Oldřich John; Jana Stará

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Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 3, 445--457

Persistent URL: http://dml.cz/dmlcz/106319

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

25(3) 1984

ON THE REGULARITY OF THE WEAK SOLUTION OF CAUCHY PROBLEM FOR NONLINEAR PARABOLIC SYSTEMS VIA LIOUVILLE PROPERTY

O. JOHN, J. STARÁ

Dedicated to the memory of Svatopluk FUČÍK

Abstract: It is proved that Liouville property of initial value problem for parabolic quasilinear system - i.e. the fact that every bounded weak solution of the system with frozen coefficients and with zero initial data in R_+^{n+1} is zero - implies the $C^{0,\alpha}$- regularity of all bounded weak solutions of initial value problem up to the t=0 part of the boundary. Moreover, if each bounded weak solution of a parabolic system is $C^{0,\alpha}$-regular, then Liouville property holds. Similar results for interior parabolic regularity were proved in [12], for elliptic systems in [5],[6],[7],[8],[9],[10],[11].

Key words: Quasilinear parabolic system, initial value problem, regularity up to the boundary, parabolic Liouville property.

Classification: 35K55

Introduction. It is well known that the bounded weak solution of a quasilinear parabolic system need not be Hölder-continuous. In [12] there was proved that Hölder-continuity of a solution in the interior of the domain is guaranteed if for the system in question certain Liouville type theorem (see Definition 4) holds.

We shall prove here that Hölder-continuity up to the part of the boundary contained in the hyperplane t=0 is a consequence of a similar Liouville type theorem for solutions on halfspace with constant initial data (see Definition 5) under the assumption that the initial data are sufficiently smooth.

There is a counterexample due to M. Struwe (see [4]) showing that a bounded weak solution starting from the smooth initial data can develop a singularity. However, in this counterexample the parabolic system does not satisfy the conditions imposed here because of the quadratic dependence of the right hand side on the gradient of the solution.

We are deeply indebted to M. Struwe for fruitful discussions.

I. Notations and definitions. Let Ω be a domain in $\mathbb{R}^n$. Denote for a $T_0 \in (0, \infty\rangle$

$$Q^+ = \{z=(t,x) \in \mathbb{R}^{1+n};\ t \in (0,T_0), x \in \Omega\},$$
$$\Gamma = \{z=(0,x) \in \mathbb{R}^{1+n};\ x \in \Omega\},$$
$$Q^- = \{z=(t,x) \in \mathbb{R}^{1+n};\ (-t,x) \in Q^+\}$$

and

$$Q = Q^+ \cup \Gamma \cup Q^-.$$

By $L_p(Q)$, $W_p^k(Q)$, $C^{0,\alpha,\alpha/2}(Q)$ will be denoted the corresponding Lebesgue and Sobolev spaces and the spaces of Hölder-continuous functions.

Let the nonlinear parabolic system in the form

$$\frac{\partial u^i}{\partial t} - \frac{\partial}{\partial x_\alpha}\left(a_{ij}^{\alpha\beta}(z;u)\frac{\partial u^j}{\partial x_\beta}\right) = -f^i(z) + \frac{\partial}{\partial x_\alpha} g_i^\alpha(z),$$

$$i,j=1,\dots,m; \quad \alpha,\beta=1,\dots,n$$

be given. For the sake of simplicity we rewrite it in the matrix form

(1) $$u_t - \mathrm{div}_x(A(z;u)D_x u) = -f(z)+\mathrm{div}_x g(z).$$

First, we introduce the concept of the weak solution of both the system (1) and of the Cauchy problem for this system. (Functions A, f, g, u_o are supposed to be defined on the corresponding sets.)

