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ON CONTINUOUS IMAGES OF ALMOST TYCHONOFF CUBES

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Abstract: A class $\mathcal{P}$ of Tychonoff spaces is called representative (for a class $\mathcal{Q}$) provided that any Tychonoff space (any space of $\mathcal{Q}$) is isomorphic with a closed subspace of a product of spaces belonging to $\mathcal{P}$. For a given class $\mathcal{P}$ of Tychonoff spaces the property of being representative is closely connected with the properties of continuous images of almost Tychonoff cubes with respect to the continuous maps into the spaces of $\mathcal{P}$. In this paper we study the properties of continuous images of almost Tychonoff cubes and apply the obtained results to the investigations connected with representativeness of classes of Tychonoff spaces.

Key words: Almost Tychonoff cube, dyadic space, almost compact space, almost dyadic space, representative class, weight, tightness, pseudocharacter, caliber, Souslin number, radial space, pseudoradial space, character

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Throughout this paper all topological spaces are supposed to be Tychonoff. $|A|$ denotes the cardinality of A and if A is a subset of a space X, then $\bar{A}$ denotes the closure of A in X. Recall that the tightness $t(X)$ of a space X is the smallest infinite cardinal τ such that if $x \in X$, $A \subset X$ and $x \in \bar{A}$, then there exists a set $B \subset A$ with $|B| \leq \tau$ and $x \in \bar{B}$. The pseudocharacter $\psi(X)$ of a space X is the smallest infinite cardinal τ such that for any $x \in X$ there exists a collection $\mathcal{U}$ of open subsets of X with $|\mathcal{U}| \leq \tau$ and $\bigcap\{U: U \in \mathcal{U}\} = \{x\}$. By $w(X)$ we always denote the weight of X. A subset of a space is said to be canonical closed provided that it is the closure of an open subset of X. βX denotes the Čech - Stone compactification of X. If τ is a cardinal, then τ^+ denotes the smallest cardinal greater than τ.
Recall that a space X is said to be almost compact provided that

the remainder $\beta X - X$ contains precisely one point. Obviously, any almost compact space is locally compact. Let τ be a cardinal, I the closed unit interval. The subspace $I^{\tau} - \{a\}$ of the Tychonoff cube I^{τ} where a is an arbitrary point of I^{τ} is denoted by S_{τ} and called an almost Tychonoff cube. It is well known that for $\tau > \aleph_0$ $\beta S_{\tau} = I^{\tau}$ i. e. for $\tau > \aleph_0$ the space S_{τ} is almost compact.

Definition 1. A space X is said to be almost dyadic provided that βX is a dyadic space and the remainder $\beta X - X$ contains exactly one point.

Clearly, any almost dyadic space is almost compact and for any $\tau > \aleph_0$ S_{τ} is almost dyadic. The space of all countable ordinals is almost compact without being almost dyadic.

The following three statements are easy to prove.

Proposition 1. If a subspace Y of a space Z is an almost compact space, then either Y is closed in Z or $\overline{Y} = \beta Y$ (and, consequently, $|\beta Y - Y| = 1$).

Proposition 2. Let X be an almost compact (almost dyadic) space and $f\colon X \longrightarrow Y$ a continuous map with $f(X) = Y$. Then the space Y is either compact (dyadic) or almost compact (almost dyadic).

Proposition 3. Let X be an almost compact space, $f\colon X \longrightarrow Y$ a continuous map, $f(X) = Y$ and Y be not compact. Then f is a perfect map.

Theorem 1. Let X be an almost dyadic space. Then the following hold:

(a) There is an uncoutable closed discrete subspace of X.

(b) X is not normal.

(c) X contains a canonical closed subspace which is non-metrizable and dyadic.

(d) X contains a subspace homeomorphic with $D^{\aleph_1}$ (the product of $\aleph_1$ discrete doubletons).

(e) $t(X) > \aleph_0$

(f) $\psi(X) > \aleph_0$ (not all points of X are of type G_{δ}).

(g) $\aleph_1$ is the caliber of X and, particularly, the Souslin number of the space X is countable (for the definition of caliber see e. g. [1]).

(h) X is not collectionwise Hausdorff and not even $\aleph_1$ - weakly collectionwise Hausdorff.

(i) The set of all dim Y where Y is a subspace of X is unbouded.

(j) X is not radial (for the definition of radial spaces see e. g. [1]).

(k) X is not countably compact.

