Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 4, 597--614

Persistent URL: http://dml.cz/dmlcz/106328

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25,4 (1984)

ON SYSTEMS, PERIODS AND SEMIPOSITIVE MAPPINGS

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Abstract: We study the periodical behaviour of discrete systems induced by symmetric graphs which cover some models investigated before. We introduce a class of transition mappings which imply restricted periods of systems.

Key words: Symmetric graph, discrete system, period.

Classification: 05C99, 90A08

Introduction. In this paper we present a particular model of discrete systems which covers some models studied before. We give a sufficient condition (Theorem 1.2) for the system to have a restricted period. The formulation of Theorem 1.2 is, in fact, a postulation of the method of the proofs of [1] and [3]. Two properties of mappings occur to be important: adjoincy and semipositivity. While the former is a known property, the latter is introduced in the paper. It appears to be a common property of nonderecreasing real functions $R \longrightarrow R$ and linear positive semidefinite functions $R^k \longrightarrow R^k$. (R^k is the k-dimensional Euclidean space, $R=R^1$ real numbers.) The semipositivity is studied in Sections 2 - 4.

We conclude the Introduction with a survey of some known results. These may be easily illustrated by a social influence model.

Let $\mathcal{S}$ be a society of m members and $\mathcal{O}$ be the set of their possible opinions. The opinion of the i-th member at time t is denoted as $x_i(t)$. The members change their opinions simultaneously in discrete steps, and the opinion $x_i(t+1)$ depends only on opinions of other members at time t. If the set $\mathcal{O}$ is finite, the system must behave periodically after some finite number of steps. We investigate possible periods of such systems. Some special cases have been considered so far.

Model A. [1]. The set $\mathcal{O}$ of opinions is a finite subset of real numbers. Every member is equipped with a nondecreasing function $f_i : R \longrightarrow \mathcal{O}$. The next opinion $x_i(t+1)$ is given by $x_i(t+1) = f_i(\sum_{j=1}^{m} w_{ji}x_j(t))$ where $w_{ji} \in R$ is the influence of the j-th member on the i-th member.

<u>Theorem A</u> [1]. If $w_{ji} = w_{ij}$ for all i, j, then the period of Model A is at most 2.

<u>Model B</u>. [2]. The set $\mathcal{O} = \{o_1, \dots, o_k\}$ is a discrete set of possible alternatives. In time t+1 every member accepts the majority opinion with respect to influences w_{ij}. That is $x_i(t+1) = o_l$ for which the sum $\sum_{x_j(t) = o_j} w(i,j)$ attains the maximum. (If the sum is maximal for more alternatives, say $o_{i_1}, \dots$ $\dots, o_{i_r}$ with $i_1 < i_2 < \dots < i_r$, the member accepts the alternative o_{i_r}.)

<u>Theorem B</u> [2]. If $w_{ji} = w_{ij}$ for all i, j, then the period of Model B is at most 2.

<u>Model C</u>.[3]. A generalization of Model B. As an addition, there are real numbers α_l for every alternative o_l which are interpreted as attractivity of the alternative. Here $x_i(t+1) =$

$= o_1$ for which the expression $\alpha_1 \cdot \sum_{x_j(t) = o_j} w_{ji}$ is maximum. Moreover, a permutation π_i is assigned to each member. In case of a tie (as above), the member accepts the opinion o_{i_s} for which $\pi_i(i_s) = \max(\pi_i(i_1), \ldots, \pi_i(i_r))$.

<u>Theorem C</u> [3]. If $w_{ji} = w_{ij}$ for all i, j, then the period of Model C is at most 2.

Further examples are given in Section 5.

1. <u>Systems and periods</u>. A <u>space</u> is a set S with two binary operations + and $\cdot$ where + is a mapping $S \times S \longrightarrow S$ and $\cdot$ is a mapping $S \times S \longrightarrow R$ which satisfy the only axiom $(u+v) \cdot w = u \cdot w + v \cdot w$ for every $u, v, w \in S$.

The Euclidean space R^k or, more generally, a real Hilbert space are examples of a space if $u \cdot v$ denotes the scalar product. However, in general, we do not require either commutativity or associativity of the operations + and $\cdot$. We will use the notation

$\sum_{i=1}^{m} u_i = (\ldots((u_1+u_2)+u_3+\ldots)+u_n$, and $u \cdot v$ will be abbreviated as uv.

