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ON THE EQUATION $y' = f(t, y)$ IN BANACH SPACES

Bogdan RZEPECKI

Abstract: In this note we are interested in the study of the differential equation $y' = f(t,y)$, $y(0) = x_0$ with applying the method of Euler polygons whenever f is a bounded continuous function with values in a Banach space. We prove for our equation Kneser's type results and theorems on the existence of extremal solutions and their continuous dependence on initial data.

Key-words: Differential equations in Banach spaces, Euler polygons, structure of the set solutions, extremal solutions, measure of noncompactness.

Classification: 34G20

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1. Introduction. Throughout this paper $I = [0,a]$, $(E, \|\cdot\|)$ is a Banach space with the zero element Θ, $B = \{x \in E: \|x - x_0\| \le r\}$, $f: I \times B \to E$ is a bounded continuous function, and $\|f(t,x)\| \le M$ on $I \times B$. Moreover, let $J = [0,T]$ with $T \le \min(a, r/M)$.

Let us consider the differential equation

$$(PC_x) \qquad y' = f(t,y), \ y(0) = x$$

where $x \in B$. In particular, by (PC) we shall denote the problem (PC_x) with $x = x_0$.

A function $y: J \to B$ is said to be a solution of (PC_x) on J, if it is a differentiable function on J, $y(0) = x$, and $y'(t) = f(t,y(t))$ for t in J.

Many papers related to the problem (PC) have been published, see e.g. [6]. Using the method of Euler polygonals we shall give Kneser's type results for (PC) (the set of solutions of (PC) is a nonempty continuum in the space of continuous functions from J to E) provided in particular some regularity Ambrosetti-Szufla type conditions (cf. [1],[18]) with respect to a measure of noncompactness defined in an axiomatic way. Employing the partial orderings induced by cones, existence of extremal solutions of (PC) and their continuous dependence on initial data are also proved.

2. <u>Notations and basic definitions</u>. Denote by $C(J)$ the space of all continuous functions from J to E, endowed with the usual supremum norm. Further, we will use standard notations. The closure of a subset X of E, its convex hull and its closed convex hull be denoted, respectively, by $\overline{X}$, conv (X) and $\overline{\text{conv}}$ (X). If X and Y are subsets of E and t, s are real numbers, then $tX + sY$ is the set of all $tx + sy$ such that $x \in X$ and $y \in Y$. $f[J \times X]$ will denote the image of $J \times X$ under f, and $f(t,X)$ is the set of all $f(t,x)$ with $x \in X$.

We introduce the following definitions:

<u>Definition 1</u>. Let $\varepsilon > 0$, $0 \leq p \leq T$, and let $w: J \longrightarrow E$ be a function with $w(0) = x_0$ and $\| w(t') - w(t'') \| \leq M|t' - t''|$ for $t', t'' \in J$. Let $r_i = p + i\varepsilon$ for $i = 1,2,\dots,k-1$, where k is an integer ≥ 1 such that $T - p = k\varepsilon$ (without loss of generality we assume that p/ε and T/ε are integers). By an Euler polygonal line for (PC) on J we mean any function g defined by

$$g(t) \equiv g(t;\varepsilon,p,w) = \begin{cases} w(t) \text{ for } 0 \le t \le p; \\ w(p) \text{ for } p \le t \le r_1; \\ g(r_i) + (t - r_i)f(r_i, g(r_i)) \\ \qquad \text{for } r_i \le t \le r_{i+1} \\ (i = 1,2,\dots,k-1). \end{cases}$$

<u>Definition 2</u>. Let n be a positive integer. A function $u: J \longrightarrow E$ is said to be 1/n-approximate solution of (PC) on J, if it satisfies the following conditions:

(i) $u(0) = x_0$;

(ii) $\| u(t') - u(t'')\| \le M|t' - t''|$ for $t', t'' \in J$;

(iii) $\sup_{t \in J} \| u(t) - x_0 - \int_0^t f(s,u(s))ds \| < 1/n$.

<u>Definition 3</u>. Let H be a subset of B. By $S_n(H)$ ($n = 1,2,\dots$) we denote the set of all 1/n-approximate solutions u of (PC) on J such that for every $t \in J$, $u(t) \in H$ and there exists $h_t \in [0,t]$ with $u(t) \in x_0 + h_t \cdot \overline{\text{conv}}\,(f[J \times H])$.

<u>Definition 4</u>. We say that the function f has the Peano property with respect to H if any sequence (v_n) with $v_n \in S_n(H)$ contains a subsequence which converges in C(J).

<u>Definition 5</u>. By S(H) we denote the set of all solutions of (PC) on J with their values in H.

Moreover, throughout this paper Φ and Ψ are functions defined in the following way.

<u>Definition 6</u>. Let $\mathcal{U}$ be the space of all bounded sequences of E with the usual supremum norm $\| \cdot \|$. We denote by $\Phi : \mathcal{U} \longrightarrow [0,\infty)$ a function with the following properties:

1^0 $\Phi(X) = 0$ for a convergent sequence $X \in \mathcal{U}$;

2^0 if $\Phi(X) = 0$ then X is a compact sequence of E;

3^0 $|\Phi(X_1) - \Phi(X_2)| \le L\|X_1 - X_2\|$ for $X_1, X_2 \in \mathcal{U}$;

4° $\Phi(\{x\} \cup X) = \Phi(X)$ and $\Phi(x + X) = \Phi(X)$ for $x \in E$ and $X \in \mathcal{U}$.

Definition 7. Let $\mathcal{V}$ be the family of all nonempty bounded subsets of E. We denote by $\Psi: \mathcal{V} \to [0,\infty)$ a function with the following properties:

1° $\Psi(X + \{x_n : n \geq 1\}) \leq \Psi(X)$ for $X \in \mathcal{V}$ and any convergent sequence (x_n) of E;

2° $\Psi(\overline{X}) = \Psi(X)$, $\Psi(\mathrm{conv}\, X) = \Psi(X)$ and $\Psi(\{\Theta\} \cup X) = \Psi(X)$ for $X \in \mathcal{V}$;

3° $\Psi(tX) \leq t \cdot \Psi(X)$ for $t \geq 0$ and $X \in \mathcal{V}$;

4° $\Psi(X_1) \leq \Psi(X_2)$ whenever $X_1 \subset X_2$;

5° if $\Psi(X) = 0$ then $\overline{X}$ is compact.

