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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
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ON THE RADIUS OF A SET IN A HILBERT SPACE

Josef DANEŠ

Abstract: An infinite dimensional Hilbert space extension of Jung theorem and its application to measures of noncompactness are given.

Key words: radius of a set, measure of noncompactness.

Classification: 46G05, 52A40

Introduction. In [1] we have proved the inequality $\chi(M) \leq (1 - \delta(1))\ \alpha(M)$ between the Hausdorff and Kuratowski measures of noncompactness of any bounded subset M of a normed linear space X, where $\delta(.)$ is the modulus of convexity of the space X. If $X = H$ is a Hilbert space, then $1 - \delta(1) = \sqrt{3}/2$, so that $\chi(M) \leq (\sqrt{3}/2)\ \alpha(M)$ for any bounded subset M of H. Here we show that the constant $\sqrt{3}/2$ can be replaced by $1/\sqrt{2}$ and that this last constant is the best possible provided H is infinite dimensional. This result is an easy consequence of an infinite dimensional generalisation of the Jung theorem given here. The infinite dimensional Jung theorem for Hilbert spaces is supplied by three proofs. The first proof is based on lemma 4 (a "mushroom" lemma as its proof suggests) which gives an information concerning the distribution of points of a bounded subset M of H near the boundary of the smallest ball containing M . The second proof uses the classical Jung theorem and the reflexivity of H and from this point of view is the most natural one. The third (and shortest) one is due to H. Steinlein [4] and it is published here by his kind permission.

The results of this paper have been communicated on the summer school on "Nonlinear Functional Analysis and Mechanics", Stará Lesná, High Tatras, Czechoslovakia, Sept. 23 - 27 (1974); see [2].

<u>Notation</u>. In what follows, H is a real Hilbert space (it is easy to see that all results below remain true for complex Hilbert spaces). For M a non-empty bounded subset of H , $B(M,r)$ is the closed r-ball centered at M (that is, the set of all points x in H with $\inf \{\|x - y\| : y \in M\} \leq r$), $C(M,r) = \{x \in H : B(x,r) \supset M\}$, $d(M)$ the diameter of M , $r(M) = \inf \{r > 0 : C(M,r) \neq \emptyset\}$ the radius of M , $\chi(M) = \inf \{r > 0 : M$ has a finite r-net in $H\}$ the Hausdorff measure of noncompactness of M and $\alpha(M) = \inf \{d > 0 : M$ can be covered by a finite number of sets of diameter $\leq d\}$ the Kuratowski measure of noncompactness of M ; $\mathrm{cocl}(M)$ denotes the closed convex hull of M and $\mathrm{sp}(M)$ the linear span of M . Furthermore, h denotes the Hausdorff (pseudo-) metric in the space of all non-empty bounded subsets of H .

<u>Lemma 1</u>. If M and N are non-empty bounded subsets of H , then:

1) the set $C(M,r)$ is closed, convex and coincides with the set $\bigcap \{B(x,r) : x \in M\}$;

2) $0 \leq r \leq R$ implies $C(M,r) \subset C(M,R)$;

3) $M \subset N$ implies $C(M,r) \supset C(N,r)$;

4) $C(M,d(M)) \supset \mathrm{cocl}(M) \supset M$;

5) $B(C(M,r),a) \subset C(M,r+a)$ for all $r, a \geq 0$;

6) $B(x,r) \cap B(y,r) \subset B(\frac{x+y}{2},(r^2 - \|x-y\|^2/4)^{1/2})$ for all $x, y \in H$ and $r \geq 0$ with $\|x-y\| \leq 2r$;

7) $d(C(M,r)) \leq 2(r^2 - r(M)^2)^{1/2}$ for all $r \geq r(M)$;

7') $d(C(M,r)) \searrow 0$ as $r \searrow r(M)$.

Proof. The proof is easy and we shall prove only 7), for example. Let x, y in $C(M,r)$ be given. Then, by 6), $(x+y)/2 \in C(M,(r^2 - \|x-y\|^2/4)^{1/2})$ and hence $r(M) \leq$ $\leq (r^2 - \|x-y\|^2/4)^{1/2}$ which implies the result.

Lemma 2. Let M be a non-empty bounded subset of H. Then:

1) $\bigcap \{C(M,r) : r > r(M)\}$ consists of a unique point which we call the center of M and denote by $c(M)$; hence $C(M,r(M)) = \{c(M)\}$;

2) $r_n \longrightarrow r(M)+$ and $x_n \in C(M,r_n)$ $(n \geq 1)$ imply $x_n \longrightarrow c(M)$.

Proof. Use lemma 1, 7) and the Cantor lemma.

Theorem 1. If M is a non-empty bounded subset of H, then there exists a unique smallest ball containing it, namely the ball $B(c(M),r(M))$.