Let m be an integer. A <u>system</u> $\mathcal{S}$ is a triple, $\mathcal{S} = (\{S_i\}, \{a_{ij}\}, \{f_i\})$, where S_i, $i = 1, \ldots, m$, are spaces, $a_{ij}: S_i \longrightarrow S_j$ and $f_i: S_i \longrightarrow S_i$, i, j = 1,...,m are mappings. The state $x(t) = (x_1(t), \ldots, x_m(t))$ of the system $\mathcal{S}$ in time $t \in \{0, 1, \ldots\}$ is an element of $S_1 \times S_2 \times \ldots \times S_m$. We shall refer to $x_i(t) \in S_i$ as to the state of the i-th element in time t. The state $x(t+1) = (x_1(t+1), \ldots, x_m(t+1)$ is given by

$$x_i(t+1) = f_i\left(\sum_{j=1}^{m} a_{ji}(x_j(t))\right), \quad i = 1, \ldots, m.$$

The state $x_i(t)$ can be interpreted as an opinion of the i-th member, a_{ij} as the influence of the i-th member on the j-th member, the sum $\sum_{j=1}^{m} a_{ji}(x_j(t))$ as the total influence on the i-th member in time t, and f_i as a mapping which creates a new opinion $x_i(t+1)$ from the total influence.

We say that a system $\mathcal{S}$ has the <u>period</u> T, $T>0$, for some initial state $x(0)$, if $x(t_0+T) = x(t_0)$ for some t_0 and T is the smallest integer with this property.

Let S and Q be spaces. A pair of mappings $g: S \longrightarrow Q$ and $h: Q \longrightarrow S$ is said to be <u>adjoint</u> (<u>co-adjoint</u>) if $g(u)\cdot v = u\cdot h(v)$ ($g(u)\cdot v = h(v)\cdot u$) for every $u\in S$ and $v\in Q$.

Clearly, if the operation " $\cdot$ " is commutative then g and h are adjoint iff they are co-adjoint. Let us remark that the mappings $u \longrightarrow Au$ and $v \longrightarrow A^T v$ are (co-)adjoint for any real matrix A.

Let S be a space and $f: S \longrightarrow S$ be a mapping. We say that f is <u>semipositive</u> (<u>seminegative</u>) if for every $n \geq 2$ and every $u_1, \dots, u_n \in S$

$$\sum_{i=1}^{m} (u_i f(u_i) - u_{i-1} f(u_i)) \geq 0 \qquad (\leq 0).$$

and the equality holds only in case $f(u_1) = f(u_2) = \dots = f(u_n)$. (The indices are taken mod n.)

We say that a mapping f is <u>positive</u> (<u>negative</u>) if it is semipositive (seminegative) and injective.

<u>Theorem 1.1</u>. Let $\mathcal{S} = (\{S_i\}, \{a_{ij}\}, \{f_i\})$ be a system such that

(i) a_{ij} and a_{ji} are co-adjoint for every $i,j=1,\dots,m$

(ii) f_i is semipositive for all $i=1,\dots,m$ or

f_i is seminegative for all $i=1,\dots,m$.

Then the only possible periods of the system $\mathcal{S}$ are 1 or 2.

This theorem is a special case of Theorem 1.2. □

In fact, we can consider a more general system such that the state x(t) depends not only on the state x(t-1) but also on the states x(t-2),...,x(t-q) for some fixed $q \geq 1$. More precisely, a system $\mathcal{S}$ is a triple $\mathcal{S} = (\{S_i\}, \{a_{ij}^l\}, \{f_i\})$, i,j=1,...,m, l=1,...,q, and the state of the i-th element in time t is given by

$$x_i(t) = f_i\left(\sum_{l=1}^{q} \sum_{j=1}^{m} a_{ji}^l(x_j(t-1))\right), \quad i=1,\dots,m.$$

Theorem 1.2. Let $\mathcal{S} = (\{S_i\}, \{a_{ij}^l\}, \{f_i\})$ be a system such that

(i) a_{ij}^l and a_{ji}^{q-l+1} are co-adjoint,
for all i,j=1,...,m and l=1,...,q,

(ii) f_i is semipositive for all i=1,...,m or
f_i is seminegative for all i=1,...,m,

then the only possible periods of the system $\mathcal{S}$ are divisors of q+1.