4. Compactness type conditions. Here we shall employ measures of noncompactness to impose conditions on f.

The notion of a measure of noncompactness was defined in many ways (see e.g. [2],[5],[15]). At first, Kuratowski [9] has introduced the function α which is a kind of a measure of noncompactness. (The measure $\alpha(X)$ of a nonempty bounded subset X of E is defined as the infimum of all $\varepsilon > 0$ such that there exists a finite covering of X by sets of diameter $\leq \varepsilon$.) Ambrosetti [1] proved the existence theorem for the problem (PC) under the assumption of uniform continuity of f with $\alpha(f(t,X)) \leq k \cdot \alpha(X)$ for all $t \in I$ and any $X \subset B$. A similar result, but without the assumption of the uniform continuity, has been proved by Szufla [18] under the condition $\alpha(f[I \times X]) \leq k \cdot \alpha(X)$ for any $X \subset B$. Further extension of Ambrosetti theorem, for uniformly continuous f, has been proved by Goebel and Rzymowski [7], and others (see [2].[3].[6]).

Next we suppose that $g: I \times [0,\infty) \to [0,\infty)$ is a continuous function such that $u(t) \equiv 0$ is the unique solution of the differential equation

$$u' = g(t,u), \ u(0) = 0$$

on the interval I.

Let Φ and Ψ be the functions defined in Sec. 3. We introduce the following conditions:

(I). $\liminf_{h \to 0_+} h^{-1} [\Phi(X + hf(t,X)) - \Phi(X)] \leq g(t, \Phi(X))$ for any sequence X of B and all $t \in I$, and assume in addition that f is uniformly continuous on $I \times B$.

(II). $\Psi(f[I \times X]) \leq k \cdot \Psi(X)$ for any subset X of B, where k is a nonnegative constant.

Let us note that the Kuratowski's measure of noncompactness α is an example of Φ and Ψ with the properties listed in definitions of Sec. 3. Other examples of such Φ, Ψ may be found in [2],[5],[15].

Lemma 1. Assume that the condition (I) is satisfied. Let $v_n: J \to B$ $(n = 1,2,\dots)$ be functions such that:

(1) $(v_n(0))$ is a convergent sequence;

(2) $\| v_n(t') - v_n(t'') \| \leq C|t' - t''|$ for each n and t', t'' in J;

(3) $\| v_n(t'') - v_n(t') - \int_{t'}^{t''} f(s,v_n(s))ds \| \leq K/n$ for each n and t', t'' in J.

Then, $\{v_n : n \geq 1\}$ is a conditionally compact subset of $C(J)$.

Proof. Define for $t \in J$ the function $p(t) = \Phi(V(t))$, where $V(t) = \{v_n(t) : n \geq 1\}$. Evidently $p(0) = 0$, p is continuous on J, and

$D_+p(t)$ $(= \liminf_{h \to 0_+} h^{-1} (p(t + h) - p(t))) \leq$

$$\leq \liminf_{h\to 0_+} h^{-1} [\Phi(V(t+h)) - \Phi(V(t) + hf(t,V(t)))] + g(t,p(t))$$

for $t \in J$.

Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|f(t',x') - f(t'',x'')\| < \varepsilon/4L$ whenever $|t' - t''| < \delta$ and $\|x' - x''\| < \delta$. By assumption (2), $\|v_n(s) - v_n(t)\| < \delta$ $(n = 1,2,\ldots)$ if $|s - t| < \delta/C$ with $t,s \in J$. Thus, for sufficiently small h, $0 < h < \min(\delta, \delta/C)$ and $n > n_0 = 8KL/\varepsilon h$ and $t \in J$ we obtain

$$\|v_n(t+h) - v_n(t) - hf(t,v_n(t))\| \leq \|v_n(t+h) - v_n(t) - \int_t^{t+h} f(s,v_n(s))ds\| + \int_t^{t+h}\|f(s,v_n(s)) - f(t,v_n(t))\|\,ds < \frac{K}{n} + \frac{\varepsilon h}{4L} < \frac{\varepsilon h}{8L} + \frac{\varepsilon h}{4L} < \frac{\varepsilon h}{2L};$$

hence

$$\Phi(V(t+h)) - \Phi(V(t) + hf(t,V(t))) = \Phi(\{v_n(t+h) : n > n_0\}) - \Phi(\{v_n(t) + hf(t,v_n(t)) : n > n_0\}) \leq L \cdot \sup_{n > n_0} \|v_n(t+h) - v_n(t) - hf(t,v_n(t))\| \leq \varepsilon h/2$$

and therefore

$$\lim_{h\to 0+} h^{-1} [\Phi(V(t+h)) - \Phi(V(t) + hf(t,V(t)))] = 0.$$

Consequently, $D_+p(t) \leq g(t,p(t))$ for $0 \leq t < T$.

From the Theorem 1.4.1 of [10] the following result may be deduced:

Let p, L be nonnegative continuous functions defined on I and $I \times [0,\infty)$, respectively. Denote by φ the maximal solution of the differential equation $u' = L(t,u)$, $u(0) = 0$ on the interval J. Assume that $p(0) = 0$, $L(t,0) \equiv 0$ on J, and the inequality $D_+p(t) \leq L(t,p(t))$ is satisfied on $[0,T)$. Then, $p(t)$ $\varphi(t)$ for $t \in I$.

Now, by the above result, we get $p(t) = 0$ on J. This implies that $\{v_n(t): n \geq 1\}$ $(0 \leq t \leq T)$ is conditionally compact and the Ascoli Theorem completes the proof.

