), after some calculations, one has

$$\int_0^T \frac{1}{m} \left|\frac{\partial u^m}{\partial t}\right|^2_{\Omega(t)} dt + \int_0^T \|u^m\|^2_{\Omega(t)} dt + |\sqrt{\rho^m}(T) u^m(T)|^2_{\Omega(T)} +$$

$$+ |\sqrt{\rho^m}(0), u^m(0)|^2_{\Omega(0)} < c,$$

hence

$$(3.21) \quad \begin{array}{l} \frac{1}{m} \int_0^T \left|\frac{\partial u^m}{\partial t}\right|^2_{\Omega(t)} dt < c; \quad \int_0^T \|u^m\|^2_{\Omega(t)} dt < c \\ |u^n(T)|^2_{\Omega(t)} < c; \quad |u^m(0)|^2_{\Omega(0)} < c. \end{array}$$

By virtue of (3.14),(3.21) one gets

$$\lim_{m\to\infty} u^m \underset{V(\hat{\Omega})}{=} u \text{ in the weak topology}$$

$$\lim_{m\to\infty} \rho^m \underset{L^\infty(\hat{\Omega})}{=} \rho \text{ in the weak* topology}$$

To passing to the limit in the non linear terms of (3.11), we need the convergence of u_h^m in a suitable strong topology for example in $L^2(\hat{\Omega})$. To do this we will prove appropriate estimates.

3.3. Time difference quotients. We denote by $\bar{u}^m(x,t)$ the extension to R^3 of u^m for every $t \in (0,T)$; moreover, we put $\bar{u} = 0$ for $t<0$, $t>T$. We let

$$u_h^m = \frac{1}{h} \int_{t-h}^t \bar{u}^m(x,s)\, ds \quad (h>0)$$

We can replace in (3.11) φ by u_h^m and get

$$\int_0^T \frac{1}{m} \left(\frac{\partial u^m}{\partial t}, \frac{\bar{u}^m(t) - \bar{u}^m(t-h)}{h}\right)_{\Omega(t)} dt - \frac{1}{h} \int_0^T (\rho^m(t) u^m(t), \bar{u}^m(t) -$$

$$- \bar{u}^m(t-h))_{\Omega(t)} dt + \int_0^T \Big\{ ((u^m, u_h^m))_{\Omega(t)} - (\rho^m u^m, u^m \cdot \nabla u_h^m)_{\Omega(t)} +$$

$$+ (\rho^m u^m \cdot \nu_t, u^m \cdot u_h^m)_{\Gamma(t)} - \frac{1}{2}(\rho^m |u^m|^2 \cdot \nu_t, u_h^m)_{\Gamma(t)} -$$

$$- (\alpha, u_h^m)_{\Gamma(t)} - \frac{1}{2}(\rho^m u^m, u_h^m \quad \frac{\partial \psi}{\partial t})_{\Gamma(t)} - \frac{1}{2m}(\Delta \rho^m u^m, u_h^m)_{\Omega(t)} -$$

$$- (\rho^m f, u_h^m)_{\Omega(t)}\} dt + (\rho^m(T) u^m(T), u_h^m(T))_{\Omega(T)} = 0$$

By virtue of (3.14),(3.21), Jensen inequality and the smoothness of $\hat{\Gamma}$ one has

$$\left| \frac{1}{m} \int_0^T \left(\frac{\partial u^m}{\partial t}, \frac{\bar{u}^m(t) - u^m(t-h)}{h} \right)_{\Omega(t)} dt \right| \leq$$

$$\leq \frac{1}{m} \int_0^T \left| \frac{\partial u^m}{\partial t} \right|_{\Omega(t)} \left| \frac{\bar{u}^m(t) - \bar{u}^m(t-h)}{h} \right|_{\Omega(t)} dt \leq \frac{c}{m} \int^T \left| \frac{\partial u^m}{\partial t} \right|^2_{\Omega(t)} dt \leq c$$

$$\left| \int_0^T ((u^m, \frac{1}{h} \int_{t-h}^t \bar{u}^m(x,s) ds))_{\Omega(t)} dt \right. \leq$$

(3.31) $$\leq c \int_0^T \| u^m \|_{\Omega(t)} \| \frac{1}{h} \int_{t-h}^t \bar{u}^m(x,s) ds \|_{\Omega(t)} dt \leq$$

$$\leq c \int_0^T \| u^m \|_{\Omega(t)} \cdot \frac{1}{\sqrt{h}} \left(\int_{t-h}^t \| \bar{u}^m(s) \|^2_{R^3} ds \right)^{\frac{1}{2}} dt \leq \frac{c}{\sqrt{h}}$$

$$\left| \int_0^T (\rho^m u^m, u^m \cdot \nabla \frac{1}{h} \int_{t-h}^t \bar{u}^m(x,s) ds)_{\Omega(t)} dt \right| \leq$$

$$\leq c \int_0^T \| u^m \|_{\Omega(t)} \| u^m \|_{\Omega(t)} \cdot \frac{1}{\sqrt{h}} \left(\int_{t-h}^t \| \bar{u}^m \|^2_{R^3} ds \right)^{1/2} dt \leq \frac{c}{\sqrt{h}}$$

