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A NOTE ON CHROMATIC NUMBER OF DIRECT PRODUCT OF GRAPHS

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Abstract: The case when the chromatic number $\chi(G\times H)$ of direct product of graphs equals to min $\{\chi(G), \chi(H)\}$ is discussed.

Key words: Graph, chromatic number, direct product of graphs.

Classification: 05C15, 05C20

The direct product $G\times H$ of two (finite, simple) graphs G, H is defined by

$V(G\times H) = V(G)\times V(H)$

and

$(x,y)\in V(G\times H)$ is adjacent to $(x',y')\in V(G\times H)$ if and only if $(x,x')\in E(G)$ and $(y,y')\in E(H)$.

In [1]

(1) $\chi(G\times H) = \min\{\chi(G), \chi(H)\}$

is conjectured (χ denotes the chromatic number) or, equivalently, if $\chi(G) = \chi(H) = k$, then $\chi(G\times H) = k$. It is clear that $\leq$ holds in (1). It is proved in [1] that (1) holds if $\chi(G) = \chi(H) = k$ and

(2) each vertex of H is contained in a complete k-1 graph.

It is not difficult to prove that (1) holds if

(3) there exists a homomorphism $\varphi: G \longrightarrow H$.

(A mapping $\varphi: V(G) \longrightarrow V(H)$ is a homomorphism if

$(x,y) \in E(G) \Longrightarrow (\varphi(x), \varphi(y)) \in E(H)$.)

In general, it is not even known whether $\lim_{k \to \infty} f(k) = \infty$ where $f(k) = \min \{\chi(G \times H) \mid \chi(G) = \chi(H) = k\}$. In [3] it is proved that either $f(k) \leq 16$ for every k or $\lim_{k \to \infty} f(k) = \infty$.

In this note we give another sufficient condition for (1) and show examples of graphs which satisfy this condition but do not satisfy either (2) or (3).

<u>Theorem</u>: Let $\chi(G) = \chi(H) = k$. If

(4) for every pair e_1, e_2 of non-incident edges of G there is an edge e_3 of G incident to both e_1 and e_2,

then $\chi(G \times H) = k$.

<u>Proof</u>. Suppose that $\chi(G \times H) < k$. Let $c: V(G \times H) \longrightarrow \{1,\dots,k-1\}$ be a coloration of $G \times H$. For each vertex $x \in V(H)$ choose an edge $e_x = (y_x, z_x) \in E(G)$ such that $c(y_x, x) = c(z_x, x)$. Define $\bar{c}: V(H) \longrightarrow \{1,\dots,k-1\}$ by $\bar{c}(x) = c(y_x, x)$. As $\chi(H) = k$ there is an edge $(x,x') \in E(H)$ such that $\bar{c}(x) = \bar{c}(x')$.

There are three possibilities:

a) $e_x = e_{x'}$ (say $y_x = y_{x'}$, $z_x = z_{x'}$). Then $c(y_x, x) = c(z_{x'}, x')$ and (y_x, x) is adjacent to $(z_{x'}, x')$ in $G \times H$.

b) $e_x, e_{x'}$ are incident (say $z_x = y_{x'}$). Then $c(y_x, x) = c(y_{x'}, x')$ and (y_x, x) is adjacent to $(y_{x'}, x')$ in $G \times H$.

c) There is an edge incident to both e_x and $e_{x'}$ (say $(z_x, y_{x'}) \in E(G)$). Then $c(z_x, x) = c(y_{x'}, x')$ and (z_x, x) is adjacent to $(y_{x'}, x')$ in $G \times H$.

In all three cases we get a contradiction to $\chi(G \times H) < k$.

Remark: Let G_k, $k \geq 4$ be a graph with the vertex set
$V(G_k) = \{x_1,\dots,x_{k-1},y_1,\dots,y_{k-1},z\}$ and the edge set

$$E(G_k) = \{(x_i,x_j) \mid i,j = 1,\dots,k-1,\ i \neq j\} \cup \{(x_i,y_j) \mid i,j = 1,\dots,k-1,\ i \neq j\} \cup \{(y_i,z) \mid i = 1,\dots,k-1\}.$$

Clearly, G_k is the k-critical graph which satisfies (4), see [2].

It is not very difficult to prove that every 4-chromatic graph contains either G_4 or a 4-chromatic subgraph which satisfies (2). Since there is a homomorphism $\varphi : H \to G_4$ for any 4-chromatic graph H which does not satisfy (2), the Theorem does not yield any new result for 4-chromatic graphs.

On the other hand, let $k \geq 5$ and let H be any k-chromatic graph each vertex of which is contained in a triangle but which does not contain the complete graph K_4. Then $\chi(G_k \times H) = k$ by the Theorem but neither G_k nor H satisfy (2) and there is neither a homomorphism $\varphi : G \to H$ nor $\varphi : H \to G$.

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