Commentationes Mathematicae Universitatis Carolinae

Jenö Szigeti

A note on reflective subcategories defined by partial algebras

Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 2, 319--323

Persistent URL: http://dml.cz/dmlcz/106304

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
25,2 (1984)

A NOTE ON REFLECTIVE SUBCATEGORIES DEFINED BY PARTIAL ALGEBRAS

Jenö SZIGETI

Abstract: By using a generalized partial F-algebra a full subcategory of a certain comma category will be defined. Then a sufficient condition will be given to provide the reflectivity of this subcategory.

Key words: F-algebra, generalized partial F-algebra, comma category, free completion of a g.p.a.

Classification: 18A25, 18A40, 18B20

---

1. **Preliminaries.** Given an endofunctor $F:\underline{A}\longrightarrow\underline{A}$ on the category $\underline{A}$ one can define the category $\underline{A}(F)$ of F-algebras (see e.g. in [1 - 5]). Let $U:\underline{A}(F)\longrightarrow\underline{A}$ denote the canonical forgetful functor.

A generalized partial F-algebra (a g.p.a.) in the sense of Koubek and Reiterman is a diagram $Fa\xleftarrow{p}x\xrightarrow{q}a$ in $\underline{A}$ (cf.[5]). In the present paper we shall consider the full subcategory $(a\downarrow U)^{\langle p,q\rangle}$ of the comma category $(a\downarrow U)$ which can be defined in a natural way by means of a g.p.a. Thus the free completion problem for the g.p.a. $Fa\xleftarrow{p}x\xrightarrow{q}a$ (see [3, 5]) will be equivalent to the existence of an initial object in $(a\downarrow U)^{\langle p,q\rangle}$. The main aim of this note is to establish conditions providing the reflectivity of $(a\downarrow U)^{\langle p,q\rangle}$ in $(a\downarrow U)$. Since the reflection functor sends $(a\downarrow U)$-initial objects to $(a\downarrow U)^{\langle p,q\rangle}$-initial ones we shall also obtain criteria for the existence of the free completion.

2. Reflective subcategories in $(a{\downarrow}U)$. Given a g.p.a. $Fa \xleftarrow{p} x \xrightarrow{q} a$ define the objects of the full subcategory $(a{\downarrow}U)^{\langle p,q\rangle}$ in $(a{\downarrow}U)$ by requiring the commutativity of (2./1).

(2./1)
$$\begin{array}{ccccc} Fa & \xleftarrow{\quad p \quad} & x & \xrightarrow{\quad q \quad} & a \\ {\scriptstyle Ff}\downarrow & & & & \downarrow{\scriptstyle f} \\ Fb & & \xrightarrow{\qquad u \qquad} & & b \end{array}$$

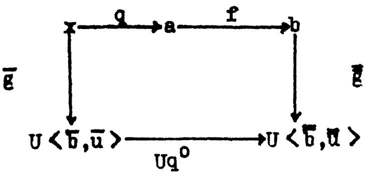
In other words: $\langle\langle b,u\rangle, f\rangle \in |(a{\downarrow}U)^{\langle p,q\rangle}|$ iff (2./1) commutes. Now we are ready to formulate our main result.

2.1. Theorem. Let $\langle\langle b,u\rangle, f\rangle \in |(a{\downarrow}U)|$ be an object and suppose that $\underline{A}(F)$ has coequalizers of all pairs. If there are initial objects in the comma categories $(x{\downarrow}U)$ and $(b{\downarrow}U)$ then there exists an initial object in $(\langle\langle b,u\rangle, f\rangle{\downarrow}E)$, where E is the natural $(a{\downarrow}U)^{\langle p,q\rangle} \longrightarrow (a{\downarrow}U)$ embedding.

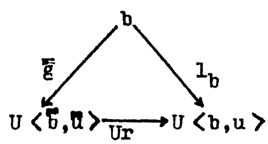
Proof. Let $x \xrightarrow{\bar g} \bar b \xleftarrow{\bar u} F\bar b$ and $b \xrightarrow{\bar{\bar g}} \bar{\bar b} \xleftarrow{\bar{\bar u}} F\bar{\bar b}$ represent initial objects in $(x{\downarrow}U)$ and $(b{\downarrow}U)$ respectively. Clearly, there exist unique $\underline{A}(F)$-morphisms $p^o, q^o : \langle \bar b, \bar u\rangle \longrightarrow \langle \bar{\bar b}, \bar{\bar u}\rangle$ and $r :$
$: \langle \bar{\bar b}, \bar{\bar u}\rangle \longrightarrow \langle b,u\rangle$ making the diagrams (2./2-4) commute.

(2./2)
$$\begin{array}{ccccccc} x & \xrightarrow{p} & Fa & \xrightarrow{Ff} & Fb & \xrightarrow{u} & b \\ {\scriptstyle \bar g}\downarrow & & & & & & \downarrow{\scriptstyle \bar{\bar g}} \\ U\langle \bar b, \bar u\rangle & & & \xrightarrow{\qquad Up^o \qquad} & & & U\langle \bar{\bar b}, \bar{\bar u}\rangle \end{array}$$

(2./3)



(2./4)



Form the coequalizer $\langle \bar{b},\bar{u}\rangle \overset{rop^o}{\underset{roq^o}{\rightrightarrows}} \langle b,u\rangle \xrightarrow{e^o} \langle b^o,u^o\rangle$ in $\underline{A}(F)$.