Proof. (a) Since $|\beta X - X| = 1$ X is not first-countable at the point $p \in \beta X - X$. At the same time βX is a dyadic space and applying [7; p. 293, (i)] we obtain that there exists a discrete subspace M of X with $|M| = \aleph_1$ such that p is the only accumulation point of M in βX.

(b) Take an infinite closed discrete subspace M of X and arbitrary disjoint infinite subsets A, B of M. Then A, B are closed in X and their closures in βX are not disjoint ($|\beta X - X| = 1$).

(c) The space βX is dyadic and X is an open subspace of βX. Therefore for each $x \in X$ there exists a neighbourhood V_x with the compact βX-closure F_x contained in X. Since F_x is a canonical closed subset of the dyadic space βX F_x is a dyadic space. Let for each $x \in X$ F_x be metrizable. Then X is a first-countable dense subspace of βX so that (see [6]) βX is second-countable contradicting (a). Consequently, there exists $x \in X$ such that F_x is non-metrizable.

(d) Since any non-metrizable dyadic space contains a subspace homeomorphic with $D^{\aleph_1}$ (see [7]) (d) follows from (c).

(e) and (f) Since the tightness and the pseudocharacter are monotonic with respect to arbitrary subspaces and
$t(D^{\aleph_1}) > \aleph_0$, $(D^{\aleph_1}) > \aleph_0$ the statements (e) and (f) follow from (d).

(g) Since βX is a dyadic space the caliber of βX and any its open subspace is $\aleph_1$. Especially, the Souslin number of X is countable.

(h) Immediate from (g) and (a).

(i) For any $n \in N$ there exists a subspace Y of $D^{\aleph_1}$ with dim $Y \geq n$.

(j) It follows from (d) since $D^{\aleph_1}$ is not radial (see [2]).

(k) Immediate from (a).

Proposition 4. Let X be an almost dyadic space, $f\colon X \longrightarrow Y$ a continuous map and the space Y satisfy at least one of the following conditions:

(1) Y is normal

(2) Y does not contain an uncountable closed discrete subspace

(3) $t(Y) \leq \aleph_0$

(4) $\psi(Y) \leq \aleph_0$ (all points of Y are of type G_δ)

(5) Y does not contain a subspace homeomorphic with $D^{\aleph_1}$

(6) Y is collectionwise Hausdorff

(7) There exists $n \in N$ such that for any subspace Z of Y dim $Z \leq n$.

(8) Y is radial

(9) Y is countably compact

Then $\overline{f(X)}$ is a dyadic space and the map f is extendable to a continuous map $\beta X \longrightarrow Y$.

Proof. According to Proposition 2 the space f(X) is almost dyadic or dyadic. In the second case the proof is completed. Let f(X) be almost dyadic. Since the properties (1) - (9) are closed-hereditary, by Theorem 1 $\overline{f(X)} \neq f(X)$ and then by Proposition 1 $\overline{f(X)} = \beta(f(X))$ is a compact space. Thus, f can be extended to a continuous map $\beta X \longrightarrow Y$.

In particular, from Proposition 4 it follows:

Corollary 1. If a continuous image Y of an almost dyadic space is normal or does not contain an uncountable closed discrete subspace, then Y is a dyadic space.

Theorem 2. Let $\mathcal{E}$ be the class of all spaces satisfying at least one of the conditions (1) - (9) of Proposition 4 and $X = \prod\{X_\alpha : \alpha \in A\}$ a product of spaces belonging to $\mathcal{E}$. Then no almost dyadic space is homeomorphic with a closed subspace of the space X.

Proof. Immediate from Proposition 4 and [9; 17.2.2.].

Evidently, any topological property which is closed-hereditary and possessed by no almost dyadic space can participate in the formulations of Proposition 4 and Theorem 2. Hence, we have the following general result:

Theorem 3. Let $\mathcal{P}$ be a class of all spaces which do not contain a closed almost dyadic subspace and $X = \prod\{X_\alpha : \alpha \in A\}$ a product of spaces belonging to $\mathcal{P}$. Then no almost dyadic space (particularly, S_τ for $\tau > \aleph_0$) is homeomorphic with a closed subspace of X.

By the proof of Proposition 4 it is evident that if $f: X \longrightarrow Y$ is a continuous map, X an almost dyadic space and Y has a hereditary topological property which is not possessed by any almost dyadic space, then f(X) is a dyadic space. This together with Theorem 1 yield:

<u>Theorem 4.</u> Let X be an almost dyadic space, $f: X \longrightarrow Y$ a continuous map with $f(X) = Y$ and one of the following conditions be fulfilled:

(1′) Y is hereditary normal.

