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GRAPHS AND ASSOCIATIVE TRIPLES IN QUASITRIVIAL GROUPOIDS

T. KEPKA, J. KRATOCHVÍL

Abstract: Several equalities and inequalities concerning the numbers of occurrences of three-element subgraphs in directed graphs are used to find the lower bound for the number of associative triples in finite quasitrivial groupoids.

Key words: Graph, quasitrivial groupoid.

Classification: 20N99

For a positive integer n, let $C_3(n)$ denote the maximal number of 3-cycles in an n-element tournament. It is well known that $C_3(n) = (n^3 - n)/24$ for n odd and $C_3(n) = (n^3 - 4n)/24$ for n even (see [1] and [3]). On the other hand, tournaments are in a close connection with commutative quasitrivial groupoids and the equivalent result is the lower bound for the number of associative triples (see [2]). Namely, if G is an n-element commutative quasitrivial groupoid then G contains at least $(3n^3 + n)/4$ (resp. $(3n^3 + 4n)/4$) associative triples of elements, provided n is odd (resp. even). The aim of this short note is to show that the same is true for non-commutative quasitrivial groupoids (see Theorem 1 (iii),(iv)), however, in this case, the equivalent combinatorial structure is that of directed graphs. Several equalities and inequalities concerning the numbers of occurrences of three-element subgraphs in a directed graph are found and the main result is then derived.

1. Quasitrivial groupoids and graphs. Throughout this note, a graph is a directed graph without loops and multiple edges, i.e. a finite non-empty set together with an antireflexive binary relation (possible empty).

Let K be a graph. Then $V = V(K)$ will designate the set of vertices, $E = E(K)$ that of edges and $v(K) = \text{card } V$. Further, for any $a \in V$, let $f(a) = f(K,a) = \text{card}\{b \in V;\ (a,b) \in E,\ (b,a) \notin E\}$, $g(a) = \text{card}\{b \in V;\ (a,b) \notin E,\ (b,a) \in E\}$, $h(a) = \text{card}\{b \in V;\ (a,b) \in E,\ (b,a) \in E\}$ and $k(a) = \text{card}\{b \in V;\ (a,b) \notin E,\ (b,a) \notin E\}$. Now, we put $w(1) = w(K,1) = \sum_{a \in V}(f(a)^2 - f(a))/2$, $w(2) = \sum (g(a)^2 - g(a))/2$, $w(3) = \sum (h(a)^2 - h(a))/2$, $w(4) = \sum (k(a)^2 - k(a))/2$, $w(5) = \sum f(a)g(a)$, $w(6) = \sum f(a)h(a)$, $w(7) = \sum f(a)k(a)$, $w(8) = \sum g(a)h(a)$, $w(9) = \sum g(a)k(a)$, $w(10) = \sum h(a)k(a)$.

We shall say that a graph K is commutative (resp. anticommutative) if $h(a) = k(a) = 0$ (resp. $f(a) = g(a) = 0$) for every $a \in V$. Thus commutative graphs are nothing else than tournaments and anticommutative graphs are in fact the simple undirected graphs.

Consider the following three-element graphs $L(1),\dots,L(16)$ where $V(L(i)) = \{1,2,3\}$ for each $1 \le i \le 16$ and $E(L(1)) = \{(1,2),(1,3),(2,3)\}$, $E(L(2)) = \{(1,2),(1,3),(2,3),(3,2)\}$, $E(L(3)) = \{(1,2),(1,3)\}$, $E(L(4)) = \{(1,2),(2,1),(1,3),(2,3)\}$, $E(L(5)) = \{(1,3),(2,3)\}$, $E(L(6)) = \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$, $E(L(7)) = \emptyset$, $E(L(8)) = \{(1,2),(2,3),(3,1)\}$, $E(L(9)) = \{(1,2),(2,3)\}$, $E(L(10)) = \{(1,2),(2,3),(1,3),(3,1)\}$, $E(L(11)) = \{(1,2),(2,3),(3,2)\}$, $E(L(12)) = \{(1,2),(1,3),(3,1)\}$,

$E(L(13)) = \{(1,2),(2,1),(2,3),(3,2)\}$, $E(L(14)) = \{(1,3),(3,1)\}$, $E(L(15)) = \{(1,2),(2,1),(2,3),(3,2),(3,1)\}$, $E(L(16)) = \{(1,3)\}$. These graphs are pair-wise non-isomorphic and every three-element graph is isomorphic to one of them. Now, for a graph K and $1 \le i \le 16$, we denote by $q(i) = q(K,i)$ the number of induced subgraphs of K isomorphic to L(i). Obviously, if $v(K) \ge 3$ then K is commutative (resp. anticommuative) iff $q(2) = \ldots = q(7) = q(9) = \ldots = q(16) = 0$ (resp. $q(1) = \ldots = q(5) = q(8) = \ldots = q(12) = q(15) = q(16) = 0$).