Proof. Let all f_i be semipositive and let the system have a period T for some initial state. We can assume $T_0=0$ in the definition of the period. Then

$$A = \sum_{i,j=1}^{m} \sum_{l=1}^{q} \left(\sum_{t=1}^{T} a_{ji}^l (x_j(t-l+1))x_i(t+1) - \right.$$
$$\left. - \sum_{t=1}^{T} a_{ji}^l(x_j(t-l+1))x_i(t-q) \right. =$$
$$= \sum_{i,j=1}^{m} \sum_{l=1}^{q} \left(\sum_{t=1}^{T} a_{ji}^l(x_j(t-l+1))x_i(t+1) - \right.$$
$$\left. - \sum_{t=1}^{T} a_{ij}^{q-l+1}(x_i(t-q))x_j(t-l+1) \right) =$$
$$= \sum_{i,j=1}^{m} \sum_{l=1}^{q} \left(\sum_{t=1}^{T} a_{ji}^l(x_j(t-l+1))x_i(t+1) - \right.$$
$$\left. - \sum_{t=1}^{T} a_{ij}^l(x_i(t-l+1))x_j(t+1) \right) = 0.$$

(In the second equality we used that a^{l}_{ij} and a^{q-l+1}_{ji} are co-adjoint, in the third one the fact that T is the period.) Put $v_i(t) = \sum_{l=1}^{q} \sum_{j=1}^{m} = a^{l}_{ji}(x_j(t-l+1))$. Then we have for every i=1, ...,m

$$
\begin{aligned}
B_i &= \sum_{t=1}^{T} \Big(\sum_{l=1}^{q} \sum_{j=1}^{m} a^{l}_{ji}(x_j(t-l+1)x_i(t+1) - \\
&\quad - \sum_{l=1}^{q} \sum_{j=1}^{m} a^{l}_{ji}(x_j(t-l+1)x_i(t-q)) = \\
&= \sum_{t=1}^{T} (v_i(t)f_i(v_i(t)) - v_i(t)f_i(v_i(t-q-1)) = \\
&= \sum_{t=1}^{T} (v_i(t)f_i(v_i(t)) - v_i(t+q+1)f_i(v_i(t)) \geq 0
\end{aligned}
$$

by the semipositivity of f_i. As $\sum_{i=1}^{m} B_i = A = 0$ we have

$$\sum_{t=1}^{T} (v_i(t)f_i(v_i(t)) - v_i(t+q+1)f_i(v_i(t)) = 0$$

Using the semipositivity of f_i we have $x_i(t+1) = f_i(v_i(t)) = f_i(v_i(t+q+1)) = x_i(t+c+2)$ for t=1,...,T. Thus, T is a divisor of q+1 as it is the period. □

2. Properties of the class of semipositive mappings. The aim of this section is to show some basic properties of the class of semipositive mappings.

Let S be a space and $f,g: S \to S$ be mappings. The sum f+g is the mapping $S \to S$ defined by $(f+g)(x) = f(x) + g(x)$ for every $x \in S$.

Proposition 2.1. If f and g are semipositive mappings, then f+g is semipositive provided " • " is commutative. □

Let $S_1 \times S_2$ be the cartesian product of two spaces S_1 and S_2 with the operations + and • defined by

$(u_1,u_2) + (v_1,v_2) = (u_1+v_1,u_2+v_2)$

$$(u_1,u_2)\cdot(v_1,v_2) = u_1\cdot v_1 + u_2\cdot v_2$$

Proposition 2.2. The product $S_1\times S_2$ of two spaces is a space. □

The product $f_1\times f_2: S_1\times S_2 \to S_1\times S_2$ is defined by $(f_1\times f_2)(u_1,u_2) = (f_1(u_1), f_2,(u_2))$.

Proposition 2.3. The product $f_1\times f_2$ of two semipositive mappings is semipositive. □

Let $f: S \to S$ be a one-one mapping. Denote f^{-1} its inverse.

Proposition 2.4. Let " $\cdot$ " be commutative. Then the mapping f^{-1} is semipositive iff f is. □

Proposition 2.5. Let $f: S \to T$ and $h: T \to S$ be a pair of adjacent mappings and $f: S \to S$ be a semipositive mapping. Then the mapping $\bar{f} = gfh$ is semipositive as well.