Proof. See lemma 2.

The following lemma will not be used in the following but it is interesting in itself.

Lemma 3. Let M and N be non-empty bounded subsets of H. Then

1) $M \subset B(N,a)$ implies $C(M,r + a) \supset C(N,r)$ and $r(M) \leq r(N) + a$;
2) $|r(M) - r(N)| \leq h(M,N)$;
3) $\|c(M) - c(N)\| \leq (h(M,N).(h(M,N) + r(M) + r(N))))^{1/2}$;
4) $r(\cdot)$ and $c(\cdot)$ are continuous with respect to the Hausdorff pseudo-metric ($r(\cdot)$ is nonexpansive and $c(\cdot)$ is locally Hölder of order 1/2).

Proof. 1) is trivial, 2) follows from 1) and 3) is a consequence of 1) and lemma 1, 7). The assertion 4) follows from 2) and 3).

Lemma 4. Let M be a non-empty bounded subset of H, $c = c(M)$ and $r = r(M)$. Then

$$c \in \operatorname{cocl}(M \cap (B(c,r) \setminus B(c,r - e)))$$

for each e $(0,r)$.

Proof. We may assume that $c = 0$. Assume, on the contrary, that $0 \notin N = \operatorname{cocl}(M \cap (B(0,r) \setminus B(0,r - e)))$ for some $e \in (0,r)$. Since N is a closed convex set and $0 \notin N$, there exists a hyperplane $E = \{ y + v \ : \ (v,y) = 0 \}$ $(H \ni v \neq 0)$ strictly separating 0 and N. Setting $E_1 = \{tv + y \ : t \leq 1, (y,v) = 0\}$ and $E_2 = \{ tv + y \ : \ t \geq 1, (y,v) = 0 \}$, we have $0 \in E_1$ and $N \subset E_2$. Let $0 < s < \min \{2, e/\|v\|\}$ be arbitrary and set $c' = sv$ and $r' = \max \{r - e + s \|v\|, (r^2 - (2-s)s \|v\|^2)^{1/2} \}$. It is clear that $r' \in (0,r)$.

We shall show that $M \subset B(c',r')$. Let x in H be given. Consider two cases:

1) $x \in E_1 \cap M$. Then $\|x\| \leq r - e$ and hence $\|x - c'\| \leq$

$\leq \|x\| + \|c'\| \leq (r - e) + s\|v\| \leq r'$, i.e. $x \in B(c',r')$.

2) $x \in E_2 \cap M$. Then $x = y + tv$ for some $t \geq 1$ and y with $(v,y) = 0$. We have

$$\|x - c'\|^2 = \|x\|^2 + \|c'\|^2 - 2(c',x) = \|x\|^2 + s^2\|v\|^2 - 2st\|v\|^2 \leq r^2 + s^2\|v\|^2 - 2s\|v\|^2 = r^2 - (2-s)s\|v\|^2 \leq \leq r'^2,$$

i.e. $x \in B(c',r')$.

We have shown that $M \subset B(c',r')$ with $r' < r(M)$, which is a contradiction. The proof of the lemma is finished.

__Remark.__ In the notation of lemma 4, the inclusion $c \in \mathrm{cocl}(M \cap \partial B(c,r))$ is generally false. If $\dim(H) > 1$ and M is not required to be closed, one easily finds counterexamples. If M is required to be closed, the counterexamples exist only in infinite dimensional spaces. For example, if H is infinite dimensional, take $M = \{(1 - 1/n)e_n : n \geq 1\}$ where $\{e_n : n \geq 1\}$ is an infinite orthonormal set in H. It is easy to see that $c(M) = 0$, $r(M) = 1$, but $M \cap \partial B(0,1) = \emptyset$ (moreover, $\mathrm{cocl}(M) \cap \partial B(0,1) = \emptyset$).

__Theorem 2__. (The generalized Jung theorem.) Let M be a non-empty bounded subset of H. Then $r(M) \leq d(M)/\sqrt{2}$.

__First proof__. We may assume that $c(M) = 0$. Let $r = r(M)$, $d = d(M)$ and take $e \in (0,r)$ arbitrarily. By lemma 4, we have

$$0 \in \mathrm{cocl}(M \cap (B(0,r) \setminus B(0,r-e))).$$

Let $a > 0$ be arbitrary. Then there are an integer $n > 0$, positive numbers $t_1,\dots,t_n$ and points $x_1,\dots,x_n$ in M such that