Lemma 2. Assume that the condition (II) is satisfied and $T \leq \min(a, r_0/(b_0 + M))$ with $kT < 1$, where $0 \leq r_0 \leq r$ and $b_0 \geq 0$. Let (x_n), (z_n) be convergent subsequences of E with $\|x_n - x_0\| \leq r - r_0$ and $\|z_n\| \leq b_0$. Then there exist subsets $H_{(i,j)}$ $(i, j = 1,2,\ldots)$ of B such that

$$H_{(i,j)} = x_i + \overline{\bigcup\{t \cdot \operatorname{conv}(z_j + f[J \times H_{(i,j)}]): 0 \leq t \leq T\}}$$

and

$$H_0 = \overline{\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} H_{(i,j)}}$$

is a conditionally compact subset of B.

Proof. Applying arguments analogous to [4] (cf. [18], p. 797), we conclude that there exist our subsets $H_{(i,j)} \subset B$. Let us put: $X = \{x_n: n \geq 1\}$, $Z = \{z_n: n \geq 1\}$. Using properties of Ψ, we get

$$\begin{aligned}\Psi(H_0) &\leq \Psi(\overline{X} + \bigcup\{t \cdot \operatorname{conv}(Z + f[J \times H_0]): 0 \leq t \leq T\}) \leq \\ &\leq \Psi(\bigcup\{t \cdot \operatorname{conv}(Z + f[J \times H_0]): 0 \leq t \leq T\}) \leq \\ &\leq \Psi(\operatorname{conv}(\{\Theta\} \cup T \cdot \operatorname{conv}(Z + f[J \times H_0]))) = \\ &= \Psi(T \cdot \operatorname{conv}(Z + f[J \times H_0])) \leq T \cdot \Psi(Z + f[J \times H_0]) \leq \\ &\leq T \cdot \Psi(f[J \times H_0]) \leq kT \cdot \Psi(H_0).\end{aligned}$$

Hence $\Psi(H_0) = 0$ and therefore $\overline{H_0}$ is compact.

5. Kneser's type result. In the Proposition and Theorem 1 below, we assume in addition that the following condition is satisfied:

(+). There exists a closed subset H of B such that $f|_{J \times H}$ is uniformly continuous and $x_0 + t \cdot \operatorname{conv}(f[J \times H]) \subset H$ for each t in J.

The proof of our result is similar to that of the Theorem in [14].

<u>Proposition.</u> The sets $S_n(H)$ $(n = 1,2,\dots)$ are nonempty and connected in $C(J)$.

<u>Proof.</u> Let $g(\cdot) = g(\cdot, \varepsilon, p, w)$ be an Euler polygonal line for (PC) on J. Obviously, $g[J] \subset B$. For $r_i \leq t \leq r_{i+1}$ (here $r_i = p + i\varepsilon$, $i = 1,2,\dots,k-1$, and k is an integer ≥ 1 such that $T - p = \varepsilon k$) we have:

$$g(t) = w(p) + \sum_{m=1}^{i-1} (r_{m+1} - r_m) f(r_m, g(r_m)) + (t - r_i) f(r_i, g(r_i))$$

and

$$\| g(t) - x_0 - \int_0^t f(s,g(s))ds \| \leq \| w(p) - x_0 - \int_0^p f(s,w(s))ds \| + \int_p^{r_1} \| f(s,g(s)) \| ds + \sum_{m=1}^{i-1} \int_{r_m}^{r_{m+1}} \| f(r_m, g(r_m)) - f(s,g(s)) \| ds + \int_{r_i}^{t} \| f(r_i, g(r_i)) - f(s,g(s)) \| ds.$$

Denote by $u_\varepsilon(\cdot)$ the Euler polygonal line $g(\cdot, \varepsilon, p, w)$ with $p = 0$. Evidently, $u_\varepsilon(t) \equiv x_0$ for $0 \leq t \leq \varepsilon$ and since $u_\varepsilon(t) \in x_0 + (t - \varepsilon)\cdot \mathrm{conv}\,(f[J \times H])$ for $t \geq \varepsilon$, we infer by (+) that $u_\varepsilon[J] \subset H$.

Fix an index n. Proceeding similarly as in [14], by uniform continuity of $f|_{J\times H}$, we conclude that there exists $\varepsilon(n) > 0$ such that

$$\sup_{t \in J} \| u_\varepsilon(t) - x_0 - \int_0^t f(s,u_\varepsilon(s))ds \| < 1/n$$

for each $\varepsilon < \varepsilon(n)$. Consequently, $u_\varepsilon \in S_n(H)$ whenever $\varepsilon < \varepsilon(n)$.

Assume that $w \in S_n(H)$. An argument similar to that in [14] implies that there is a positive $\varepsilon_w \leq \varepsilon(n)$ such that for $\varepsilon < \varepsilon_w$ and $0 \leq p \leq T$:

$$\sup_{t \in J} \| g(t; \varepsilon, p, w) - x_0 - \int_0^t f(s, g(s; \varepsilon, p, w))ds \| < 1/n.$$

Furthermore, for $0 \le t \le p$ there exists $h_t \in [0, t]$ and

$$g(t; \varepsilon, p, w) = w(t) \in x_0 + h_t \cdot \overline{\text{conv}}\,(f[J \times H]).$$

If $r_i \le t \le r_{i+1}$, then

$$\begin{aligned} g(t; \varepsilon, p, w) \in w(p) + \sum_{m=1}^{i-1} \varepsilon f[J \times H] + (t - r_i) f[J \times H] \subset \\ \subset x_0 + h_p \cdot \overline{\text{conv}}\,(f[J \times H]) + \sum_{m=1}^{i-1} \varepsilon \cdot \overline{\text{conv}}\,(f[J \times H]) + \\ + (t - r_i) \cdot \overline{\text{conv}}\,(f[J \times H]) = \\ = x_0 + (h_p + t - p - \varepsilon) \cdot \overline{\text{conv}}\,(f[J \times H]) \end{aligned}$$

and $h_p + t - p - \varepsilon \le t$. From this we deduce that Euler polygonal line $g(\cdot, \varepsilon, p, w) \in S_n(H)$ for each $\varepsilon < \varepsilon_w$ and $0 \le p \le T$.