Analogously one obtains

$$\left| \int_0^T (\rho^m u^m, \frac{\partial \psi}{\partial t})_{\Gamma(t)} dt \leq c/\sqrt{h}; \quad \left| \int_0^T (\alpha, \bar{u}_h^m)_{\Gamma(t)} dt \right| \leq c/\sqrt{h}$$

(3.32) $$\left| \int_0^T \frac{1}{2m}(\Delta \rho^m u^m, u_h^m)_{\Omega(t)} dt \leq c/\sqrt{h}; \quad \left| \int_0^T (\rho^m f, \bar{u}_h^m)_{\Omega(t)} dt \leq c/\sqrt{h}$$

$$\left| \int_0^T (\rho^m |u^m|^2 \cdot \nu_t, u_h^m)_{\Gamma(t)} \leq c/\sqrt{h}; \quad \left| \int_0^T (\rho^m u^m \cdot \nu_t, u^m \cdot u_h)_{\Gamma(t)} dt) \right| < c/\sqrt{h}.$$

Finally we will estimate

$$- \frac{1}{h} \int_0^T (\rho^m(t) u^m(t), \bar{u}^m(t) - \bar{u}^m(t-h))_{\Omega(t)} dt.$$

First we integrate (3.12) from t - h to t and obtain

(3.33) $$\rho(t) - \rho(t-h) = \frac{1}{m} \int_{t-h}^t \Delta \rho^m ds - \int_{t-h}^t \nabla \cdot (\rho^m u^m) ds$$

obviously $x \in C(t) = \bigcap_{s} \Omega(s)$ with $s \in (t-h,t)$. Multiplying (3.33) by $\bar{u}^m \cdot \bar{u}^m/2$ and integrating, one obtains

$$\frac{1}{2}\int_h^T ((\rho^m(t) - \rho^m(t-h))u^m, u^m)_{C(t)}dt =$$

$$(3.34)\quad = \frac{1}{2m}\int_h^T ((\int_{t-h}^t \Delta\rho^m ds)u^m, u^m)_{C(t)}dt + $$
$$+ \frac{1}{2}\int_h^T ((\int_{t-h}^t \nabla(u^m\rho^m)ds)u^m, u^m)_{C(t)}dt \le c\sqrt{h}$$

Next setting $\Delta\Omega(t) = \Omega(t)\setminus\Omega(t-h)$; $\Delta\Omega(t+h) = \Omega(t+h)\setminus\Omega(t)$ and bearing in mind the smoothness of $\hat{\Gamma}$ one has

$$-\frac{1}{h}\int_0^T(\rho^m(t)u^m(t), \bar{u}^m(t) - \bar{u}^m(t-h))_{\Omega(t)}dt =$$

$$-\frac{1}{h}\int_0^T |\sqrt{\rho^m(t)}u^m(t)|^2_{\Omega(t)}dt + \frac{1}{2h}\int_0^T |\sqrt{\rho^m(t)}u^m(t)|^2_{\Omega(t)}dt +$$

$$\frac{1}{2h}\int_0^T |\sqrt{\rho^m(t)}\bar{u}^m(t-h)|^2_{\Omega(t)}dt -$$

$$\frac{1}{2h}\int_0^T |\sqrt{\rho^m(t)}(u^m(t) - \bar{u}^m(t-h)|^2_{\Omega(t)}dt \le$$

$$-\frac{1}{2h}\int_0^T |\sqrt{\rho^m(t)}u^m(t)|^2_{\Delta\Omega(t+h)}dt -$$

$$(3.35)\quad \frac{1}{2h}\int_0^T |\sqrt{\rho^m(t)}u^m(t)|^2_{\Omega(t)\cap\Omega(t+h)}dt +$$

$$\frac{1}{2h}\int_h^T |\sqrt{\rho^m(t-h)}\bar{u}^m(t-h)|^2_{\Delta\Omega(t)}dt +$$

$$\frac{1}{2h}\int_h^T |\sqrt{\rho^m(t-h)}u^m(t-h))u^m(t-h)|^2_{\Omega(t)\cap\Omega(t-h)}$$

$$\frac{1}{2h}\int_h^T((\rho^m(t) - \rho^m(t-h)u^m(t-h), u^m(t-h))_{C(t)}dt +$$

$$\frac{1}{2h}\int_h^T((\rho^m(t) - \rho^m(t-h))\bar{u}^m(t-h), \bar{u}(t-h))_{\Omega(t)\setminus C(t)}dt -$$

