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A CONSTRUCTION OF BRUCK LOOPS

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Abstract: A new construction of Bruck loops is presented.

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Let p be an odd prime number. A possible analogue of 3-elementary commutative Moufang loops (which are closely related with distributive Steiner quasigroups alias Hall triple systems) could be the class of p-elementary Bruck loops. Commutative Moufang loops are usually constructed by means of triadditive mappings (see e.g. [1],[2],[5] and [8]) and one can ask whether a similar method will work for Bruck loops, too. This short note is meant as a modest contribution to the question.

1. Introduction. By a (left) Bruck loop we mean a loop satisfying the identities $(x.yx)z = x(y.xz)$ and $(xy)^{-1} = x^{-1}y^{-1}$, so that a Bruck loop is a (left) Bol loop in which the mapping $x \longrightarrow x^{-1}$ is an automorphism (some properties and constructions of Bruck loops are collected in [3],[4],[6] and [7]). As proved in [6], Bol loops, and hence Bruck loops, are monoassociative and we can consider the variety $\mathcal{B}_p$ of p-elementary Bruck loops

(a Bruck loop G belongs to this variety iff every non-trivial monogenic subloop of G is a p-element group). Then $\mathcal{B}_3$ is just the variety of 3-elementary Commutative Moufang loops and the varieties $\mathcal{B}_p$ are in a close connection with the varieties of p-elementary left distributive left symmetric quasigroups (see [7]).

A ternary ring $G(+,T)$ is an abelian group $G(+)$ together with a triadditive mapping T of G^3 into G. Consider the following equations for ternary rings:

(1) $T(T(x,y,z)u,v) = T(u,T(x,y,z),v) = T(u,v,T(x,y,z)) = 0$;

(2) $T(x,y,z) = T(x,z,y)$;

(3) $T(x,y,y) = T(y,y,x)$;

(4) $T(x,y,z) = T(y,z,x)$;

(5) $3T(x,y,z) = 3T(y,z,x)$.

1.1. <u>Lemma.</u> Let $G = G(+,T)$ be a ternary ring.

(i) If G satisfies (4) then G satisfies (3) and (5).

(ii) If G satisfies (2) and (3) then G satisfies (5).

(iii) If G satisfies (2) and (3) and the group G(+) contains no element of order 3 then G satisfies (4).

Proof. Suppose that G satisfies both (2) and (3). We have $T(x,y,z) + T(x,z,y) = T(y,z,x) + T(z,y,x)$ by (3), and hence $2T(x,y,z) = T(y,z,x) + T(z,y,x)$ by (2). Similarly, $T(x,y,z) + T(y,x,z) = 2T(z,y,x)$ and $3T(x,y,z) = T(y,z,x) + T(z,y,x) + T(x,y,z) = T(y,x,z) + T(x,y,z) + T(z,y,x) = 3T(z,y,x)$.

2. <u>A construction.</u> Throughout this section, let $G(+,T)$ be a ternary ring satisfying the identities (1) and (2). We define a new binary operation (multiplication) on the underlying set G by $xy = x + y + T(x,y,x+y)$ for all $x,y \in G$. In this way,

we obtain a groupoid G.

2.1. <u>Lemma</u>. (i) $x0 = 0x = x$ and $x(-x) = (-x)x = 0$ for every $x \in G$. (ii) $(-x).xy = y$ and $(-x)(-y) = -xy$ for all $x,y \in G$.

Proof. Obvious.

2.2. <u>Lemma</u>. $(x.yx)z = x(y.xz)$ for all $x,y,z \in G$.

Proof. We have $x.yx = 2x + y + 2T(x,x,x) + 3T(x,x,y) + T(x,y,y) + T(y,y,x) + T(y,x,x)$, $(x.yx)z = 2x + y + z + 2T(x,x,x) + 3T(x,x,y) + T(x,y,y) + T(y,y,x) + T(y,x,x) + 2T(x,z,z) + T(y,z,z) + T(y,y,z) + 4T(x,x,z) + 2T(x,y,z) + 2T(y,x,z)$, $y.xz = x + y + z + T(x,x,z) + T(x,z,z) + T(y,y,x) + T(y,x,x) + T(y,x,z) + T(y,y,z) + T(y,x,z) + T(y,z,z)$ and $x(y.xz) = 2x + y + z + 4T(x,x,z) + 2T(x,z,z) + T(y,y,x) + T(y,x,x) + T(y,y,z) + T(x,y,y) + 3T(x,x,y) + T(y,z,z) + 2T(x,x,x) + 2T(x,y,z) + 2T(y,x,z)$ by (1) and (2).

2.3. <u>Lemma</u>. G is a loop.

Proof. By 2.1, G is a left quasigroup with a neutral element and it suffices to show that G is a right quasigroup. If $ba = ca$ for some $a,b,c \in G$ then $d = b - c = T(c,a,a+c) - T(b,a,a+b)$, $T(c,a,a+c) = T(b-d,a,b-d+a) = T(b,a,a+b)$ by (1), and so $b = c$. Finally, $(b-a+T(a-b,a,b))a = b$ for all $a,b \in G$.

2.4. <u>Proposition</u>. G is a Bruck loop.

Proof. The result is an immediate consequence of the preceding lemmas.

2.5. <u>Lemma</u>. $xy.z - x.yz = T(y,z,x) - T(x,y,z)$ for all $x,y,z \in G$.

Proof. Easy.

2.6. <u>Proposition</u>. (i) The loop G is centrally nilpotent of class at most 2.

(ii) G is a Moufang loop iff the ternary ring satisfies (3).

(iii) G is a group iff the ternary ring satisfies (4).

Proof. (i) An easy observation.

(ii) Use 2.5 and the fact that a (left) Bol loop is a Moufang loop iff it is right alternative.

(iii) Use 2.5.

Put $w(0) = 0$ and $w(n) = \sum_{i=1}^{n} i(i-1) = (n-1)n(n+1)/3$ for every positive integer n.

2.7. Lemma. $x^n = nx + w(n)T(x,x,x)$ for all $x \in G$ and all non-negative integers n.

Proof. By induction on n.

2.8. Proposition. Let $p \neq 3$ be a prime and suppose that the group G(+) is p-elementary. Then the loop G is p-elementary.

Proof. An easy consequence of 2.7.

3. Example. Let p be a prime and $G(+) = Z_p^3$, Z_p being the p-element field of integers modulo p. Define a new binary operation $*$ on G by $x * y = (x_1+y_1, x_2+y_2, x_3+y_3+x_1y_2(x_2+y_2))$. Then $G(*)$ is a Bruck loop and it is not a Moufang loop. Moreover, if $p \neq 3$ then $G(*)$ is p-elementary.

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