Let K be a graph and $p = (p_1,\ldots,p_{16}) \in Z^{16}$, Z being the ring of integers. Put $q(K,p) = \sum_{i=1}^{16} p_i q(i)$.

A groupoid is a non-empty set with a binary operation (usually denoted multiplicatively). A groupoid G is said to be commutative (resp. anticommuative) if $xy = yx$ (resp. $xy \neq yx$) for all $x,y \in G$ such that $x \neq y$. A groupoid G is said to be quasitrivial if $xy \in \{x,y\}$ for all $x,y \in G$.

For a groupoid G, let $A(G) = \{(x,y,z); x,y,z \in G,\ x.yz = xy.z\}$ and $a(G) = \mathrm{card}\, A(G)$. If G is quasitrivial then clearly $(x,x,y),(x,y,x),(y,x,x) \in A(G)$ for all $x,y \in G$.

Let C be a class of groupoids and n a positive integer such that C contains at least one groupoid with n elements. We define $a(C,n) = \min a(G)$, $G \in C$, card $G = n$ and $b(C,n) = \max a(H)$, $H \in C$, H non-associative, card $H = n$; $b(C,n) = n^3$ if there exists no such $H \in C$.

Let G be a finite quasitrivial groupoid. Define a graph $L = L(G)$ as follows: $V(L) = G$ and $(a,b) \in E(L)$ iff $a \neq b$ and $ab = a$. Conversely, let K be a graph. Define a quasitrivial groupoid $H = H(K)$ as follows: The underlying set of H is the set V(K) and for $a,b \in V(K)$ we have $ab = a$ if $(a,b) \in E(K)$ and $ab = b$ in the opposite case. Then $G \longrightarrow L(G)$ and $K \longrightarrow H(K)$ are bijective correspondences between finite quasitrivial groupoids and graphs preserving underlying sets and injective homomorphisms. They induce by restriction bijective correspondences between finite commutative (resp. anticommutative) quasitrivial groupoids and commutative (resp. anticommuative) graphs.

For $1 \leq i \leq 16$, let $P_i = 27 - a(H(L(i)))$ and $P = (P_i)$. It is easy to verify that $P = (0,\dots,0,6,3,3,2,2,2,2,1,1)$. For a graph K, let $q(K) = q(K,P)$. Notice that this number is even provided K is commutative (resp. anticommutative). The following proposition is obvious:

Proposition 1. Let G be a finite quasitrivial groupoid and $n = \operatorname{card} G$. Then $a(G) = n^3 - q(L(G))$.

2. Several equalities and inequalities. In this section, let K be a graph, $n = v(K)$ and $p = (p_i) \in Z^{16}$. We have the following ten obvious equalities:

$w(1) = q(1) + q(2) + q(3)$,

$w(2) = q(1) + q(4) + q(5)$,

$w(3) = 3q(6) + q(13) + q(15)$,

$w(4) = 3q(7) + q(14) + q(16)$,

$w(5) = q(1) + 3q(8) + q(9) + q(10)$.

$w(6) = 2q(4) + q(10) + q(12) + q(15)$.

$w(7) = 2q(5) + q(9) + q(11) + q(16)$,

$w(8) = 2q(2) + q(10) + q(11) + q(15)$,

$w(9) = 2q(3) + q(9) + q(12) + q(16)$,

$w(10) = q(11) + q(12) + 2q(13) + 2q(14)$.

From this we get the following equality:

(1) $\quad 2w(1) - 2w(2) + w(6) + w(7) - w(8) - w(9) = 0$.

Moreover, it is easy to see that:

$q(1) = w(1) - w(8)/2 - w(9)/2 + q(9)/2 + q(10)/2 + q(11)/2$
$\quad + q(12)/2 + q(15)/2 + q(16)/2,$

$q(2) = w(8)/2 - q(10)/2 - q(11)/2 - q(15)/2,$

$q(3) = w(9)/2 - q(9)/2 - q(12)/2 - q(16)/2,$

$q(4) = w(6)/2 - q(10)/2 + q(12)/2 - q(15)/2,$

$q(5) = w(7)/2 - q(9)/2 - q(11)/2 - q(16)/2,$

$q(6) = w(3)/3 - q(13)/3 - q(15)/3,$

$q(7) = w(4)/3 - w(10)/6 + q(11)/6 + q(12)/6 + q(13)/3 - q(16)/$

$q(8) = -w(1)/3 + w(5)/3 + w(8)/6 + w(9)/6 - q(9)/2 - q(10)/2$
$\quad - q(11)/6 - q(12)/6 - q(15)/6 - q(16)/6,$

