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A CURIOUS GENERALIZATION OF LOCAL UNIFORM ROTUNDITY

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Abstract: It is shown that several well known generalizations of local uniform rotundity can be viewed as directionalizations of local uniform rotundity. From this vantage point, a new and natural generalization of local uniform rotundity is identified. Some fundamental properties of this new notion are established.

Key words: rotundity, local uniform rotundity.

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Recall that a Banach space X is said to be locally uniformly rotund (LUR) if whenever x is in S, the unit sphere of X, and $\{x_n\}$ is a sequence in S such that $\|x + x_n\| \to 2$, then $x_n \to x$. This geometrical notion was introduced and studied in [2]. Several generalizations of LUR have been considered in the literature (see [4] and the references given there); perhaps most notable are weak local uniform rotundity, midpoint lócal uniform rotundity and, of course, rotundity, which appeared long before LUR. For the sake of completeness, recall that a Banach space X is said to be weakly locally uniformly rotund (WLUR) if X satisfies the definition of LUR with $x_n \to x$ replaced by $x_n \to x$ weakly; a dual space X* is said to be weak* locally uniformly rotund (W*LUR) if X* satisfies the definition of LUR with $x_n \to x$ replaced by $x_n \to x$ weak*; and a Banach space X is said to be rotund (R) if whenever x and y are in S such that $\|x + y\| = 2$, then $x = y$.

In this note, it is shown that the properties WLUR, W*LUR and R can all be viewed as directionalizations of the property LUR by restricting the directions of the chords of the unit sphere to lie within some member of a prescribed family of subsets of the space. From this viewpoint, a new, natural and rather curious generalization of LUR is identified. An investigation is then undertaken and examples are cited that establish some stability properties of this new notion as well as give its relationships to other rotundity conditions.

The perspective developed here and the subsequent analysis very much parallel that done in [3] where directionalizations of uniform rotundity were considered. Consequently, proofs here that parallel those in [3] are omitted and the reader is advised to consult [3] and the references cited there.

Definition 1. Let X be a Banach space and let $\mathcal{A}$ be a nonempty collection of nonempty subsets of $X\setminus\{0\}$. For x in S, $0 < \varepsilon \le 2$ and A in $\mathcal{A}$, define $\delta(x,\varepsilon,A) = \inf\{1 - \|\frac{1}{2}(x + y)\|: y \in S, \|x - y\| \geq \varepsilon \text{ and } x - y = \alpha z \text{ for } z \in A\}$. Then X is said to be $LUR_{\mathcal{A}}$ if and only if $\delta(x,\varepsilon,A) > 0$ for every x in S, $0 < \varepsilon \le 2$ and A in $\mathcal{A}$.

Proposition 2. For a Banach space X, let $\mathcal{A}$ be the collection of all norm closed and bounded nonempty subsets of $X\setminus\{0\}$, let $\mathcal{B}$ be the collection of all weakly closed and bounded nonempty subsets of $X\setminus\{0\}$, let $\mathcal{C}$ be the collection of all weak* closed and bounded (equivalently, weak* compact) nonempty subsets of $X^*\setminus\{0\}$, and let $\mathcal{D}$ be the collection of all norm compact subsets of $X\setminus\{0\}$. Then

(i) X is LUR if and only if X is $LUR_{\mathcal{A}}$,

(ii) X is WLUR if and only if X is $LUR_{\mathcal{B}}$,

(iii) X* is W*LUR if and only if X* is $LUR_{\mathcal{C}}$,

(iv) X is R if and only if X is $LUR_{\mathcal{D}}$.

Proof: That (i) is true follows easily since S is in $\mathcal{A}$, and X is LUR if and only if $\delta(x,\varepsilon,S) > 0$ for each x in S and $0 < \varepsilon \le 2$. Statements (ii) and (iii) are proved as in the proof of Theorem 2.2 of [3].

To prove (iv), suppose X is not R. Then there exist x and y in S with $\|x + y\| = 2$ and $x \neq y$. Hence $\delta(x, \|x - y\|, \{x - y\}) = 0$ and X is not $LUR_{\mathcal{D}}$. Conversely, if X is not $LUR_{\mathcal{D}}$, then there exist D in $\mathcal{D}$, x in S and $0 < \varepsilon \leq 2$ with $\delta(x,\varepsilon,D) = 0$. Choose $\{y_n\}$ in S such that $\|x + y_n\| \to 2$, $\|x - y_n\| > \varepsilon$ and $x - y_n = \alpha_n z_n$ where z_n is in D. Since D is in $\mathcal{D}$, by passing to subsequences, it may be assumed that $\alpha_n \to \alpha$ where $\alpha \neq 0$ and $z_n \to z$ where z is in D. Letting $y = x - \alpha z$ and noting $y_n \to y$, it follows that $\|x - y\| > \varepsilon$, y is in S and $\|x + y\| = 2$. Thus X is not R. This completes the proof.