Let us put:

$$U = \{u_\varepsilon : 0 < \varepsilon < \varepsilon(n)\},$$
$$V_w = \{g(\cdot, \eta_w, p, w) : 0 \le p \le T\}$$

with $w \in S_n(H)$ and $\eta_w < \varepsilon_w$. Modifying the proof from [14] (cf. [21], p. 664) we prove that the sets U, V_w are connected in $C(J)$. Now, we set

$$W = \bigcup \{U \cup V_w : w \in S_n(H)\}.$$

Since $w(\cdot) = g(\cdot, \eta_w, T, w) \in V_w$ for $w \in S_n(H)$, we get $S_n(H) \subset W$. As $U \subset S_n(H)$ and $V_w \subset S_n(H)$, $W \subset S_n(H)$. Further, we observe that $g(\cdot, \eta_w, 0, w) = u_{\eta_w}(\cdot) \in U \cap V_w$. Therefore $U \cup V_w$ is connected in $C(J)$. Consequently, $S_n(H) = W$ is a connected subset of $C(J)$.

Now, we are in position to prove

<u>Theorem 1</u>. If the function f has the Peano property with respect to H, then the set $S(H)$ is nonempty, compact and connected in the space $C(J)$.

<u>Proof</u>. The integral mean-value theorem may be stated as follows: $\int_0^t \omega(s)ds \in t \cdot \overline{\text{conv}}\,(\{\omega(s) : 0 \le s \le t\})$. Hence $S(H) \subset \bigcap_{n \ge 1} S_n(H)$, and consequently $S(H) = \bigcap_{n \ge 1} \overline{S_n(H)}$.

Assume that f has the Peano property with respect to H. Then (see Lemma 1 in [14]) S(H) is nonempty and compact. By Lemma 2 of [14] with $X_n = \overline{S_n(H)}$ and the facts above, $\bigcap_{n \geq 1} \overline{S_n(H)}$ is a connected subset of C(J). This completes the proof.

<u>Example 1</u>. Suppose that the condition (I) is satisfied.

Let $T = \min(a, r/M)$. We have: $x_0 + t \cdot \operatorname{conv}(f[J \times B]) \subset B$ for $t \in J$. It is obvious that Lemma 1 is applicable to any sequence (v_n) with $v_n \in \overline{S_n(B)}$. Therefore f has the Peano property with respect to B. Thus all assumptions of Theorem 1 are satisfied, S(B) (the set of all solutions of (PC) defined on J) is a continuum in C(J).

Let $T = \min(a, r_0/M)$ with $0 \leq r_0 \leq r$. Let Q be a nonempty subset of the closed ball of E with the center in x_0 and of radius $r - r_0$. Denote by S(x) the set of all solutions of (PC_x) on J. We shall prove below that if the set Q is connected then

$$S = \bigcup \{S(x) : x \in Q\}$$

is a connected subset of C(J).

In fact, let us assume that S is not connected. Thus there are nonempty sets S_1, S_2 such that $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$ and $S_i = S \cap \overline{S_i}$ $(i = 1,2)$. Define the sets Q_i $(i = 1,2)$ by

$$Q_i = \{x \in Q: \text{ for some } y \in S_i \text{ we have } y(0) = x\}.$$

Note $Q_i \neq \emptyset$ and $Q = Q_1 \cup Q_2$. We show that $Q_1 \cap Q_2 = \emptyset$ and $Q_i = Q \cap \overline{Q_i}$.

Suppose on the contrary that $Q_1 \cap Q_2 \neq \emptyset$. Let $x \in Q_1 \cap Q_2$. Put $Y_i = S(x) \cap S_i$ $(i = 1,2)$. Evidently $Y_i \neq \emptyset$, $Y_1 \cup Y_2 = S(x)$ and $Y_1 \cap Y_2 = \emptyset$. Furthermore,

$$\overline{Y_i} \subset \overline{S(x)} \cap \overline{S_i} = S(x) \cap \overline{S_i} \subset S \cap \overline{S_i} \subset S_i,$$

hence $S(x) \cap \overline{Y_i} \subset Y_i$. Therefore S(x) is disconnected, in contradiction with the connectedness of S(x).

To show that $Q_1 = Q \cap \overline{Q_1}$ let $x \in Q \cap \overline{Q_1}$. Then $x \in Q$ and there exist $x_n \in Q$, $y_n \in S_1$ $(n = 1,2,\dots)$ such that $y_n(0) = x_n$ and $\|x_n - x\| \to 0$ as $n \to \infty$. By Lemma 1 we conclude that $\bigcup_{n=1}^{\infty} S(x_n)$ is a conditionally compact subset of $C(J)$. Hence (y_n) contains a subsequence which converges uniformly on J to some function y_0 and $y_0 \in S(x)$. Since $y_0 \in \overline{S_1}$ and $S \cap \overline{S_1} \subset S_1$ it follows that $y_0 \in S_1$. This implies $x \in Q_1$, and the proof is complete.

By the facts above, Q is disconnected and contradicting our assumption. This proves that the set S is connected in $C(J)$.

Moreover, notice that using Lemma 1 we obtain: S is compact in $C(J)$ whenever Q is a compact subset of E.

Example 2. Let the condition (II) be satisfied and $T \le \min(a, r/M)$ with $kT < 1$.