Definition 1. The function $u \in W^{o,1}_{2,loc}(Q) \cap L_\infty(Q)$ is said to be a weak solution of the system (1) in Q if

(2) $\forall \varphi \in C_o^\infty(Q): \int_Q [u\varphi_t - A(z;u)D_x u D_x \varphi] dz = \int_Q [f\varphi + gD_x\varphi] dz.$

Definition 2. The function $u \in W^{o,1}_{2,loc}(Q^+ \cup \Gamma) \cap L_\infty(Q^+)$ is called a weak solution of the Cauchy problem for the system (1) in Q^+ with the initial value u_o if

(3) $\forall \varphi \in C^\infty(\overline{Q^+})$, supp $\varphi \subset Q^+ \cup \Gamma$

$$\int_{Q^+} [u\varphi_t - A(z;u)\, D_x u D_x \varphi]\, dz = \int_{Q^+} [f\varphi + gD_x\varphi]\, dz - \int_\Gamma u_o(x)\varphi(0,x)\, dx .$$

In what follows, the functions A, f, g, u_o satisfy the conditions

(4) $A(z;p)$ is continuous on $(Q^+ \cup \Gamma) \times \mathbb{R}^m$;

(5) $(A(z;p)\xi, \xi) > 0$ for all $(z;p) \in (Q^+ \cup \Gamma) \times \mathbb{R}^m$, $\xi \neq 0$;

(6) $f \in L_{s,loc}(Q^+ \cup \Gamma)$ with $s > n/2 + 1$;

(7) $g \in L_{q,loc}(Q^+ \cup \Gamma)$ with $q > n + 2$;

(8) $u_o \in W^1_{r,loc}(\Gamma) \cap L_\infty(\Gamma)$ with $r > n$.

Now, the properties (Li),(Lb) of the Liouville type are defined. They concern the behaviour of a weak solution of (1) in the whole space $\mathbb{R}^{1+n}$ (resp. of a weak solution of the Cauchy problem for (1) in $\mathbb{R}^{1+n}_+$) in case $f=0$, $g=0$, $u_o=0$ and A being frozen in an arbitrary point $z_o \in Q^+$ (resp. $z_o \in \Gamma$).

Definition 3 (Li). We shall say that the system (1) has Liouville property (Li) if the following assertion holds:

Let z_o be a generic point of Q^+ and let the function u be a weak solution of the system

$$(9) \qquad u_t - \mathrm{div}_x(A(z_o;u)\, D_x u) = 0$$

on $\mathbb{R}^{1+n}$. Then u is constant.

<u>Definition 4</u> (Lb). We shall say that the system (1) has Liouville property (Lb) if the following assertion holds:

Let z_o be an arbitrary point of Γ . Let u be a weak solution of the Cauchy problem for the system (9) in R_+^{1+n} with the initial value $u_o=0$. Then u is zero.

We should like to prove that each system (1) satisfying both (Li) and (Lb) is regular in the following way:

<u>Definition 5</u> (Re). Let u be a weak solution of the Cauchy problem for (1) with the initial value u_o satisfying (8). Then there exists $\alpha \in (0,1)$ such that $u \in C_{loc}^{0,\alpha/2,\alpha}(Q^+ \cup \Gamma)$.

<u>Remark</u>. Cauchy problem for (1), being regular in the sense of Definition 5, is regular with the maximal exponent corresponding to the regularity of u_o and right hand side. It can be proved in the following way:

The function $u \in C_{loc}^{0,\alpha/2,\alpha}$ substituted to $A(z;u)$ in (1) enables us to treat it as a linear system with Hölder-continuous coefficients. Applying Schauder estimates we obtain that the maximal coefficient α_1 of hölderianity of the solution u is determined by the quality of f, g and u_o.

II. <u>Main theorem</u>

<u>Theorem</u>. Let the system (1) have the properties (Li) and (Lb). Then it has the property (Re).

__Sketch of the proof.__ We extend the coefficients and the right hand side functions of (1) to the whole cylinder Q. After that the weak solution u of the Cauchy problem for (1) in Q^+ can be shifted and prolonged in a suitable manner to the weak solution w of the extended system on the whole Q. The (Li) and (Lb) imply that w is in $C^{0,\alpha/2,\alpha}_{loc}(Q)$ with an $\alpha \in (0,1)$. Thus the assertion of the Theorem follows immediately.