(2′) Y is hereditary collectionwise Hausdorff.

(3′) $t(Y) \leq \aleph_0$.

(4′) $\psi(Y) \leq \aleph_0$.

(5′) Any discrete subspace of Y is countable.

(6′) Y does not contain a subspace homeomorphic with $D^{\aleph_1}$.

Then Y is a compact space with countable base.

<u>Proof.</u> By Theorem 1 and the observations preceding Theorem 4 Y is a dyadic space and it is well known that a dyadic space possessing one of the properties (1′) - (6′) is metrizable.

Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of topological spaces. We shall say that the class $\mathcal{P}$ is representative for the class $\mathcal{Q}$ provided that any space belonging to the class $\mathcal{Q}$ is homeomorphic with a closed subspace of a product of spaces belonging to $\mathcal{P}$ (in the categorical language - $\mathcal{Q}$ is a subclass of the epireflective hull of $\mathcal{P}$ in the category of all Tychonoff spaces). A class $\mathcal{P}$ will be said to be representative provided that it is representative for the class of all Tychonoff spaces.

Theorem 2 implies now the following:

<u>Corollary 2.</u> The class $\mathcal{E}$ of all spaces satisfying at least one of the conditions (1) - (9) of Proposition 4 is not representative.

<u>Remark.</u> J. Vermeer showed in [13] that the class of all normal spaces is not representative.

Using the cardinals greater than $\aleph_0$ and $\aleph_1$ we can generalize the preceding results in such a way that we obtain rather strong necessary conditions for the property of being a representative class.

We outline these generalizations.

Proposition 5. Let τ, λ be cardinals with $\tau > \lambda \geqq \aleph_0$ and $f\colon S_\tau \longrightarrow Y$ a continuous map. Then at least one of the following assertions is true:

(a) The map f is perfect (and, particularly, $f(S_\tau)$ is closed in Y), $f(S_\tau)$ contains a closed discrete subspace of the cardinality τ (evidently, closed also in Y), $f(S_\tau)$ contains a topological copy of I^{λ^+} and $f(S_\tau)$ is almost dyadic (and, consequently, non-normal, non-collectionwise Hausdorff, etc.)

(b) $\overline{f(S_\tau)}$ is a dyadic space and the map f is extendable to a continuous map $\beta S_\tau \longrightarrow Y$.

Proof. If $\overline{f(S_\tau)}$ is compact, then, evidently, the assertion (b) holds. Let $\overline{f(S_\tau)}$ be not compact. Then applying Propositions 1 - 3 we obtain that $f(S_\tau)$ is almost dyadic and f is perfect. It is well known (see e. g. [7, p. 293, (i)]) that the space S_τ contains a discrete closed subspace A of the cardinality τ. Then f(A) is a closed discrete subspace of $f(S_\tau)$ and for all $y \in f(A)$ $f_{-1}(y)$ is finite ($f|_A$ is perfect and A is discrete). Consequently, $|f(A)| = |A| = \tau$ and this implies that the weight $w(\beta Z)$ of the compact space βZ where $Z = f(S_\tau)$ is not less than τ. At the same time βZ is a continuous image of I^τ. E. V. Ščepin (Е.В. Щепин [15]) proved that under these conditions $w(\beta Z) \geqq \tau$ implies that βZ contains a topological copy of the cube I^{λ^+}. It is easy to see that then Z also contains a topological copy of I^{λ^+}.

From Proposition 5 it immediately follows:

Theorem 5. If a class $\mathcal{P}$ of spaces is representative, then for any cardinal $\tau > \aleph_0$ there exists a space $X_\tau \in \mathcal{P}$ with the following properties: X_τ is not normal, X_τ is not collectionwise Hausdorff, X_τ contains a closed discrete subspace of the cardinality τ and X_τ contains a subspace homeomorphic with I^τ.

Corollary 3. If a class $\mathcal{P}$ is representative and closed-hereditary, then $\mathcal{P}$ contains all discrete spaces and all Tychonoff cubes I^τ.

Remark. If $\mathcal{P}$ is a left fitting class (i. e. if $f: X \longrightarrow Y$ is perfect and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$), then using the results of [8] we obtain the following simple characterization of representativeness: $\mathcal{P}$ is representative if and only if it contains all almost Tychonoff cubes.