Perusing the collections $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ in the preceding proposition, a natural collection not found there is the collection of weakly compact subsets.

Definition 3. Let X be a Banach space and let $\mathcal{W}$ be the collection of all weakly compact nonempty subsets of $X\setminus\{0\}$. Then X is said to be locally uniformly rotund in weakly compact sets of directions (LURWC) if X is $LUR_{\mathcal{W}}$.

From Proposition 2, it is immediate that LURWC lies in strength between WLUR and R, the properties WLUR and LURWC coincide in reflexive Banach spaces, and the properties LURWC and R coincide in spaces with the Schur property. Examples will be cited later that show the property LURWC is in fact distinct from the known properties WLUR and R.

The next two facts are easily obtained by the techniques used in [3].

Proposition 4. For a Banach space X, each of the following statements is equivalent to the statement that X is LURWC.

(i) If x is in S and $\{x_n\}$ is a sequence in S such that $\|x + x_n\| \to 2$ and $x_n \to z$ weakly, then $z = x$.

(ii) If x and $\{x_n\}$ are in X such that $2(\|x\|^2 + \|x_n\|^2) - \|x + x_n\|^2 \to 0$ and $x_n \to z$ weakly, then $z = x$.

Proposition 5. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces and let $T: X_1 \to X_2$ be a continuous linear injection. If X_2 is LURWC, then $\|\cdot\|$, defined for x in X_1 by

$$\|x\| = (\|x\|_1^2 + \|Tx\|_2^2)^{1/2},$$

is an equivalent norm on X_1 that is LURWC.

Note that every subspace of a Banach space that is LURWC is also LURWC. The next two facts describe the stability of LURWC under the forming of substitution spaces and quotient spaces. Since the proofs of these facts are analogous to the proofs of Theorems 2.10 and 2.12 of [3], they are omitted.

Proposition 6. Let X be a full function space on an index set T and let $\{X_t: t \in T\}$ be a collection of Banach spaces.

(i) If X is UR^C, where C is the set of evaluation functionals on X (see [3]), and if every X_t is LURWC, then the substitution space $P_X X_t$ is LURWC.

(ii) If X is reflexive, then $P_X X_t$ is LURWC if and only if X and X_t are all LURWC.

Proposition 7. If X is a Banach space that is LURWC and if Y is a reflexive subspace of X, then the quotient space X/Y is LURWC.

It should be noted that Proposition 6 says, in particular, that the ℓ^p-sum for $1 < p < \infty$ of LURWC spaces is likewise LURWC. Also, note that the property LURWC is not inherited by quotient spaces in general (for example $m(\Gamma)$ is always a quotient of some $\ell^1(T)$, and $\ell^1(T)$ has an equivalent LURWC (= R here) norm, but $m(\Gamma)$ has no equivalent norm that is even R if Γ is uncountable).

The same sort of program that has just been carried out to generalize local uniform rotundity in a natural way, namely, restricting the directions of the chords of the unit sphere to lie in certain subsets, can be considered to naturally generalize midpoint local uniform rotundity.

Recall that a Banach space X is said to be midpoint locally uniformly rotund (MLUR) if whenever x is in S and $\{x_n\}$ and $\{y_n\}$ are sequences in S such that $\|2x - (x_n + y_n)\| \to 0$, then $x_n - y_n \to 0$. This condition was initially studied in [1]. Also, call X weakly midpoint locally uniformly rotund (WMLUR) if X satisfies the definition of MLUR with $x_n - y_n \to 0$ replaced by $x_n - y_n \to 0$ weakly.

It follows immediately from the definitions that WMLUR lies in strength between MLUR and R. A straightforward application of the Principle of Local Reflexivity yields that a Banach space X is WMLUR if and only if every element of the unit sphere of X is an extreme point of the unit ball of the second dual space X** when X is considered as a subspace of X** under the canonical embedding. As a consequence, the properties WMLUR and R coincide in reflexive Banach spaces.