$$\frac{c}{h}\int_h^T |u^m(t) - u^m(t-h)|^2_{\Omega(t)}dt \le$$

$$-\frac{1}{2h}\int_0^{T-h} |\sqrt{\rho^m(t)}u^m(t)|^2_{\Omega(t+h)\cap\Omega(t)}dt +$$

$$\frac{1}{2h}\int_0^{T-h} |\sqrt{\rho^m(t)}u^m(t)|^2_{\Omega(t+h)\cap\Omega(t)}dt + \frac{c}{\sqrt{h}} -$$

$$\frac{c}{\sqrt{h}}\int_h^T |u^m(t) - u^m(t-h)|^2_{\Omega(t)}dt = \frac{c}{\sqrt{h}} -$$

$$\frac{c}{h}\int_h^T |u^m(t) - u^m(t-h)|^2_{\Omega(t)}dt.$$

From (3.32),(3.32),(3.35) we conclude

(3.36) $$\int_{h}^{T} |u^m(t) - u^m(t-h)|^2_{\Omega(t)} dt \le c\sqrt{h}.$$

By the classical characterization of M. Riesz and A. Kolmogorov of compact sets in $L^2(\hat{\Omega})$ (see [6]) we can prove that the set $\{u^m\}$ of u^m satisfying (3.21),(3.36) is relatively compact in $L^2(\hat{\Omega})$.

From (3.21) and the relative compactness of $\{u^m\}$ in $L^2(\hat{\Omega})$ we can choose a subsequence again denoted by u^m such that

$$\lim_{m\to\infty} \int_0^T (\rho^m u^m, u^m \cdot \nabla\varphi)_{\Omega(t)} dt = \int_0^T (\rho u, u \cdot \nabla\varphi)_{\Omega(t)} dt$$

$\forall \varphi \in D(\hat{\Omega})$.

Now it remains to prove

(3.37) $$\begin{aligned} &\lim_{m\to 0} \int_0^T (\rho^m u^m \cdot \nu_t, u^m \varphi)_{\Gamma(t)} dt = \int_0^T (\rho u \cdot \nu_t, u \cdot \varphi)_{\Gamma(t)} dt \\ &\lim_{m\to 0} \int_0^T (\rho^m |u^m|^2 \cdot \nu_t, \varphi)_{\Gamma(t)} dt = \int_0^T (\rho |u|^2 \cdot \nu_t, \varphi)_{\Gamma(t)} d \end{aligned}$$

First we will need the following compactness theorem (see [10]).

<u>Theorem 3.</u> <u>Let</u> $B_0 \subset B_1 \subset B_2$ <u>be three reflexive Banach spaces and the injection of</u> B_0 <u>into</u> B_1 <u>is compact. For given</u> $h > 0$ <u>we define the space</u>

$$W = \{\varphi \mid \varphi(t) \in L^2(0, ;B_0);\ \sup_{h} \frac{1}{\sqrt{h}} \int_0^{T-h} \|\varphi(t+h) - \varphi(t)\|^2_{B_2} dt < \infty\}.$$

<u>Then the injection of</u> W <u>into</u> $L^2(0,T;B_1)$ <u>is compact.</u>

Now geometric notions are needed below.

$\omega_i(t,\delta)$ means the interior boundary strip of $\Omega(t)$ with width δ that is

$$\omega_i(t,\delta) = \{x \mid x \in \Omega(t), \mathrm{dist.}(x, \Gamma(t)) < \delta\}.$$

Then let $\{t_j\}$ be a countable dense subset of $(0,T)$. For positive integers j, k, ℓ we put $G_{j,k,\ell} = (t_j, t_k) \times \Omega^{\ell}(t_j)$ where

$\Omega^{\ell}(t_j) = \Omega(t_j) \setminus \omega_1(t_j, 1/\ell) = \{x | x \in \Omega(t_j), \text{dist.}(x, \Gamma(t)) > > 1/\ell\}$.

We denote by S the totality of $G_{j,k,\ell}$ such that $G_{j,k,\ell}$ is non-void open set of $\hat{\Omega}_1$. An element G of S is called a slab of type S. For $G \in S$ the following lemma is an immediate consequence of (3.21),(3.37) and Theorem 3.