__Proof.__ Let u be a weak solution of the Cauchy problem for (1) with the initial condition u_0. Put

(10) $$v(z) = u(z) - u_0(x).$$

Substituting to (3) we check immediately that v satisfies the integral identity

(11) $$\forall \varphi \in C^{\infty}(\overline{Q^+}),\ \operatorname{supp} \varphi \subset Q^+ \cup \Gamma$$

$$\int_{Q^+} [v \varphi_t - A(z; v+u_0) D_x v\, D_x \varphi]\, dz = \int_{Q^+} [f\varphi + G\, D_x \varphi]\, dz,$$

where

(12) $$G(z) = g(z) - A(z; v(z) + u_0(x))\, D_x u_0(x).$$

Denote for $z_0 = (t_0, x_0) \in \mathbb{R}^{1+n}$, $R > 0$

(13) $$Q(z_0,R) = \{z = (t,x);\ t \in (t_0 - R^2, t_0),\ |x - x_0| < R\} =$$
$$= (t_0 - R^2, t_0) \times B(x_0,R).$$

In the next lemma we show that the function G has the quality needed in what follows.

__Lemma 1.__ Let the assumptions (4) - (8) hold. Then for each $b > 0$, $M > 0$ there exists $C > 0$ such that for each $Q(z_0,R) \subset\subset Q^+ \cup \Gamma$ with $z_0 \in Q^+$, $R < 1$, dist $(Q(z_0 R), \partial Q^+ \setminus \Gamma) > b$ and for each $v \in L_\infty(Q^+)$, $\|v\|_\infty < M$ it is

(14) $$R^{-(n+\lambda)} \int_{Q(z_0,R)} |G(z)|^2 dz \leq C$$

where

(15) $$\lambda = \min \left\{ \frac{2}{q} [q - (n+2)], \frac{2}{r} (r-n) \right\} > 0.$$

To prove it, we use the assumptions (4) - (8) and Hölder inequality.

Now, we extend the system (1) to the whole domain Q. Put

(16) $$A_e(z;p) = \begin{cases} A(z;p+u_0(x)), & z \in Q^+, \\ A((0;x);p+u_0(x)), & z \in Q^-, \end{cases}$$

$$f_e(z) = \begin{cases} f(z), & z \in Q^+, \\ 0 & , z \in Q^-, \end{cases}$$

$$G_e(z) = \begin{cases} G(z), & z \in Q^+, \\ 0 & , z \in Q^-. \end{cases}$$

It can be easily verified that

(17) $A_e(z;p)$ is continuous on $Q \times \mathbb{R}^m$,

(18) $(A_e(z;p)\xi, \xi) > 0$ for all $(z;p) \in Q \times \mathbb{R}^m$, $\xi \neq 0$,

(19) $f_e \in L_{s,loc}(Q)$ with the same s as for f,

(20) the assertion of Lemma 1 remains valid for the function G_e and $Q(z_0,R) \subset\subset Q$.

We formulate the next obvious result as

Lemma 2. The function

(21) $$v_e(z) = \begin{cases} v(z) & \text{on } Q^+ \\ 0 & \text{on } Q^- \end{cases}$$

is a weak solution of the system

(22) $$w_t - \operatorname{div}_x(A_e D_x w) = - f_e + \operatorname{div}_x G_e$$

on Q.

Denote further

(23) $$h_{z_0,R} = \frac{1}{\mu Q(z_0,R)} \int_{Q(z_0,R)} h(z)\, dz,$$

(24) $$\fint_{Q(z_0,R)} |h(z)|^2 dz = R^{-n-2} \int_{Q(z_0,R)} |h(z)|^2 dz.$$

Definition 6. Let w be a weak solution of (21) in Q. A point $z_0 \in Q$ is said to be a regular point of w if

(25) $$\lim_{R \to 0} \fint_{Q(z_0,R)} |w(z) - w_{z_0,R}|^2 dz = 0.$$

Lemma 3. Each point of Q is a regular point of the weak solution v_e of the system (22).