Proof. Let $u,v\in T$. If we set $w=f(h(v))$, then $u\cdot\bar{f}(v) = u\cdot g(w) = h(u)\cdot w = h(u)\cdot f(h(u))$. Hence
$\sum (x_i\bar{f}(x_i) - x_{i-1}\bar{f}(x_i)) = \sum (h(x_i)f(h(x_i)) - h(x_{i-1})f(h(x_i)))$
for every choice of $x_1,\dots,x_n$. The sum is nonnegative as it corresponds to the choice $h(x_1),\dots,h(x_n)\in S$ for f which is semipositive. If $f(h(x_1)) = \dots = f(h(x_n))$ then obviously $\bar{f}(x_1) = \dots = \bar{f}(x_n)$. Thus $\bar{f}$ is semipositive. □

Beginning from here, we will deal with Euclidean spaces only. Let us introduce some necessary notation. For $x\in R^k$ let x^j be the j-th component of x. If $f: R^k \to R^k$ is a mapping, we write $f^j(x)$ instead of $(f(x))^j$. We shall use the symbol $\sum$ instead of $\sum_{i=1}^{n}$. Using the sum the subscript $i-1$ in x_{i-1} is always taken mod n. The axiom of semipositivity will be used in the form $\sum (x_i-x_{i-1})f(x_i)\geq 0$.

Proposition 2.6. Let c be a positive real number and f: $:R^k \longrightarrow R^k$ be (semi)positive. Then the mapping cf defined by $(cf)(x) = cf(x)$ is (semi)positive as well. □

Proposition 2.7. Let V be a linear subspace of R^k and $p: R^k \longrightarrow V$ be the orthogonal projection on V. Then the composition pf is (semi)positive on V for any f (semi)positive on R^k.

Proof. For every $y \in R^k$ there is a unique decomposition $y = p(y)+y'$ where $y' \in V^{\perp}$ (the orthogonal complement of V). Clearly $xy=xp(y)+xy'=xp(y)$ as $xy'=0$ from the orthogonality. Hence

$$\sum (x_i-x_{i-1})pf(x_i) = \sum (x_i-x_{i-1})f(x_i) \geq 0.$$

Obviously pf is injective iff f is. □

Let us remark that the mappings $p: R^k \longrightarrow V$ and $id: V \longrightarrow R^k$ are adjacent. Thus pf is semipositive also by Proposition 2.5.

3. Linear mappings. A symmetric square matrix A is called positive (semi)definite if $xAx > 0$ (≥ 0) for every vector x, $x \neq 0$.

Theorem 3.1. Let A be a real square matrix of size k. Then A is positive semidefinite iff the mapping $x \longmapsto Ax$ is semipositive.

Proof. 1. Sufficiency. Let A be positive semidefinite and $x_1, \ldots, x_n \in R^k$, $n \geq 2$. Then

$$\sum (x_i-x_{i-1})(Ax_i) = \frac{1}{2} \sum (x_i-x_{i-1})A(x_i-x_{i-1}) \geq 0.$$

If $\sum (x_i-x_{i-1})(Ax_i)=0$ then $(x_i-x_{i-1})A(x_i-x_{i-1}) = 0$ for every $i=1,2,\ldots,n$. As every positive semidefinite matrix A equals B^TB for some B, we have $0 = (x_i-x_{i-1})B^TB(x_i-x_{i-1}) = (B(x_i-x_{i-1}))^2$. Hence $B(x_i-x_{i-1})=0$ and also $A(x_i-x_{i-1}) =$

$= B^T B(x_i - x_{i-1}) = 0$. Thus $Ax_1 = Ax_2 = \dots = Ax_n$ and the mapping $x \longmapsto Ax$ is semipositive.

2. Necessity. Let the mapping $x \longmapsto Ax$ be semipositive. We are to prove

(a) $xAx \geq 0$ for every $x \in R^k$,

(b) A is symmetric.

Denote o the zero vector. Then $xAx = (x-o)Ax + (o-x)Ao \geq 0$ for every $x \in R^k$. Thus (a) holds. Let A be a nonsymmetric matrix such that the mapping $x \longmapsto Ax$ is semipositive. Let us first assume k=2. Then there exist a positive semidefinite matrix B and a real $c > 0$ such that

$c(A+B) = \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix} = D$. The mapping $x \longmapsto Dx$ is semipositive due to Propositions 2.1, 2.6 and the first part of the proof. Clearly $\varepsilon \neq 0$ as D is nonsymmetric. Let a be a real number and n be an integer such that

$$na^2 + (2n-1)n + \varepsilon an^2 < 0.$$

Consider 2n vectors $x_1, x_2, \dots, x_{2n} \in R^2$ defined by $x_i = (ia, i)$ for $i=1,\dots,n$, $x_{n+i} = ((n-i)a, n+i)$ for $i=1,\dots,n-1$, and $x_{2n} = (0,0)$.