By Lemma 2 there exists a compact subset B_0 of B such that

$$B_0 = x_0 + \overline{\bigcup \{t \cdot \operatorname{conv}(f[J \times B_0]) : 0 \le t \le T\}}.$$

Since $f|_{I \times B_0}$ is uniformly continuous and

$$x_0 + t \cdot \operatorname{conv}(f[J \times B_0]) \subset x_0 + \bigcup \{t \cdot \operatorname{conv}(f[J \times B_0]) : 0 \le t \le T\} \subset B_0,$$

the condition (+) is satisfied for $H = B_0$. Let $X = (v_n)$ be a sequence with $v_n \in \overline{S_n(B_0)}$. The set $\overline{X}$ is a closed equicontinuous subset of $C(J)$. Since B_0 is compact, Ascoli's Theorem proves that $\overline{X}$ is compact in $C(J)$. Thus f has the Peano property with respect to B_0. Therefore all assumptions of Theorem 1 are satisfied, and we are done.

For more results of Kneser type we refer to Szufla, e.g. [19], [20].

6. <u>Extremal solutions</u>. Throughout this section it will be assumed that K is a solid cone in E (i.e., K is a closed subset of E with nonvoid interior such that x, y, z, $-z \in K$ and t, $s \geq 0$ imply $tx + sy \in K$ and $z = \Theta$). The partial orderings on E induced by K are $x \leq y$ if $y - x \in K$ and $x < y$ if $y - x \in \operatorname{Int} K$ (the interior of K).

We say that a function $f: J \times B \longrightarrow E$ is nondecreasing if $x_1 \leq x_2$ implies $f(t,x_1) \leq f(t,x_2)$ for each t in J.

The following theorem ([17], Th. 70.1, p. 224) on "strong differential inequalities" will be needed for later use:

Let $f: J \times B \longrightarrow E$ be a nondecreasing function. Suppose that u, v are continuous functions from $[0,T)$ into B satisfying the following conditions:

(1) $u(0) \leq v(0)$ (or, $u(0) < v(0)$);

(2) $u_+'(t) \leq f(t,u(t))$ for $0 \leq t < T$;

(3) $f(t,v(t)) < v_+'(t)$ (or, resp. $f(t,v(t)) \leq v_+'(t)$) for $0 \leq t < T$.

Under our assumptions $u(t) < v(t)$ for $0 < t < T$.

A solution y_x^0 of the problem (PC_x) on the interval J is called a maximal solution, if for every solution y of (PC_x) existing on J, the relation $y(t) \leq y_x^0(t)$ holds on J.

<u>Theorem 2</u>. Suppose that the function f is nondecreasing and the condition (I) (resp. (II)) (see Sec. 4) is satisfied. Let $0 \leq r_0 \leq r$, $b_0 > 0$ and $b = r_0/(b_0 + M)$. Let $J = [0,T_0]$, where $T_0 < T = \min(a,b)$ (resp. $T \leq \min(a,b)$ with $kT < 1$).

Then, for every $x \in E$ such that $\| x - x_0 \| \leq r - r_0$, (PC_x) has a unique maximal solution y_x^0 on J_0. Moreover, the mapping

$$x \longmapsto y_x^0$$

from $\mathfrak{X} = \{x \in E : \|x - x_0\| \leq r - r_0,\ x_0 < x\}$ into $C(J)$ is continuous at the point x_0.

Proof. Consider the differential equations

(PC_m) $\quad y' = z_m + f(t,y),\ y(0) = x\ (m = 1,2,\dots)$

where $\|x - x_0\| \leq r - r_0$ and (z_m) is a sequence of E such that $\Theta < z_m$ and $\|z_m\| \leq 1/m$.

Denote by $B(x,r_0)$ the closed ball of E with the center in x and of radius r_0. The functions $f_m(t,x) = z_m + f(t,x)$ ($m = 1,2,\dots$) are bounded on $I \times B(x,r_0)$ by $b_0 + M$, $\Phi(X + hf_m(t,X)) \leq \Phi(X + hf(t,X))$ and $\Psi(f_m[I \times X]) \leq \Psi(f[I \times X])$. Therefore, by the examples of Sec. 5 and our lemmas, if the condition (I) (resp., (II) with $kT < 1$) is satisfied, then (PC_m) has a solution y_m defined on J and $\{y_m : m \geq 1\}$ is a conditionally compact subset of $C(J)$.

Assume that $(y_{i(m)})$ is a subsequence of (y_m) which converges uniformly to the limit y^0. Evidently, y^0 is a solution of (PC_x) on J. Let y be any solution of (PC_x). We have:

$$y(0) = x = y_{i(m)}(0),$$

$$y'(t) = f(t,y(t)),$$

$$f(t,y_{i(m)}(t)) < z_{i(m)} + f(t,y_{i(m)}(t)) = y'_{i(m)}(t);$$

therefore, by the result on "strong differential inequalities", $y(t) < y_{i(m)}(t)$ ($m = 1,2,\dots$) for $0 < t < T$. Now, taking the limit as $m \to \infty$, conclude $y(t) \leq y^0(t)$ for $0 \leq t < T$. Consequently, y^0 is the desired maximal solution of (PC_x) on J_0.

Let (x_i) be a sequence of $\mathfrak{X}$ which converges to x_0. Denote by m_i, m_0 the maximal solutions of (PC_{x_i}) and (PC), respectively. To show that

$$\sup_{t \in J_0} \|m_i(t) - m_0(t) - \qquad\qquad \text{as } i \to \infty,$$

let $(m_{k(i)})$ be any subsequence of (m_i). In view of Lemma 1 (resp. Lemma 2), $(m_{k(i)})$ contains a convergent subsequence whenever the condition (I) (resp., (II)) holds.

Let $(m_{n(i)})$ be a convergent subsequence of $(m_{k(i)})$ and $m_{n(i)}(t) \to m(t)$ $(i \to \infty)$ uniformly on J_o. By

$$m_o(0) = x_o < x_{n(i)} = m_{n(i)}(0),$$
$$m_o'(t) = f(t, m_o(t)),$$
$$m_{n(i)}'(t) = f(t, m_{n(i)}(t)),$$

we may apply the theorem on "strong differential inequalities" to obtain $m_o(t) < m_{n(i)}(t)$ $(i = 1, 2, \ldots)$ for $0 < t < T$. Hence $m_o(t) \leq m(t)$ for $0 \leq t < T$ and since m is a solution of (PC), $m(t) \equiv m_o(t)$ on J_o.