Proof. Let $z_0 = (t_0,x_0) \in Q$ be fixed, $Q(z_0,R) \subset\subset Q$. To prove that z_0 is regular we substitute first

(26) $$T = (t-t_0)R^{-2},\ X = (x-x_0)R^{-1},\ Z = (T,X),$$
$$v_R(T,X) = v_e(t_0+R^2T,\ x_0+RX).$$

For an arbitrary constant vector $H \in \mathbb{R}^m$ we get

(27) $$\fint_{Q(z_0,R)} |v_e(z) - (v_e)_{z_0,R}|^2 dz \le \fint_{Q(z_0,R)} |v_e(z) - H|^2 dz =$$
$$= \fint_{Q(0,1)} |v_R(Z) - H|^2\, dZ.$$

(The first inequality in (27) is due to the fact that the functional

$$I(H) = \fint_{Q(z_0,R)} |w(z) - H|^2\, dz$$

attains its minimum on $\mathbb{R}^m$ in the point $H = w_{z_0,R}$.)

Thus, z_0 is a regular point of v_e if there exists a sequence $\{v_{R_n}\}$ ($R_n \to 0+$ as $n \to \infty$) such that

(28) $$v_{R_n} \to p \text{ in } L_2(Q(0,1)),$$

(29) p is a constant vector function.

To prove (28) and (29) we go back to the system (22), substituting there for t, x, v_e from (26) and using the notation

(30) $$A_R(Z) = A_e(t_o+TR^2, x_o+XR; v_R(Z))$$
$$f_R(Z) = f_e(t_o+TR^2, x_o+XR),$$
$$G_R(Z) = G_e(t_o+TR^2, x_o+XR),$$

we see that $v_R(Z)$ weakly solves the system

(31) $$(w)_T - \mathrm{div}_X(A_R(Z)D_X w) = -f_R + \mathrm{div}_X G_R \quad \text{on } (Q)_R,$$

where $(Q)_R$ is the image of Q in the mapping (26).

$R>0$ going to zero, the set $(Q)_R$ expands to the whole space $\mathbb{R}^{1+n}$. Thus, choosing $K>0$, we obtain that

(32) $$\exists\, R(K)>0 : Q(0,K) \subset\subset (Q)_R \text{ for all } R<R(K).$$

It follows that each v_R $(R<R(K))$ is the solution of the system

(33) $$\forall \varphi \in C_o^\infty(Q(0,K))$$

$$\int_{Q(0,K)} [v_R \varphi_T - A_R(Z) D_X v_R D_X \varphi]\, dZ = \int_{Q(0,K)} [R^2 f_R \varphi + R\, G_R D_X \varphi]\, dZ$$

The class of systems (33) can be interpreted as a class of linear parabolic systems with bounded measurable coefficients $\{A_R\}_{R<R(K)}$. Because of the estimate

(34) $$\| v_R \|_{L_\infty(Q(0,K))} \leq \| v_e \|_{L_\infty(Q)}$$

and the continuity of $A_e(z,p)$ we can deduce that the coefficients A_R, $R<R(K)$, are equibounded and that all the systems of the class have the same constant γ of ellipticity.

To prove that $\{v_R\}_{R<R(K)}$ is a compact set in $L_2(Q(0,K/2))$

we take use of the Caccioppoli type estimate (see [1], (3.1)). Taking account of the possibility to estimate L_2-norm of $D_X v_R$ by means of the L_2-norm of v_R itself (see e.g. [3], Lemma 2.1) over the larger domain, we get finally

$$(35) \quad \| v_R \|^2_{W_2^{1/2,1}(Q(0,K/2))} \leq c(1+ \| v_R \|^2_{L_2(Q(0,K))}), \quad R < R(K).$$