Then $\sum_{i=1}^{2n} (x_i - x_{i-1}) Dx_i = \sum_{i=1}^{2n} (x_i^1 - x_{i-1}^1)x_i^1 + \sum_{i=1}^{2n}(x_i^2 - x_{i-1}^2)x_i^2 + \varepsilon \cdot \sum_{i=1}^{2n}(x_i^2 - x_{i-1}^2)x_i^1 = \frac{1}{2}\sum_{i=1}^{2n}(x_i^1 - x_{i-1}^1)^2 + \frac{1}{2}\sum_{i=1}^{2n}(x_i^2 - x_{i-1}^2)^2 + \varepsilon \cdot \sum_{i=1}^{2n}(x_i^2 - x_{i-1}^2)x_i^1 = na^2 + \frac{1}{2}((2n-1) + (2n-1)^2) + \varepsilon \cdot (a+2a+\dots$
$\dots na+(n-1)a+\dots+a) = na^2 + (2n-1)n + \varepsilon \cdot an^2 < 0.$

The case $k > 2$ can be reduced to the case k=2. □

Corollary 3.2. A square matrix A is positive definite iff the mapping $x \longmapsto Ax$ is positive. □

Let us say that a mapping $f:R^k \longrightarrow R^k$ has the property P_n for some integer n if $\Sigma (x_i - x_{i-1}) f(x_i) \geq 0$ for every $x_1, \ldots$ $\ldots, x_n \in R^k$.

Hence, a semipositive mapping f has property P_n for all n.

<u>Proposition 3.3</u>. For every integer $n \geq 2$ there exists a linear mapping f having property P_n which is not semipositive.

Proof. Let f be the linear mapping defined by the matrix $\begin{pmatrix} n & 0 \\ 1 & n \end{pmatrix}$. The mapping f is not semipositive by Theorem 3.1 as the matrix is not symmetric. It has the property P_n by the following lemma.

<u>Lemma 3.4</u>. For every integer $n \geq 2$ and every $x_i, y_i \in R$, $i = 1, \ldots, n$, we have

$$\frac{n-1}{2} \left(\sum_{i=1}^{n} (x_i - x_{i-1})^2 + \sum_{i=1}^{n} (y_i - y_{i-1})^2 \right) + \sum_{i=1}^{n} (y_i - y_{i-1}) x_i \geq 0.$$

Proof. Let us set $s_i = x_i - x_{i-1}$, $r_i = y_i - y_{i-1}$ for $i = 1, \ldots, n$. Then $s_n = -(s_1 + \ldots + s_{n-1})$ and $r_n = -(r_1 + \ldots + r_{n-1})$. We can write the expression in Lemma as

$$\frac{n-1}{2} \left(\sum_{i=1}^{n-1} s_i^2 + \left(\sum_{i=1}^{n-1} s_i \right)^2 + \sum_{i=1}^{n-1} r_i^2 + \left(\sum_{i=1}^{n-1} r_i \right)^2 \right) + \sum_{i=1}^{n-1} r_i x_i - x_n \cdot \sum_{i=1}^{n-1} r_i \geq \frac{n-1}{2} \sum_{i=1}^{n-1} (s_i^2 + r_i^2) + \sum_{i=1}^{n-1} \left(r_i \cdot \sum_{j=1}^{i} s_j \right) \geq$$

$$\geq \frac{1}{2} \sum_{1 \leq j \leq i < n-1} (r_i + s_j)^2 \geq 0. \quad \square$$

4. <u>Constructions based on monotonous mappings</u>. The next lemma can be found in [8].

<u>Lemma 4.1</u>. Let $u_1 \leq u_2 \leq \ldots \leq u_n$ and $v_1 \leq v_2 \leq \ldots \leq v_n$ be real numbers, and π be a permutation of the set $\{1, 2, \ldots, n\}$. Then

(i) $\sum_{i=1}^{m} u_i v_i \geq \sum_{i=1}^{m} u_i v_{\pi(i)}$

(ii) The inequality (i) is sharp iff there are some i, j such that $u_i < u_j$ and $v_{\pi(i)} > v_{\pi(j)}$. □

Corollary 4.2. Let $u_1 \leq \ldots \leq u_s < u_{s+1} \leq \ldots \leq u_n$ and $v_1 \leq \ldots \leq v_s < v_{s+1} \leq \ldots \leq v_n$ be real numbers, π a cyclic permutation of $\{1,2,\ldots,n\}$. Then

$$\sum_{i=1}^{m} u_i v_i > \sum_{i=1}^{m} u_i v_{\pi(i)}.$$

Proof. Put $I = \{1,\ldots,s\}$ and $J = \{s+1,\ldots,n\}$. As π is the cyclic permutation there are i and j such that $i \in I$, $j \in J$, $\pi(i) \in J$ and $\pi(j) \in I$. Thus $u_i < u_j$ and $v_{\pi(i)} > v_{\pi(j)}$. □

Theorem 4.3. Let $f: R \to R$ be a mapping. Then

(i) f is positive iff f is increasing,

(ii) f is semipositive iff f is nondecreasing.