$q(14) = w(10)/2 - q(11)/2 - q(12)/2 - q(13$

and consequently

$$
\begin{aligned}
q(K,p) = {} & w(1)(p_1-p_8/3) + w(3)p_6/3 + w(4)p_7/3 + w(5)p_8/3 \\
& + w(6)p_4/2 + w(7)p_5/2 + w(8)(-p_1/2+p_2/2+p_8/6) \\
& + w(9)(-p_1/2+p_3/2+p_8/6) + w(10)(-p_7/6+p_{14}/2) \\
& + q(9)(p_1/2-p_3/2-p_5/2-p_8/2+p_9) + q(10)(p_1/2-p_2/2 \\
& -p_4/2-p_8/2+p_{10}) + q(11)(p_1/2-p_2/2-p_5/2+p_7/6-p_8/6 \\
& +p_{11}-p_{14}/2) + q(12)(p_1/2-p_3/2-p_4/2+p_7/6-p_8/6 \\
& +p_{12}-p_{14}/2) + q(13)(-p_6/3+p_7/3+p_{13}-p_{14}) + \\
& + q(15)p_1/2-p_2/2-p_4/2-p_6/3-p_8/6+p_{15}) + \\
& + q(16)(p_1/2-p_3/2-p_5/2-p_7/3-p_8/6+p_{16}).
\end{aligned} \tag{2}
$$

Now, using (1) and (2), we have the following result:

<u>Proposition 2</u>. (i) $q(K) = -2w(1) + 2w(5) + w(8) + w(9) + w(10)$.

(ii) $q(K) = -w(1) - w(2) + 2w(5) + w(6)/2 + w(7)/2 + w(8)/2 + w(9)/2 + w(10)$.

(iii) $q(K) = -2w(1) + 2w(5)$, provided K is commutative.

(iv) $q(K) = w(10)$, provided K is anticommutative.

Proposition 3. (i) $q(K) \leq (n^3 - n)/4$.

(ii) $q(K) \leq (n^3 - 4n)/4$, provided n is even.

Proof. For any $a \in V$, let $r(a) = (f(a)+g(a))^2/2-2f(a)g(a)$, $s(a) = (h(a)+k(a))^2/2-2h(a)k(a)$ and $t(a) = f(a)+g(a)+(f(a) + g(a)+h(a))^2/2-(f(a)-g(a))^2-r(a)-s(a)$. Then $t(a)/2 = 2f(a)g(a)+h(a)k(a)+f(a)h(a)/2+f(a)k(a)/2+g(a)h(a)/2+g(a)k(a)/2-(f(a)^2-f(a))/2-(g(a)^2-g(a))/2$, and hence, by Proposition 2(ii), $q(K) = \sum_{a \in V} t(a)/2$. On the other hand, for any $a \in V$, we have $f(a)+g(a) \leq n-1$, $f(a)+g(a)+h(a)+k(a) = n-1$, $0 \leq (f(a)-g(a))^2$, $0 \leq r(a)$, $0 \leq s(a)$ and $t(a) \leq (n^2-1)/2$. Consequently, $q(K) \leq (n^3-n)/4$. In the rest of the proof, suppose that n is even. If $f(a)+g(a)$ is even then $h(a)+k(a)$ is odd, $h(a) \neq k(a)$ and $1/2 \leq s(a)$. Moreover, $f(a)+g(a) \leq n-2$, and therefore $t(a) \leq (n^2-4)/2$. If $f(a)+g(a)$ is odd then $1/2 \leq r(a)$, $1 \leq (f(a)-g(a))^2$ and again $t(a) \leq (n^2-4)/2$.

Proposition 4. Assume that K is anticommutative.

(i) $q(K) \leq (n^3 - 2n^2 + n)/4$.

(ii) $q(K) \leq (n^3 - 2n^2 + n - 4)/4$, provided n is odd and $n = 4m + 3$ for some $m \in Z$.

(iii) $q(K) \leq (n^3 - 2n^2)/4$, provided n is even.

Proof. By Proposition 2(iv), $q(K) = \sum h(a)k(a)$. Moreover, q(K) is even and the rest is easy.

Proposition 5. Assume that $q(K) \neq 0$.

(i) $1 \leq q(K)$.

(ii) $6 \leq q(K)$, provided K is commutative.

(iii) $2n - 4 \leq q(K)$, provided K is anticommutative.

Proof. Easy.