From (34) and (35) follows

$$(36) \quad \| v_R \|^2_{W_2^{1/2,1}(Q(0,K/2))} \leq c(K), \quad R < R(K).$$

Because of the compactness of the imbedding of $W_2^{1/2,1}$ into L_2 it follows from (36) that we can choose the sequence $\{v_k\} = \{v_{R_k}\}$, $\lim_{k \to \infty} R_k = 0$, for which

(37) $\{v_k\}$ converges to a function p in $L_2(Q(0,K/2))$,

$D_X v_k \rightharpoonup D_X p$ in $L_2(Q(0,K/2))$,

$v_k \to p$ almost everywhere in $Q(0,K/2)$.

By means of the diagonal method we get the subsequence of $\{v_k\}$ (keeping the same notation for it) such that for each bounded domain $D \subset \mathbb{R}^{1+n}$ it is

(38) $v_k \to p$ and $D_X v_k \rightharpoonup D_X p$ in $L_2(D)$,

$v_k \to p$ almost everywhere on $\mathbb{R}^{1+n}$;

(in particular $p \in L_\infty(\mathbb{R}^{1+n})$).

Assumptions (6) - (8) and Lemma 1 give

(39) $R_k^2 f_k \to 0$ and $R_k G_k \to 0$ in $L_2(D)$.

($f_k = f_{R_k}$ and for the definition of f_R see (30); similarly for G_k.)

Let now φ be a fixed function of $C^{\infty}(\mathbb{R}^{1+n})$ with a compact support. We can rewrite (33) as

$$(40)\ \int_{\mathbb{R}^{1+n}}[v_k \varphi_T - A_k(Z)D_X v_k D_X \varphi]\, dZ = \int_{\mathbb{R}^{1+n}}[R_k^2 f_k \varphi + R_k G_k D_X \varphi]\, dZ.$$

According to (39), the right hand side of (40) tends to zero. Thanks to the uniform boundedness of the set $\{A_k\}$ on supp φ and the almost everywhere convergence

$$(41)\qquad \lim_{k \to \infty} A_k(Z) = A_e(z_o, p(Z)),$$

we get that the vector function p solves the equation

$$(42)\ \int_{\mathbb{R}^{1+n}}[p\varphi_T - A_e(z_o; p)D_X p\, D_X\varphi]\, dZ = 0, \quad \forall \varphi \in C^{\infty}(\mathbb{R}^{1+n})$$

supp φ is compact.

If $z_o \in Q^-$, then (25) with $w = v_e$ is trivial and z_o is a regular point of v_e.

If $z_o \in Q^+$, then (42) means that the vector function p is the weak solution of the system

$$(43)\qquad p_t - \mathrm{div}_x(A(z_o; p + u_o(x_o))D_x p) = 0$$

in $\mathbb{R}^{1+n}$. According to (Li), p is a constant vector function and thus z_o is regular, too.

If, finally, $z_o \in \Gamma$, then (42) gives that the p is a weak solution of the Cauchy problem with zero initial value for the system (43). So $p \equiv 0$ on $\mathbb{R}_+^{1+n}$, according to (Lb). From the trivial fact that $p \equiv 0$ on $\mathbb{R}_-^{1+n}$, we have again that z_o is a regular point of the solution v_e of (22) and the proof of Lemma 3 is completed.

As it was proved in [1],[2],[3], if for a weak solution of (22) all points of Q are regular, then $v_e \in C_{loc}^{0,\alpha/2,\alpha}(Q)$ and thanks to the assumptions on u_o, $u \in C_{loc}^{0,\alpha/2,\alpha}(Q^+ \cup \Gamma)$.

Remark. Let us mention now the "almost necessity" of Liouville condition:

Let $z_0 \in \Gamma$, let the system

(44) $$u_t - \operatorname{div}_x(A(z_0,u)\, D_x u) = 0$$

have the property (Re). Let u be a weak solution of the Cauchy problem for (44) on $\mathbb{R}_+^{1+n}$ with zero initial data. Let z be an arbitrary point of $\mathbb{R}_+^{1+n}$. We shall prove that (Re) implies that $u(z) = u(0) = 0$.