Proof. The part $\Rightarrow$. Assume that f is not nondecreasing. Then there exists a pair $x, x' \in R$ such that $x < x'$ and $f(x) > f(x')$. Then $xf(x) + x'f(x') < xf(x') + x'f(x)$ and hence f is not semipositive.

The part $\Leftarrow$. Let f be nondecreasing, and let $x_1,\ldots,x_n \in R$. Then $\sum (x_i - x_{i-1}) f(x_i) \geq 0$ by Lemma 4.1. Assume that not all $f(x_i)$ are equal. Then $f(x_s) < f(x_{s+1})$ for some $s < n$, and hence $x_s < x_{s+1}$ as f is nondecreasing. Thus $\sum (x_i - x_{i-1}) f(x_i) > 0$ by Corollary 4.2. □

The following example generalizes the fact that the product of increasing mappings is positive.

Example 4.4. Let $f: R^k \to R^k$ be a mapping satisfying $x^j < y^j \Rightarrow f^j(x) < f^j(y)$ for every $x, y \in R^k$ and $j=1,\ldots,k$. Then f is positive.

Proof. Let $x_1,\dots,x_n \in R^k$ be such that $x_r^{j_o} \neq x_s^{j_o}$ for some j_o, r and s. By Lemma 4.1 $\Sigma\,(x_i^j - x_{i-1}^j)f^j(x_i) \geq 0$ for every $j = 1,\dots,k$.
Moreover, $\Sigma\,(x_i^{j_o} - x_{i-1}^{j_o})f^{j_o}(x_i) > 0$ by Corollary 4.2.
Thus $\Sigma\,(x_i - x_{i-1})f(x_i) > 0$. □

Let $x \in R^k$ and π be a permutation of $\{1,\dots,k\}$. Then we denote by $\pi(x)$ the vector $(x^{\pi(1)},\dots,x^{\pi(k)})$.

Let us assign to every vector $x \in R^k$ a unique permutation π_x of $\{1,\dots,k\}$ such that $\pi_x(i) < \pi_x(j)$ if either $x_i < x_j$ or $x_i = x_j$ and $i < j$. We shall also use the notation $\bar{x} = \pi_x(x)$ and $\bar{x}^i = x^{\pi_x(i)}$. (Let us remark that the vector $\bar{x}$ arises from the vector x by ordering its components in the nondecreasing sequence.)

It is easy to see that Lemma 4.1 gives

(1) $xy \leq \bar{x}\bar{y}$ for every $x,y \in R^k$.

Let us denote $M^k = \{x \in R^k \mid x^1 \leq x^2 \leq \dots \leq x^k\}$.

We say that a mapping $g: R^k \to R^k$ is the <u>fair-extension</u> of a mapping $f: M^k \to M^k$ if $g(x) = \pi_x^{-1}(f(\pi_x(x)))$ for every $x \in R^k$.

Let us remark that the mapping g satisfies

(2) $\pi_x(i) < \pi_x(j) \Rightarrow g^i(x) \leq g^j(x)$

for every $x \in R^k$ and $i,j = 1,\dots,k$.

<u>Theorem 4.5</u>. Let $g: R^k \to R^k$ be the fair-extension of a semipositive mapping $f: M^k \to M^k$. Then g is semipositive as well.

Proof. Let $x_1,\dots,x_n \in R^k$. Denote $y_i = g(x_i)$, $i=1,\dots,n$. Then $\bar{y}_i = f(\bar{x}_i)$. It follows from (2) that

(3) $x_i y_i = \bar{x}_i \bar{y}_i$ for $i=1,\dots,n$.

As the mapping f is semipositive, we have

(4) $\Sigma\,(\bar{x}_i - \bar{x}_{i-1})\bar{y}_i \geq 0$.