In a similar way one can introduce the definition of the minimal solution of (PC_x) and formulate a result analogous to Theorem 2.

7. <u>Dependence on parameters</u>. The solution of (PC_x) is an operator (multivalued, in general) defined on some spaces of points (f, x). In [11] and [12] we characterize sufficient conditions for this operator to be continuous. Here we give a version of the Krasnoselskii and Krein result (see [8]) on continuous dependence of a solution of the differential equation $y' = F(t, y, \lambda)$ on the parameter λ.

Let $\Omega = I \times B \times \Lambda$, where Λ is some space with a limit point λ_o. We assume that $F: \Omega \to E$ is a bounded mapping and the function $(t, x) \mapsto F(t, x, \lambda)$ is continuous on $I \times B$ uniformly with respect to $(t, x, \lambda) \in \Omega$ (i.e., for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$\|F(t_1, x_1, \lambda) - F(t_2, x_2, \lambda)\| < \varepsilon$ whenever $|t_1 - t_2| < \delta$,

$\|x_1 - x_2\| < \delta$ and $\lambda \in \Lambda$).

Next, denote by S_λ the set of all solutions of the equation

$$y' = F(t,y,\lambda),\ y(0) = x_0$$

on the interval $J = [0, \min(a, r/M)]$ with $\|F(t,x,\lambda)\| \leq M$ on Ω.

Theorem 3. Suppose that

$$\liminf_{h \to 0+} h^{-1} [\Phi(X + hF(t,X,\lambda)) - \Phi(X)] \leq g(t, \Phi(X))$$

for any sequence X of B and all $(t,\lambda) \in I \times \Lambda$, where $g: I \times [0,\infty) \to [0,\infty)$ is a continuous function such that $u(t) \equiv \equiv 0$ is the unique solution of the differential equation $u' = g(t,u)$, $u(0) = 0$ on the interval I. Moreover, let

$$(*) \quad \lim_{\lambda \to \lambda_0} \int_0^t F(s,x,\lambda)ds = \int_0^t F(s,x,\lambda_0)ds$$

for every (t,x) in $I \times B$.

Under our assumptions for any $\eta > 0$ there exists a neighbourhood U of the point λ_0 such that if $\lambda \in U$, then

$$\inf_{y \in S_{\lambda_0}} \sup_{t \in J} \|y_\lambda(t) - y(t)\| < \eta$$

for every $y_\lambda \in S_\lambda$.

Proof. By Example 1, $S_\lambda \neq \emptyset$ for each $\lambda \in \Lambda$. Suppose that the theorem is false. Then, there is $\lambda > 0$ and for every n (n = = 1,2,...) there exist $\lambda_n \in \Lambda$ and $y_n \in S_{\lambda_n}$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0$ and

$$\inf_{y \in S_{\lambda_0}} \sup_{t \in J} \|y_n(t) - y(t)\| \geq \eta \quad (n = 1,2,\ldots).$$

As in the proof of Lemma 1, we obtain that $\{y_n : n \geq 1\}$ is a conditionally compact subset of C(J). Let $(y_{i(n)})$ be a convergent subsequence of (y_n) with $y_{i(n)}(t) \to y_0(t)$ $(n \to \infty)$ uniformly on J.

Now observe that by means of $(*)$ we obtain

$$\lim_{n\to\infty} \int_0^t F(s,\hat{q}(s), \lambda_n)ds = \int_0^t F(s,\hat{q}(s), \lambda_0)ds$$

for every on J piecewise constant function $\hat{q}$. Therefore (cf. [8]), one can prove that

$$\lim_{n\to\infty} \int_0^t F(s,y_{i(n)}(s), \lambda_{i(n)})ds = \int_0^t F(s,y_0(s), \lambda_0)ds$$

for $t \in J$. From this it follows that $y_0 \in S_{\lambda_0}$ and consequently

$$\inf_{y\in S_{\lambda_0}} \sup_{t\in J} \|y_{i(n)}(t) - y(t)\| \to 0 \text{ as } n \to \infty ,$$

a contradiction. The proof is therefore complete.

It is known that if a function $G: I\times B \to E$ is Lipschitz on B with a constant k, then $\alpha(G(t,X)) \le k \cdot \alpha(X)$ (here α is the Kuratowski measure of noncompactness) for $t \in I$ and $X \subset B$. From this remark and Theorem 3 we obtain the following result:

Let (λ_n) be a sequence of Λ convergent to λ_0, let $y_m \in S_{\lambda_m}$ $(m = 0,1,\dots)$ and S_{λ_0} is singleton. Further, let

$$\lim_{n\to\infty} \int_0^t F(s,x, \lambda_n)ds = \int_0^t F(s,x, \lambda_0)ds$$

for $(t,x) \in I\times B$. Assume in addition that

$\|F(t,x_1, \lambda) - F(t,x_2, \lambda)\| \le p(t) \|x_1 - x_2\|$ for $t\in I$, x_1, x_2 in B and $\lambda \in \Lambda$, where p is an integrable function such that $2\cdot\sup_{t\in I} \int_0^t p(s)ds < 1$. Then

$$\sup_{t\in I} \|y_n(t) - y_0(t)\| \to 0$$

as $n \to \infty$.

8. <u>Appendix</u>. The object of this appendix is to derive a result of Stokes type ([16]) on existence of solutions of (PC) on the half-line $t \ge 0$ via the fixed-point theorem given below.

Denote by $C[0,\infty)$ the set of all continuous functions from the nonnegative reals into E (E is our Banach space with the Kuratowski measure of noncompactness α). The set $C[0,\infty)$ will be considered as a vector space endowed with the topology of uniform convergence on compact subsets of $[0,\infty)$.