The class of all factor spaces of ordered spaces coincides with the class of pseudoradial spaces studied in [2] (for the definition see also [1]). The property of being pseudoradial is closed-hereditary and the weight of any pseudoradial dyadic space does not exceed $c = 2^{\aleph_0}$ (see [2]). Hence, I^{c^+} is not pseudoradial and by Corollary 3 we have:

Theorem 6. The class of all pseudoradial spaces is not representative.

Proposition 6. The space ω_2 of all ordinals less than ω_2 (which is countably compact, normal, almost compact) and any non-compact continuous image Y of ω_2 contain a compact subspace of the uncountable tightness (and, consequently, a compact subspace of the uncountable pseudocharacter).

Proof. Let f be a continuous map of the space ω_2 onto a non-compact space Y. Then f is perfect and, obviously, $|Y| = \aleph_2$. Define a transfinite sequence $a: \omega_1 + 1 \longrightarrow Y$ as follows: a_0 is an arbitrary point of Y. For $\lambda < \omega_1 + 1$ choose $a_{\lambda+1} \in Y - \{a_\beta : \beta \leq \lambda\}$ such that if $x \in f_{-1}(a_{\lambda+1})$ and $y \in f_{-1}(a_\lambda)$, then $x > y$. For a limit ordinal α, $0 < \alpha \leq \omega_1$ put $a_\alpha = f(x_\alpha)$ where $x_\alpha = \sup\left(\bigcup\{f_{-1}(a_\lambda) : \lambda < \alpha\}\right)$. It is easy to check that the subspace $X = \{x_\tau : \tau \in \omega_1 + 1\}$ of the space Y is isomorphic with the space $\omega_1 + 1$.

Theorem 7. The class $\mathcal{P}$ of all spaces X such that the tightness of any compact subspace of X is countable is not representative for the class of all normal countably compact spaces.

Corollary 4. The class of all spaces with the countable tightness (with the countable pseudocharacter) is not representative for the class of all normal countably compact spaces.

Recently, a new cardinal invariant of the type of tightness has been introduced (see e. g. [3]). We write $t_0(X) \leq \aleph_0$ provided that every real function on X for which the restrictions to all countable subspaces of X are continuous is a continuous function on X. V. V. Uspenskii proved in [12] that if a cardinal τ is non-measurable, then $t_0(I^{\tau}) \leqq \aleph_0$. Since any space S_{τ} is locally homeomorphic with I^{τ} we obtain:

Proposition 8. If a cardinal τ is non-measurable, then $t_0(S_{\tau}) \leqq \aleph_0$.

Theorem 8. The class $\mathcal{Q}$ of all spaces X with $t_0(X) \leqq \aleph_0$ is representative for the class of all spaces of non-measurable cardinality. Particularly, if there exists no measurable cardinal, then the class $\mathcal{Q}$ is representative.

Proof. Let Y be a space of non-measurable cardinality. Then the weight τ of Y is also non-measurable and Y can be regarded as a subspace of I^{τ}. Obviously, $Y = \bigcap\{I^{\tau} - \{p\}: p \in I^{\tau} - Y\}$ and then (see [4; ch. II.,367]) Y is a closed subspace of the space $\prod\{I^{\tau} - \{p\}: p \in I^{\tau} - Y\}$. But $t_0(S_{\tau}) = \aleph_0$.

Let τ be an uncountable cardinal. Denote by $\mathcal{S}_{\tau}$ the class of all spaces X with the following property: Any continuous map $f: S_{\tau} \longrightarrow X$ is extendable to a continuous map $\beta S_{\tau} \longrightarrow X$. Then it is easy to see that it holds:

Proposition 9. The class $\mathcal{S}_{\tau}$ is the greatest (with respect to inclusion) productive and closed-hereditary class of spaces which does not contain the space S_{τ}.

As a corollary of Proposition 9 we obtain:

Theorem 10. Let A be a set and for any $\alpha \in A$ $\mathcal{P}_{\alpha}$ be a non-representative class of spaces. Then the class $\mathcal{P} = \bigcup\{\mathcal{P}_{\alpha}: \alpha \in A\}$ is not representative.

The following questions seem to be interesting:

(1) For which spaces is the class of all spaces with the countable tightness (pseudocharacter) representative?

(2) Is the class of all spaces with the countable preudocharacter representative for the class of all spaces with the countable tightness?

(3) Is the class of all spaces with the countable tightness representative for the class of all spaces with the countable pseudocharacter?

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