Combining (1),(3) and (4) we get

$$(5)\quad \sum (x_i-x_{i-1})y_i \geq \sum (\bar{x}_i-\bar{x}_{i-1})\bar{y}_i \geq 0.$$

Suppose that

$$(6)\quad \sum (x_i-x_{i-1})y_i = 0.$$

Then (5) implies that the equality holds in (4), and hence $\bar{y}_1 = \dots = \bar{y}_n$ by the semipositivity of f. In a way of contradiction assume that not all y_i are the same. Let s be the minimum j such that at least two y_i^j, $i=1,\dots,n$, are distinct. Choose an r such that $y_r^s = \min\{y_i^s \mid i=1,\dots,n\}$ and $y_r^s < y_{r-1}^s$. Let us consider the sets I_{r-1} and I_r defined by $I_t = \{j \mid y_t^j \leq y_r^s\}$ for $t=r-1$, r. As $s \in I_r - I_{r-1}$, and I_r and I_{r-1} are of the same cardinality, there exists some $t \in I_{r-1}-I_r$ such that

$$(7)\quad y_{r-1}^s > y_{r-1}^t,$$

$$(8)\quad y_r^s < y_r^t, \text{ and}$$

$$(9)\quad s < t.$$

Since $I_{r-1} \cap \{1,\dots,s-1\} = I_r \cap \{1,\dots,s-1\}$, the conditions (2), (7) and (9) give

$$(10)\quad x_{r-1}^s > x_{r-1}^t.$$

Thus Lemma 4.1 due to (8) and (10) gives $x_{r-1}y_r < \bar{x}_{r-1}\bar{y}_r$ which contradicts (6). □

Let $f^1, f^2, \dots, f^k$ be real mappings $R \to R$ such that $f^1(x) \leq f^2(x) \leq \dots \leq f^k(x)$ for every $x \in R$. Let us define the mapping $g: R^k \to R^k$ by $g^j(x) = f^{\pi_x(j)}(x^j)$ for $x \in R^k$ and $j=1,\dots,k$. (This means that f^1 is applied to the smallest component of x, f^2 to the smallest component but one, etc.) Let us call this mapping g the <u>cross-product</u> of $f^1,\dots,f^k$.

<u>Corollary 4.6.</u> (i) The cross-product g is semipositive provided all f^i are nondecreasing.

(ii) The cross-product g is positive provided all f^i are increasing.

Proof. Let f^i be nondecreasing for $i=1,\dots,k$. Then the cartesian product $f=f^1\times \dots \times f^k$ is a semipositive mapping $M^k \to M^k$ by the Proposition 2.3 and Theorem 4.3. This yields that g is semipositive as well. If f^i are increasing the mapping g is injective and hence positive. □

Example 4.7. Let s be an integer, $1\le s\le k$. Then the mapping $g:R^k \to R^k$ defined by $g^j(x) = 1$ for $\pi_x(j) > k-s$, and $=0$ otherwise, is semipositive as it is the cross-product of constant mappings $f^1,\dots,f^k$ where $f^j(x)=0$ for $j=1,\dots,k-s$, and $f^j(x)=1$ for $j=k-s+1,\dots,k$.

Example 4.8. The mapping $g:R^k \to R^k$ defined by $g^j(x) = \pi_x(j)$ is semipositive as it is the cross-product of constant mappings $f^1,\dots,f^k$ where $f^j(x)=j$ for $j=1,\dots,k$.

Let f and g be mappings $M^k \to M^k$. We say that g is a tie-modification of f if for every $x\in M^k$ and every $I\subset\{1,2,\dots,k\}$ such that $x^i=x^j$ for $i,j\in I$ and $x^i\ne x^j$ for $i\in I$, $j\notin I$ we have

(i) $\sum_{j\in I} x^j f^j(x) = \sum_{j\in I} x^j g^j(x)$ and

(ii) $\sum_{\substack{j\in I\\ j\le i}} x^j f^j(x) \le \sum_{\substack{j\in I\\ j\le i}} x^j g^j(x)$ for every $i=1,\dots,k$.

(iii) $f(x)=f(x') \Rightarrow g(x)=g(x')$ for $x,x'\in M^k$.

Theorem 4.9. A tie-modification g of a (semi)positive mapping f is (semi)positive as well.

Proof. Let f be semipositive and $x_1,\dots,x_n\in M^k$. Condition (i) gives

(11) $x_i g(x_i) = x_i f(x_i)$ for $i=1,2,\dots,n$.

Using (ii) one can prove that

(12) $x_{i-1}g(x_i) \le x_{i-1}f(x_i)$ for $i=1,2,\dots,n$.

Thus

(13) $\sum (x_i-x_{i-1})g(x_i) \ge \sum (x_i-x_{i-1})f(x_i) \ge 0$.