$\sum_{i=1}^{M} t_i = 1$, $r - e \leq \|x_i\| \leq r$ $(i = 1,\dots,n)$ and $\|x\| \leq a$, where $x = \sum_{i=1}^{n} t_i x_i$. We have

$$\|x_i - x_j\|^2 = \|x_i\|^2 + \|x_j\|^2 - 2(x_i,x_j) \quad (i,j = 1,\dots,n),$$

and hence

$$d^2 \geq \sum_{i=1}^{n} t_i \|x_i - x_j\|^2 = \sum_{i=1}^{n} t_i \|x_i\|^2 + \|x_j\|^2 -$$
$$- 2\Big(\sum_{i=1}^{n} t_i x_i, x_j\Big) = \sum_{i=1}^{n} t_i \|x_i\|^2 + \|x_j\|^2 - 2(x,x_j) \geq$$
$$\geq \sum_{i=1}^{n} t_i (r - e)^2 + (r - e)^2 - 2ar =$$
$$= 2(r - e)^2 - 2ar.$$

As $e \in (0,r)$ and $a > 0$ are arbitrary, we obtain $2r^2 \leq d^2$. The proof of the theorem is completed.

<u>Second proof</u>. This proof uses the classical Jung theorem for the case of finite dimensional spaces H ; it says that $r(M) \leq (n/(2(n+1)))^{1/2}\, d(M)$ provided $\dim(H) = n$ and M is a non-empty bounded subset of H.

Let M be as in the theorem and consider the system $F = \{ B(x,d(M)/\sqrt{2}) : x \in M \}$. The assertion of the theorem is equivalent to the non-emptiness of the intersection of all sets of the system F. Since F consists of weakly compact (and non-empty) subsets of H, it is sufficient to prove that F possesses the finite intersection property.

Let $x_1,\dots,x_n \in M$. By the classical Jung theorem, $x_1,\dots,x_n \in B(x,r')$ for some $x \in sp\{x_1,\dots,x_n\}$, where $r' = ((n+1)/(2(n+2)))^{1/2}\, d(\{x_1,\dots,x_n\}) \leq d/\sqrt{2}$. Hence $x \in \bigcap_{i=1}^{n} B(x_i,d/\sqrt{2}) \neq \emptyset$ and the proof is completed.

Third proof. ([4]) Assume $d < r\sqrt{2}$. Then $d(1 - (d/2r)^2)^{1/2} < r$. Choose $r' > r$ and $d' > d$ with $d' < r\sqrt{2}$ and $d'(1 - (d'/2r')^2)^{1/2} < r$. Then there exist $r'' \in (r,r')$, $x_0 \in H$ and $x_1 \in M \setminus \{x_0\}$ such that $M \subset B(x_0,r'')$ and $r'' - \|x_1 - x_0\| < d' - d$. Set $x_2 = x_0 +$ $+ (r''/\|x_1 - x_0\|)(x_1 - x_0)$ and $x_3 = x_0 +$ $+ ((r'' - d'^2/2r'')/\|x_1 - x_0\|)(x_1 - x_0)$. Then we have

$$M \subset B(x_1,d) \cap B(x_0,r'') \subset B(x_2,d') \cap B(x_0,r'') \subset$$

$$\subset B(x_3,d'(1 - (d'/2r'')^2)^{1/2}) \subset B(x_3,d'(1 - (d'/2r')^2)^{1/2})$$

which contradicts $d'(1 - (d'/2r')^2)^{1/2} < r$.

Remark. The constant $1/\sqrt{2}$ in theorem 2 is the best possible, provided H is infinite dimensional. Indeed, let $M = \{e_1, e_2, \ldots\}$ be an orthonormal infinite set in H. Then $d(M) = \sqrt{2}$ and $r(M) = 1$, because $r(M) \geqslant r(\{e_1, \ldots, e_n\}) = (n/(n+1))^{1/2}$ for all $n > 0$ and $B(0,1) \supset M$. (See also the remark following lemma 4.)

Theorem 3. $\chi(M) \leqslant \alpha(M)/\sqrt{2}$ for each bounded subset M of H.

Proof. Let $d > \alpha(M)$ and $M = M_1 \cup \ldots \cup M_n$ with $d(M_i) \leqslant d$ for $i = 1, \ldots, n$. By theorem 2,

$$M = \bigcup_{i=1}^{n} M_i \subset \bigcup_{i=1}^{n} B(c(M_i), d(M_i)/\sqrt{2}) \subset$$

$$\subset \bigcup_{i=1}^{n} B(c(M_i), d/\sqrt{2}),$$

so that $\chi(M) \leqslant d/\sqrt{2}$. Thus $\chi(M) \leqslant \alpha(M)/\sqrt{2}$.

Remark. The constant $1/\sqrt{2}$ in theorem 3 is the best possible provided H is infinite dimensional. Indeed, let M be as in the remark following theorem 2. Then $\alpha(M) = \sqrt{2}$

and $\chi(M) = 1$, because $d(N) = \sqrt{2}$ and $r(N) = 1$ for each infinite set $N \subset M$.

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