Let Q^+ be a set described in Sec. I which is, in addition, convex, bounded and such that the points 0 and z are contained in $\overline{Q^+}$. Using (Re) we get the existence of a constant C such that for every solution v of (44) with zero initial data the estimate

(45) $$\|v\|_{C^{0,\alpha/2,\alpha}(\overline{Q^+})} \leq c(\|v\|_{L_\infty(\mathbb{R}_+^{1+n})})$$

holds.

Putting $u_R(T,X) = u(TR^2, XR)$ we get a sequence of solutions of (44) with zero initial data and the same bound for $\|u_R\|_{L_\infty}$. Thus for all $R \geq 1$ the norms

$$\|u_R\|_{C^{0,\alpha/2,\alpha}(\overline{Q^+})}$$

are equibounded. Let $R \geq 1$, $z=(t,x) = (TR^2, XR)$. Then $(T,X) \in \overline{Q^+}$ and

$$|u(z) - u(0)| = |u_R(T,X) - u_R(0)| = c(|X|^\alpha + |T|^{\alpha/2}) =$$
$$= c\, R^{-\alpha}(|x|^\alpha + |t|^{\alpha/2}).$$

Letting $R \longrightarrow \infty$ we obtain $u(z) = u(0)$.

So we proved that the condition (Re) for the system (44) yields (Lb) in the point z_0. Similar assertion can be proved in the interior point z_0. ▾

R e f e r e n c e s

[1] S. CAMPANATO: Partial Hölder continuity of solutions of quasilinear parabolic systems of second order with linear growth, Rend. Sem. Mat. Univ. Padova, Vol. 64(1981), 60-75.

[2] M. GIAQUINTA, E. GIUSTI: Partial regularity for the solutions to nonlinear parabolic systems, Annali di Matematica Pura ed Appl. 97(1973), 253-266.

[3] M. GIAQUINTA, M. STRUWE: On the partial regularity of weak solutions of nonlinear parabolic systems, Math. Z. 179(1982), 437-451.

[4] M. STRUWE: A counterexample in regularity theory for parabolic systems, Czech. Math. Journal (to appear).

[5] M. GIAQUINTA, J. NEČAS: On the regularity of weak solutions to nonlinear elliptic systems via Liouville's type property, Comment. Math. Univ. Carolinae 20(1979), 111-121.

[6] M. GIAQUINTA, J. NEČAS: On the regularity of weak solutions to nonlinear elliptic systems of partial differential equations, J. reine angew. Math. 316(1980), 140-159.

[7] M. GIAQUINTA, J. NEČAS, O. JOHN, J. STARÁ: On the regularity up to the boundary for second order nonlinear elliptic systems, Pacific J. of Math. 99(1982), 1-17.

[8] B. KAWOHL: On Liouville theorems, continuity and Hölder continuity of weak solution to some quasilinear elliptic systems, Comment. Math. Univ. Carolinae 21 (1980), 679-697.

[9] S. HILDEBRANDT, K.O. WIDMAN: Sätze von Liouvilleschen Typ für quasilineare elliptische Gleichungen und Systeme, Nach. Akad. Wiss. Göttingen, II. Math. Klasse, 4(1979), 41-59.

[10] M. MEIER: Liouville theorems for nonlinear elliptic equations and systems, Manuscripta Math. 29(1979), 207-228.

[11] M. MEIER: Liouville theorems for nondiagonal elliptic systems in arbitrary dimensions, Math. Z. 176(1981), 123-133.

[12] O. JOHN: The interior regularity and the Liouville property for the quasilinear parabolic systems, Comment. Math. Univ. Carolinae 23(1982), 685-690.

Matematicko-fyzikální fakulta, Karlova universita, Sokolovská 83, 18600 Praha 8, Czechoslovakia

(Oblatum 19.7. 1984)