We use the following fixed-point theorem [13]:

Let $\mathfrak{X}$ be a nonempty closed convex and bounded subset of $C[0,\infty)$. Let Γ be a function which assigns to each nonempty subset X of $\mathfrak{X}$ a nonnegative real number $\Gamma(X)$ with the following properties: (1) $\Gamma(\overline{\text{conv}}\, X) = \Gamma(X)$, (2) if $\Gamma(X) = 0$ then $\overline{X}$ is compact, and (3) $\bigcap_{n\geq 1} X_n$ is nonempty compact whenever (X_n) is a decreasing sequence of nonempty closed convex subsets of $\mathfrak{X}$ and $\Gamma(X_n) \to 0$ as $n \to \infty$. Suppose that T is a continuous mapping of $\mathfrak{X}$ into itself such that $\Gamma(T[X]) \leq \varphi(\Gamma(X))$ for each nonempty subset X of $\mathfrak{X}$, where φ is a right-continuous function with $\varphi(t) < t$ for $t > 0$. Under the hypotheses, T has a fixed point in $\mathfrak{X}$.

For $X \subset C[0,\infty)$ and $t \geq 0$ we denote by $X(t)$ the set of all $x(t)$ such that $x \in X$. We state the Ascoli Theorem as follows: A subset X of $C[0,\infty)$ is conditionally compact if and only if X is almost equicontinuous and $\overline{X(t)}$ is compact in E for every $t \geq 0$.

Let us denote by α_I the Kuratowski measure of noncompactness on the space C(I). A. Ambrosetti [1] proved that

$$\alpha_I(X) = \alpha(\bigcup\{X(t) : t \in I\}) = \sup_{t \in I} \alpha(X(t))$$

for each bounded equicontinuous subset X of C(I). We shall use also the following theorem due to Kuratowski [9]: If (X_n) is a decreasing sequence of nonempty closed bounded subsets of C(I) and $\alpha_I(X_n) \to 0$ as $n \to \infty$, then $\bigcap_{n\geq 1} X_n$ is nonempty and compact subset of $C(I)$.

Theorem 4. Suppose that $f_0 : [0,\infty) \times E \to E$ is a continuous function such that $\|f_0(t,x)\| \le H(t, \|x\|)$ for $t \ge 0$ and $x \in E$, where H is continuous and monotonically nondecreasing in the second variable. Let $L : [0,\infty) \to [0,\infty)$ be a continuous function with $\int_0^\infty L(s)ds \le 1$, and $\varphi : [0,\infty) \to [0,\infty)$ a nondecreasing right-continuous function with $\varphi(t) < t$ for $t > 0$. Further, let the scalar differential equation

$$u' = H(t,u), \ u(0) = \|x_0\|$$

have a bounded solution u and $u(t) \le u_0$ for $t \ge 0$. Assume in addition that

$$\alpha(f_0[I \times X]) \le \sup_{t \in I} L(t) \cdot \varphi(\alpha(X))$$

for any compact subinterval I of $[0,\infty)$ and any $X \subset E$ bounded by u_0.

Then there exists a solution y of the equation

$$y' = f_0(t,y), \ y(0) = x_0$$

satisfying the inequality $\|y(t)\| \le u(t)$ for every $t \ge 0$.

Proof. Let us denote by $\mathfrak{X}$ the set of all $x \in C[0,\infty)$ such that $\|x(t)\| \le u(t)$ for $t \ge 0$ and

$$\|x(t_1) - x(t_2)\| \le \left| \int_{t_1}^{t_2} H(s,u_0)ds \right|$$

for $t_1, t_2 \ge 0$. Obviously, $\mathfrak{X}$ is a closed convex bounded and an almost equicontinuous subset of $C[0,\infty)$.

Put $\Gamma(X) = \sup_{t \ge 0} \alpha(X(t))$ for $X \subset \mathfrak{X}$. Since $\Gamma(X_1) \le \Gamma(X_2)$ whenever $X_1 \subset X_2$, by the corresponding properties of α, $\Gamma(\bar{X}) = \Gamma(X) = \Gamma(\operatorname{conv} X)$. If $\Gamma(X) = 0$ then $\overline{\Gamma(t)}$ is compact for every $t \ge 0$, and therefore Ascoli's theorem proves that $\bar{X}$ is compact in $C[0,\infty)$. Now, let (X_n) be a decreasing sequence of nonempty closed subsets of $\mathfrak{X}$ such that $\Gamma(X_n) \to 0$ as $n \to \infty$. Let $t > 0$. Since the set $V_n = X_n|[0,t]$ is equicontinuous and $\sup_{0\le\tau\le t} \alpha(X_n(\tau)) \to 0$ as $n \to \infty$, Ambrosetti's theorem proves that $\alpha_{[0,t]}(V_n) \to 0$ as $n \to \infty$. According to Kuratowski's theorem applied to the sequence (V_n), $\bigcap_{n\ge 1} V_n$ is nonempty compact in $C[0,t]$; hence $\{x(t): x \in \bigcap_{n\ge 1} X_n\}$ is compact. Consequently, $\bigcap_{n\ge 1} X_n$ is a nonempty and compact subset of $C[0,\infty)$.

To apply our fixed-point theorem, let us consider the continuous mapping T defined by

$$(Ty)(t) = x_0 + \int_0^t f_0(s,y(s))ds$$

for $y \in C[0,\infty)$. Modifying the reasoning from the proof of Theorem 2.1.2 [10] we infer that $T[\mathfrak{X}] \subset \mathfrak{X}$. Let X be a nonempty subset of $\mathfrak{X}$. To prove the theorem it remains to show that $\Gamma(T[X]) \le \varphi(\Gamma(X))$.

To this end, fix $t \ge 0$. Let us put $X_t = \bigcup\{X(\sigma): 0 \le \sigma \le t\}$. Since $L|_{[0,t]}$ is uniformly continuous, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that $|t' - t''| < \delta$ $(0 \le t', t'' \le t)$ implies $\alpha(X_t)|L(t') - L(t'')| < \varepsilon$. For a positive integer $m > t/\delta$, let $t_0 = 0 < t_1 < \ldots < t_m = t$ be the partition of the interval $[0,t]$ with $t_i = \frac{t}{m} + t_{i-1}$ $(i = 1,2,\ldots,m)$. Moreover, let us denote by σ_i $(i = 1,2,\ldots,m)$ a point in $I_i = [t_{i-1},t_i]$ such that $L(\sigma_i) = \sup\{L(\sigma): t_{i-1} \le \sigma \le t_i\}$.