Definition 8. Let X be a Banach space and let $\mathcal{A}$ be a nonempty collection of nonempty subsets of $X\setminus\{0\}$. For x in S, $0 < \varepsilon \leq 2$ and A in $\mathcal{A}$, define $\gamma(x,\varepsilon,A) = \inf\{\|x - \frac{1}{2}(u + v)\|: u,v \in S,\ \|u - v\| \geq \varepsilon \text{ and } u - v = \alpha z \text{ for } z \in A\}$. Then X is said to be $MLUR_{\mathcal{A}}$ if and only if $\gamma(x,\varepsilon,A) > 0$ for every x in S, $0 < \varepsilon \leq 2$ and A in $\mathcal{A}$.

Statements (i) and (ii) of the following proposition are expected; however, what is interesting here is (iii) which says that by restricting the directions of the chords of the unit sphere to lie in weakly compact subsets no new generalized notion of MLUR is obtained.

Proposition 9. For a Banach space X, let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ be defined as in Proposition 2 and let $\mathcal{W}$ be defined as in Definition 3. Then

(i) X is MLUR if and only if X is $MLUR_{\mathcal{A}}$,

(ii) X is WMLUR if and only if X is $MLUR_{\mathcal{B}}$,

(iii) X is R if and only if X is $MLUR_{\mathcal{D}}$ if and only if X is $MLUR_{\mathcal{W}}$

Proof: Statements (i) and (ii) follow as in Proposition 2 above and Theorem 2.2 of [3].

To prove (iii), note that X is $MLUR_{\mathcal{W}}$ implies X is $MLUR_{\mathcal{D}}$ trivially, and X is $MLUR_{\mathcal{D}}$ implies X is R follows as in the proof of Proposition 2. To finish, it must be shown that X is not R if X is not $MLUR_{\mathcal{W}}$. Assuming X is not $MLUR_{\mathcal{W}}$, there exist x in S, $0 < \varepsilon < 2$ and W in $\mathcal{W}$ with $\gamma(x,\varepsilon,W) = 0$. Choose sequences $\{u_n\}$ and $\{v_n\}$ in S such that $\|2x - (u_n + y_n)\| \to 0$, $\|u_n - v_n\| > \varepsilon$ and $u_n - v_n = \alpha_n w_n$ where w_n is in W. Since W is in $\mathcal{W}$, by passing to subsequences, it may be assumed that $\alpha_n \to \alpha$ where $\alpha \neq 0$ and $w_n \to w$ weakly where w is in W. Let $u = x + \frac{1}{2}\alpha w$ and $v = x - \frac{1}{2}\alpha w$. Then, since $u_n + v_n \to 2x$ and $u_n - v_n \to \alpha w$ weakly, it follows that $u_n \to u$ weakly and $v_n \to v$ weakly, and hence $\|u\| \leq 1$ and $\|v\| \leq 1$. But $\|u + v\| = 2$ and so, in fact, u and v are in S. Note $u - v = \alpha w$ and $\alpha w \neq 0$ since w is in W. This shows X is not R and the proof is complete.

For a dual space X*, a natural way to attempt to generalize MLUR would be to replace $x_n - y_n \to 0$ by $x_n - y_n \to 0$ weak* in the definition of MLUR; however, by an argument similar to the proof of Proposition 9, it can be shown that if $\mathcal{C}$ is defined as in Proposition 2, then X* satisfies this rotundity notion if and only if X* is $MLUR_{\mathcal{C}}$ if and only if X* is R.

This note concludes with the examples promised earlier. To see that LURWC is in fact distinct from the properties WLUR and R, see the examples given in [4]; the table here appends Table 2 in [4] by considering the properties LURWC and WMLUR for the examples given there. Note to fill in the table here requires no additional work since ℓ^2 is reflexive and ℓ^1 has the Schur property.

	LURWC	WMLUR
$(\ell^2, \|\cdot\|_2)$	+	+
$(\ell^2, \|\cdot\|_L)$	+	+
$(\ell^2, \|\cdot\|_W)$	+	+
$(\ell^2, \|\cdot\|_A)$	-	+
$(\ell^2, \|\cdot\|_N)$	+	+
$(\ell^1, \|\cdot\|_H)$	+	-
$(\ell^1, \|\cdot\|_E)$	+	+
$(\ell^1, \|\cdot\|_1)$	-	-

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