<u>Lemma 1.</u> $\{u^m\}$ <u>is relatively compact in</u> $L^2(t_j, t_k; H^{1-\varepsilon}(\Omega^{\ell}(t_j)) \quad \forall \varepsilon > 0$. Hence $\{u^m\}$ is relatively compact in $L^2(t_j, t_k; \partial\Omega^{\ell}(t_j))$. Moreover, one has (see [1])

<u>Lemma 2.</u> <u>Let</u> $G = (\alpha, \beta) \times \Omega$ <u>be a slab of type</u> S <u>and let</u> δ <u>be a small positive number</u>. <u>Suppose that the lateral boundary</u> $(\alpha, \beta) \times \partial\Omega$ <u>of</u> G <u>lies in the interior boundary strip</u> $\omega_1(x, \delta)$. Then for any $u \in V(\Omega))$ we have

$$\int_{\alpha}^{\beta} |u|^2_{\Gamma(t)} dt \leq c \int_{\alpha}^{\beta} |u|^2_{\partial\Omega} dt + \delta \int_{\alpha}^{\beta} \| u \|^2_{\Omega} dt.$$

Now we are in condition to prove the strong convergence of u^m in $L^2(\hat{\Gamma})$. Suppose $\varepsilon > 0$ is given. If $\delta_1 > 0$ is sufficiently small, then for each δ in $0 < \delta < \delta_1$ we can choose a finite number of slabs $G_j^{(\delta)}$ $(j = 1, 2, \dots, N_\delta)$ of type S with the following properties:

i) by K we denote the union of $G_j^{(\delta)}$, then $\hat{\Omega} \setminus K \subset \subset \omega_1(x, \delta)$;

ii) overlapping of $G_j^{(\delta)}$'s is such that any point of K is contained in at most two of $G_j^{(\delta)}$'s.

The smoothness assumptions of $\hat{\Gamma}$ make this choice possible. We suppose the $G_j^{(\delta)}$ is expressed as $(\alpha_j, \beta_j) \times \Omega_j$ (understanding the dependence on δ). We put $w = u^m - u^n$ and attempt to show

$$\int_0^T |w|^2_{\Gamma(t)} dt \longrightarrow 0 \quad \text{as} \quad m,n \longrightarrow \infty$$

Now

$$\int_0^T |w|^2_{\Gamma(t)}dt \leq \sum_1^{N_\delta}{}_j \int_{\alpha_j}^{\beta_j} |w|^2_{\partial\Omega_j}dt + \delta \sum_1^{N_\delta}{}_j \int_{\alpha_j}^{\beta_j} \|w\|^2_{\Omega_j}dt.$$

Re-choosing δ if necessary, we get

$$\int_0^T |w|^2_{\Gamma(t)}dt \leq \sum_1^{N_\delta}{}_j \int_{\alpha_j}^{\beta_j} |w|^2_{\partial\Omega_j}dt + \varepsilon .$$

From Lemma 1 one has

$$\int_{\alpha_j}^{\beta_j} |w|^2_{\partial\Omega_j}dt \longrightarrow 0 \quad \text{as} \quad m,n \longrightarrow \infty$$

consequently

$$\int_0^T |w|^2_{\Gamma(t)}dt < 2\varepsilon$$

provided that m, n are large enough.

From (3.21) one obtains (3.37).

Now passing to the limit $m \longrightarrow \infty$ in (3.11) all terms converge to the respective terms in (2.3).

Finally we prove

$$\sup_{0<t<T} |u|_{\Omega(t)} < c$$

We let

$$u^m_{\bar{t}}(t) = \begin{cases} u^m(t) & 0 \leq t \leq \bar{t} \\ 0 & t > \bar{t}. \end{cases}$$

We replace in (3.11) φ by $u^m_{\bar{t}}(t)$ and after some calculations obtain

$$|\sqrt{\varphi^m(\bar{t})}\,u^m(\bar{t})|^2_{\Omega(\bar{t})} \leq \frac{1}{m} \int_0^{\bar{t}} \left|\frac{\partial u^m}{\partial t}\right|^2_{\Omega(t)} dt +$$

$$+ \frac{1}{m}\left(\frac{\partial u^m(\bar{t})}{\partial t}, u^m(\bar{t})\right)_{\Omega(\bar{t})} + \frac{1}{2}\,|\sqrt{\rho^m_0}\,u^m(0)|^2_{\Omega(0)} +$$

$$+ c\int_t^{\bar{t}} \|u^m\|^2_{\Omega(t)}dt + c.$$

By virtue of (3.21) and the compactness of $\{u^m\}$ in $L^2(\hat{\Omega})$ one obtains

$$\lim_{m\to\infty} \left| \frac{1}{m} \left(\frac{\partial u^m(\bar t)}{\partial t}, u^m(\bar t) \right)_{\Omega(\bar t)} \right| \le$$

$$\le \lim_{m\to\infty} \frac{1}{\sqrt{m}} \left| \frac{\partial u^m(\bar t)}{\partial t} \right|_{\Omega(\bar t)} \sqrt{m}\, |u^m(\bar t)|_{\Omega(\bar t)} = 0$$

hence

$$|u(\bar t)|_{\Omega(\bar t)} < c.$$

Finally in a standard way one has $\wp$ satisfies (2.4) in the distribution sense. The proof is completed.

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