If the left-hand sum equals zero so it does the right-hand sum, and it is $f(x_1)=\dots=f(x_n)$ by the semipositivity of f. Using (iii) we complete the proof of semipositivity of g. If f is injective, then by (13) g is injective as well. □

Remark 4.10. Let us consider a particular case of tie-modification. Keeping the notation from the definition, let g satisfy (iv) instead of (i) and (ii).

(iv) $g^i(x)=\frac{1}{|I|}\sum_{j\in I} f^j(x)$ for every $i\in I$.

Clearly (iv) implies (i) and (ii) which proves that g is a tie-modification of f.

Example 4.11. For a vector $x\in R^k$ set $m(x)=\{i \mid x^i = \max\{x^1,\dots,x^k\}\}$. Then the mapping $f:R^k\longrightarrow R^k$ defined by

$$f^j(x) = \begin{cases} \frac{1}{|m(x)|} & \text{for } j\in m(x) \\ 0 & \text{otherwise} \end{cases}$$

is semipositive as it is a tie-modification of the mapping defined in Example 4.7 (for s=1).

5. Applications. In this section we show that the models A, B and C can be interpreted in our general scheme and that Theorems A, B and C follow from Theorem 1.1.

Model A. Let $S_i=R$, f_i be nondecreasing mappings, and $a_{ij}(x)=w_{ij}x$ for $i,j=1,\dots,m$. The mappings f_i are semipositive by Theorem 4.3. The mappings a_{ij} and a_{ji} are co-adjoint as $w_{ij} = w_{ji}$.

Model B. Let $S_i=R^k$, f_i be the cross-product of $g^1,\dots$ $\dots,g^k$ where $g^1=g^2=\dots=g^{k-1}=0$, $g^k=1$, and $a_{ij}(x)=w_{ij}x$ for $i,j=$ $=1,2,\dots,m$. Here the possible opinions are vectors with all components but one equal zero. The auxiliar component with value 1 indicates the choice of an alternative from $o_1,\dots,o_k$. The mappings f_i are semipositive by Example 4.7 (s=1). The mappings a_{ij} and a_{ji} are co-adjoint as $w_{ij}=w_{ji}$.

Model C. This model differs from the previous one only in the mappings a_{ij}. Here a_{ij} is the linear mapping given by the matrix $P_jBP_i^T$ where B is the diagonal matrix with entries $b_{ll}=$ $=\alpha_l w_{ij}$ and P_i is the permutation matrix of π_i. The mappings a_{ij} and a_{ji} are co-adjoint by Remark 2.1.

A particular case of Theorem 2.2 when all $S_i=R^k$, for some fixed integer k, can be interpreted as follows. A society of m members is to decide about k alternatives. The possible opinion ($x\in R^k$) of a member is formed by thinking of the alternatives with (possibly) distinct intensity (the component x^j expresses the weight of j-th alternative in one's opinion).

Clearly each of the models A, B and C is involved in this more general one: We have k=1 and general weights in the model A, while $k>1$ and weights either 0 or 1 in models B and C. One can get a lot of other examples when combining the results of Sections 2, 3 and 4. We mention only some of them in the form of brief remarks.

1. The tie rule in models B and C can be replaced by another one: If a member is influenced by a great number, say r, alternatives of the same weight, he accepts all of them with the same weight 1/r (see Ecample 4.11).

2. One may consider a model where the opinions are formed by the choice of the best s alternatives (see Example 4.7).

3. The members of the society need not have the same rule for computing their new opinions. Moreover, the opinions can consider only the best alternative, another choice of more best ones, and other may use some tie rule. The members even may differ in the dimension of their "opinion space".

4. The opinion of a member may be a ranking of the alternatives (i.e. a permutation of $\{1,2,\dots,k\}$, see Example 4.8).

5. There are several possible ways of computing the new ranking of a member:

- A member may prefer the alternatives according to the number of first places among rankings of other members. If two alternatives coincide in the number of first places, the preference is done according to the number of second places, etc.

- Another member may use some more sophisticated way based on suitable weighting of positions in rankings, then summing the weight of each alternative, putting in the first place the alternative with the maximum sum, etc.

This paper was worked out at Technical University as a Research Report 1983. In a particular case, when considered in connection with the scalar product, the semipositive mappings introduced here coincide with cyclically monotonous mappings used by Rockafellar in [4] to characterize the subgradients of convex functions. Thus, Theorem 3.1 can be derived immediately. The connection between discrete influence systems and convex functions is pointed out in [5]. The number of necessary steps before a system falls into a period has been studied in [6]. The limit behaviour of systems with infinite number of states has been studied in [7].

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(Oblatum 11.6. 1984)