For $x \in X$, we have

$$\int_0^t f_0(s,x(s))ds = \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} f_0(s,x(s))ds \in$$

$$\in \sum_{i=1}^{m} (t_i - t_{i-1})\, \overline{\mathrm{conv}}\, (\{f_0(\sigma,x(\sigma)): t_{i-1} \le \sigma \le t_i\}) \subset$$

$$\subset \sum_{i=1}^{m} (t_i - t_{i-1}) \cdot \overline{\mathrm{conv}}\, (f_0[I_i \times X_t])$$

and it follows that

$$\alpha(T[X](t)) = \alpha(\{\int_0^t f_0(s,x(s))ds : x\in X\}) \le$$
$$\le \alpha(\sum_{i=1}^{m}(t_i - t_{i-1})\cdot \overline{\mathrm{conv}}\,(f_0[I_i\times X_t])) \le$$
$$\le \sum_{i=1}^{m}(t_i - t_{i-1})L(\sigma_i)\,\varphi(\alpha(X_t)) =$$
$$= \varphi(\alpha(X_t))\cdot \int_0^t L(s)ds + \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i}(L(\sigma_i) - L(s))\,\varphi(\alpha(X_t))ds =$$
$$= \varphi(\sup\{\alpha(X(\sigma)) : 0\le \sigma \le t\}) + \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i}\varphi(\alpha(X_t))(L(\sigma_i) -$$
$$- L(s))ds \le \varphi(\Gamma(X)) + \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \varphi(\alpha(X_t))(L(\sigma_i) - L(s))ds.$$

If $\alpha(X_t) = 0$ then $\varphi(\alpha(X_t)) = 0$ and therefore $\alpha(T[X](t)) \le 0 = \varphi(0) \le \varphi(\Gamma(X))$. As $\alpha(X_t) > 0$,

$$\alpha(T[X](t)) < \varphi(\Gamma(X)) +$$
$$+ \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \varphi(\alpha(X_t))\,\varepsilon\,(\alpha(X_t))^{-1}ds < \varphi(\Gamma(X)) + \varepsilon t$$

and since ε is arbitrary, we obtain $\alpha(T[X](t)) \le \varphi(\Gamma(X))$.

Thus, $\alpha(T[X](t)) \le \varphi(\Gamma(X))$ for all $t\ge 0$. This implies $\Gamma(T[X]) \le \varphi(\Gamma(X))$, and, consequently T has a fixed point in $\mathfrak{X}$. The proof is complete.

References

[1] A. AMBROSETTI: Un teorema di esistenza per le equazioni differenziali negli spazi di Banach, Rend. Sem. Mat. Univ. Padova 39(1967), 349-360.

[2] J. BANAŚ and K. GOEBEL: Measure of noncompactness in Banach spaces, Lect. Notes Pure Applied Math. 60, Marcel Dekker, New York 1980.

[3] J. BANAŚ, A. HAJNOSZ and S. WEDRYCHOWICZ: Some generalization of Szufla's theorem for ordinary differential equations in Banach space, Bull. Acad. Polon. Sci., Sér. Sci. Math. 29(1981), 459-464.

[4] J. DANEŠ: Some fixed point theorems, Comment. Math. Univ. Carolinae 9(1968), 223-235.

[5] J. DANEŠ: On densifying and related mappings and their application in nonlinear functional analysis,

Theory of nonlinear operators, Akademie-Verlag, Berlin 1974, 15-56.

[6] K. DEIMLING: Ordinary differential equations in Banach spaces, Lect. Notes in Math. 596, Springer-Verlag, Berlin 1977.

[7] K. GOEBEL and W. RZYMOWSKI: An existence theorem for the equation $x' = f(t,x)$ in Banach space, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 28(1970), 367-370.

[8] М.А. КРАСНОСЕЛЬСКИЙ и С.Г. КРЕЙН: О принципе усреднения в нелинейной механике, Успехи Мат. Наук 10(1955), 147-152.

[9] K. KURATOWSKI: Sur les espaces complets, Fund. Math. 15 (1930), 301-309.

[10] V. LAKSHMIKANTHAM and S. LEELA: Differential and integral inequalities, Vol. 1, Academic Press, New York 1969.

[12] B. RZEPECKI: On the operator equation in Banach spaces, Demonstratio Math. 12(1979), 189-201.

[11] B. RZEPECKI: Some properties of the set of solutions on an operator equation in a Banach space, Comment. Math. 22(1978), 467-478.

[13] B. RZEPECKI: On measure of noncompactness in topological spaces, Comment. Math. Univ. Carolinae 23(1982), 105-116.

[14] B. RZEPECKI: Euler polygons and Kneser's theorem for solutions of differential equations in Banach spaces, Comment. Math. Univ. Carolinae 23(1982), 657-669.

[15] B.N. SADOVSKII: Limit compact and condensing operators, Russian Math. Surveys 27(1972), 86-144.

[16] A. STOKES: The application of a fixed-point theorem to a variety of nonlinear stability problems, Proc. Nat. Acad. Sci. U.S.A. 45(1959), 231-235.

[17] J. SZARSKI: Differential inequalities, PWN, Warszawa 1965.

[18] S. SZUFLA: Some remarks on ordinary differential equations in Banach spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 16(1968), 795-800.

[19] S. SZUFLA: Solutions sets of nonlinear equations, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 21(1973), 971-976.

[20] S. SZUFLA: Some properties of the solutions set of ordinary differential equations, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 22(1974), 675-678.

[21] S. SZUFLA: Kneser's theorem for weak solutions of ordinary differential equations in reflexive Banach spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 26(1978), 407-413.

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