3. Several examples

Example 1. Let $G = G(+)$ be a finite abelian group of order n and let M be a subset of G such that $0 \notin M$. Put

$m = \text{card } M$ and $k = \text{card } \{a \in M;\ -a \in M\}$. We define a graph $J = J(G,M)$ as follows: $V(J) = G$ and $(a,b) \in E(J)$ iff $a - b \in M$. Then $q(J) = n^2m - nm^2 - nk$ and we have the following particular cases:

(i) $n \geq 3$ is odd, $G = Z_n = \{0,1,\dots,n-1\}$ (the additive group of integers modulo n) and $M = \{1,2,\dots,(n-1)/2\}$. Then J is commutative and $q(J) = (n^3 - n)/4$.

(ii) $n \geq 4$ is even, $G = Z_n$ and $M = \{1,2,\dots,(n-2)/2\}$. Then J is not commutative and $q(J) = (n^3 - 4n)/4$.

(iii) $n \geq 5$ is odd, $n = 4r + 1$, $G = Z_n$ and $M = \{1,2,\dots,r, n-r,n-r+1,\dots,n-2,n-1\}$. Then J is anticommutative and $q(J) = (n^3 - 2n^2 + n)/4$.

(iv) $n \geq 6$ is even, $n = 4r + 2$, $G = Z_n$ and $M = \{1,2,\dots,r, n-r,n-r+1,\dots,n-2,n-1\}$. Then J is anticommutative and $q(J) = (n^3 - 2n^2)/4$.

(v) $n \geq 4$ is even, $n = 4r$, $G = Z_n$ and $M = \{1,2,\dots,r,n-r, n-r+1,\dots,n-2,n-1\}$. Then J is anticommutative and $q(J) = (n^3 - 2n^2)/4$.

Example 2. Let $n \geq 4$ be even and $M = \{1,2,\dots,(n-2)/2\}$. Define a graph $I = I(n)$ as follows: $V(I) = Z_n$ and $(a,b) \in E(I)$ iff either $a - b \in M$ or $a \in M \cup \{0\}$ and $a - b = n/2$. Then I is commutative and $q(I) = (n^3 - 4n)/4$.

Example 3. Let $n \geq 7$ be odd, $n = 4r + 3$ and $M = \{1,2,\dots, r,n-r,n-r+1,\dots,n-2,n-1\}$. Define a graph $R = R(n)$ as follows: $V(R) = Z_n$ and $(a,b) \in E(R)$ iff either $a - b \in M$ or $2r + 2 \leq a \leq n-1$ and $a - b = 2r + 1$ or $1 \leq a \leq 2r + 1$ and $a - b = 2r + 2$. Then R is anctiommutative and $q(R) = (n^3 - 2n^2 + n - 4)/2$.

Example 4. Let $n \geq 3$. Define a graph $S = S(n)$ as follows: $V(S) = Z_n$ and $(a,b) \in E(S)$ iff either $3 \leq a$ and $b \leq 2$ or $a = 0$

and b = 1. Then q(S) = 1.

Example 5. Let $n \gtrsim 3$. Define a graph T = T(n) as follows: V(T) = Z_n and $(a,b) \in E(T)$ iff either $b < a$ and $3 \leq a$ or a = 0, b = 1, or a = 1, b = 2 or a = 2, b = 0. Then T is commutative and q(T) = 6.

Example 6. Let $n \gtrsim 3$. Define a graph Q = Q(n) as follows: V(Q) = Z_n and $(a,b) \in E(Q)$ iff either a = 0, b = 1 or a = 1, b = = 0. Then Q is anticommutative and q(Q) = 2n - 4.

4. Summary. In the following theorem, let A (resp. B, C) denote the class of quasitrivial (commutative, anticommutative) groupoids.

Theorem 1. (i) a(A,1) = a(B,1) = a(C,1) = b(A,1) = = b(B,1) = b(C,1) = 1.

(ii) a(A,2) = a(B,2) = a(C,2) = b(A,2) = b(B,2) = b(C,2) = = 8.

(iii) a(A,n) = a(B,n) = $(3n^3 + n)/4$ for every odd $n \gtrsim 3$.

(iv) a(A,n) = a(B,n) = $(3n^3 + 4n)/4$ for every even $n \gtrsim 4$.

(v) a(C,n) = $(3n^3 + 2n^2 - n)/4$ for every odd $n \gtrsim 5$, n = 4m + 1.

(vi) a(C,n) = $(3n^3 + 2n^2 - n + 4)/4$ for every odd $n \gtrsim 3$, n = 4m + 3.

(vii) a(C,n) = $(3n^3 + 2n^2)/4$ for every even $n \gtrsim 4$.

(viii) b(A,n) = $n^3 - 1$ for every $n \gtrsim 3$.

(ix) b(B,n) = $n^3 - 6$ for every $n \gtrsim 3$.

(x) b(C,n) = $n^3 - 2n + 4$ for every $n \gtrsim 3$.

Proof. The result follows easily from Propositions 1,2,3, 4,5 and Examples 1,2,3,4,5,6.

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