We claim that $e^o: \langle\langle b,u\rangle,f\rangle \longrightarrow E\,\langle\langle b^o,u^o\rangle,e^o of\rangle$ is initial in $(\langle\langle b,u\rangle,f\rangle \downarrow E)$. $u^o o(Fe^o)o(Ff)op = e^o ouo(Ff)op =$
$= e^o oro\bar{\bar{g}}ouo(Ff)op = e^o orop^o o\bar{g} = e^o oroq^o o\bar{g} = e^o oro\bar{\bar{g}}ofoq =$
$= e^o ofoq$ proves that $\langle\langle b^o,u^o\rangle,e^o of\rangle \in |(a\downarrow U)^{\langle p,q\rangle}|$ as required.
For a morphism $e':\langle\langle b,u\rangle,f\rangle \longrightarrow E\langle\langle b',u'\rangle,f'\rangle$ in $(a\downarrow U)$ we have
$(U(e'orop^o))o\bar{g} = e'orop^o o\bar{g} = e'oro\bar{\bar{g}}ouo(Ff)op = e'ouo(Ff)op =$
$= u'o(Fe')o(Ff)op = u'o(Ff')op = f'oq = e'ofoq = e'oro\bar{\bar{g}}ofoq =$
$= e'oroq^o o\bar{g} = (U(e'oroq^o))o\bar{g}$.
The $(x\downarrow U)$ initiality of $\langle\langle \bar{b},\bar{u}\rangle,\bar{g}\rangle$ immediately gives that
$e'orop^o = e'oroq^o$. Hence there is a unique $\underline{A}(F)$-morphism t:
: $\langle b^o,u^o\rangle \longrightarrow \langle b',u'\rangle$ with $toe^o = e'$. But easily can be seen
that for a morphism t: $\langle b^o,u^o\rangle \longrightarrow \langle b',u'\rangle$ the condition $toe^o =$
$= e'$ is equivalent to the commutativity of (2./5).‖

$$(2./5)\qquad \begin{array}{ccc} & & E\langle\langle b^{o},u^{o}\rangle,\ e^{o}\circ f\rangle \\ & \overset{e^{o}}{\nearrow} & \\ \langle\langle b,u\rangle,f\rangle & & \Big\downarrow Et \\ & \underset{e'}{\searrow} & \\ & & E\langle\langle b',u'\rangle,f'\rangle \end{array}$$

The next theorem is an obvious consequence of 2.1.

2.2. Theorem. Let $\underline{A}(F)$ have free algebras (i.e. $\dashv U$) and coequalizers of all pairs. Then

(i) for each g.p.a. $Fa \xleftarrow{p} x \xrightarrow{q} a$ the full subcategory $(a\downarrow U)^{\langle p,q\rangle}$ is reflective in $(a\downarrow U)$;

(ii) each g.p.a. $Fa \xleftarrow{p} x \xrightarrow{q} a$ has a free completion in $\underline{A}(F)$.

2.3. Remark. The (ii) part of the above theorem improves a result of Koubek and Reiterman ([5] p. 220). Indeed, if $\underline{A}$ is cocomplete, E-co-well-powered and $F:\underline{A}\longrightarrow\underline{A}$ preserves E of an image factorization system (E,M), then $\underline{A}(F)$ has coequalizers of all pairs (see [1 - 3]).

2.4. Remark. The reflection of an $(A\downarrow U)$-object $a \xrightarrow{f} b \xleftarrow{u} Fb$ in $(a\downarrow U)^{\langle p,q\rangle}$ also can be obtained by using a certain free completion. Take the g.p.a. $Fb \xleftarrow{\tilde{p}} x \amalg Fb \xrightarrow{\tilde{q}} b$ where $x \amalg Fb$ denotes an $\underline{A}$-coproduct with injections j_x, j_{Fb} and $\tilde{p}\circ j_x = (Ff)\circ p$, $\tilde{p}\circ j_{Fb} = 1_{Fb}$ defines $\tilde{p}$ and $\tilde{q}\circ j_x = f\circ q$, $\tilde{q}\circ j_{Fb} = u$ defines $\tilde{q}$. The free completion (2./6) of this g.p.a. yields the required reflection: $k: \langle\langle b,u\rangle,f\rangle \longrightarrow \langle\langle b^{o},u^{o}\rangle,k\circ f\rangle$.

$$\begin{array}{ccccc} Fb & \xleftarrow{\tilde{p}} & x \sqcup Fb & \xrightarrow{\tilde{q}} & b \\ {\scriptstyle Fk}\Big\downarrow & & & & \Big\downarrow{\scriptstyle k} \\ Fb^{o} & & \xrightarrow{\quad u^{o} \quad} & & b^{o} \end{array} \qquad (2./6)$$

References

[1] ADÁMEK J.: Colimits of algebras revisited, Bull. Austral. Math. Soc. 17(1977), 433-450.

[2] ADÁMEK J. and KOUBEK V.: Functorial algebras and automata, Kybernetika 13(1977), 245-260.

[3] ADÁMEK J. and TRNKOVÁ V.: Varietors and machines, COINS Technical Report 78-6, Univ. of Mass. at Amherst, 1978.

[4] BARR M.: Coequalizers and free triples, Math. Z. 116(1970), 307-322.

[5] KOUBEK V. and Reiterman J.: Categorical constructions of free algebras, colimits, and completions of partial algebras, J. Pure Appl. Algebra 14(1979), 195-231.

[6] MacLANE S.: Categories for the Working Mathematician, GTM 5, Springer-Verlag 1971.

Department of Analysis, Technical University, Miskolc-Egyetemváros, 3515, Hungary

(Oblatum 10.1. 1984)