John D. O'Neill

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ON SUMMANDS OF DIRECT PRODUCTS OF ABELIAN GROUPS

J. D. O'NEILL

Abstract: In this paper we show that an infinite direct product of abelian groups can equal the direct sum of two indecomposable subgroups. This and other similar results are derived from corresponding results about direct sums of abelian groups first obtained by A.L.S. Corner and L. Fuchs.

Key words: direct product and direct sum of groups, slender, algebraically compact, rank.

Classification: 20K25, 20K26

In 1969 in [1] A.L.S. Corner showed, by example, that an infinite direct sum of rank two torsion-free reduced abelian groups can equal the direct sum of two indecomposable subgroups. We will prove that "infinite direct sum" can be replaced by "infinite direct product" in this statement. We will then prove the same thing for a variation of Corner's result obtained by L. Fuchs [Theorem 91.2 in 2]. By contrast we show in Theorem 4 that an infinite direct product of rank one torsion-free abelian groups cannot equal the direct sum of indecomposable subgroups. Finally, utilizing another example of Corner's, we present an abelian group G which can be expressed as an infinite direct product in many unusual ways.

All groups herein are abelian. The letter N will denote the set of natural numbers. All unexplained terminology may be found in [2], particularly in Chapter XIII.

A few words on topology are necessary. Suppose a group E equals $\prod_{-\infty}^{\infty} E_n$. We give it the product topology induced by the discrete topology on the E_n's. This topology is Hausdorff. If H is a subgroup of E, we designate its closure by $\bar{H}$. If $A = B \oplus C \subseteq E$, then $\bar{A} = \bar{B} \oplus \bar{C}$ when the following criterion is satisfied:

(*) if a sequence $a_1, a_2, \ldots$ of elements in A converges to 0, then $f_B(a_n)$ and $f_C(a_n)$ both converge to 0 where f_B and f_C are the projections to B and C.

In what follows we shall have use for these open neighborhoods of 0 in E: $E^n = \prod_{|k| \geq n} E_k$ for $n \geq 0$. Clearly $\bigcap E^n = 0$.

Theorems

Our first theorem was inspired by Corner's example in II of [1] (see also Theorem 91.1 in [2]).

Theorem 1. There exists a torsion-free group E with decompositions $E = \prod_{-\infty}^{\infty} E_n = \bar{B} \oplus \bar{C}$ where $\bar{B}$, $\bar{C}$, and each E_n is indecomposable and every E_n has rank 2.

Proof. We divide the proof into three parts. (a) First we construct a group $A = B \oplus C = \bigoplus_{-\infty}^{\infty} E_n$ such that B, C, and each E_n is indecomposable and every E_n has rank 2. Let

$\{p_n, q_n, r_n\}$, $n \in Z$, be a set of distinct primes, let $\{b_n, c_n\}$, $n \in Z$, be independent elements, and let $A = B \oplus C$ where $B = \langle p_n^{-\infty} b_n, q_n^{-1}(b_n+b_{n+1})$ for all $n\rangle$ and $C = \langle p_n^{-\infty} c_n$, $r_n^{-1}(c_n+c_{n+1})$ for all $n\rangle$. Then $A = \bigoplus_{-\infty}^{\infty} E_n$ where each

$$E_n = \langle p_n^{-\infty} u_n,\ p_{n+1}^{-\infty} v_{n+1},\ q_n^{-1} r_n^{-1}(u_n+v_{n+1})\rangle$$ with u_n, v_n being

suitably chosen linear combinations of b_n, c_n. This is proved in the references cited above. The proof also reveals that, if x is an element in $A \cap E_n$, its projections $f_B(x)$ and $f_C(x)$ are both in $E_{n-1} + E_n + E_{n+1}$.

(b) Secondly we let $E = \Pi E_n$ and show that E equals $\bar{B} \oplus \bar{C}$. Since $E = \bar{A}$, we may apply the criterion (*) stated above. Suppose the sequence a_1, a_2, ... in A converges to 0. We may suppose $a_n \in E^n$ for each n in N. But then $f_B(a_n)$ and $f_C(a_n)$ are both in E^{n-1} for each n and hence both converge to 0. So $E = \bar{A} = \bar{B} \oplus \bar{C}$.

(c) Finally we show that $\bar{B}$ is indecomposable (the proof for $\bar{C}$ is similar). Suppose $\bar{B} = K \oplus L$. Now B is fully invariant in $\bar{B}$, is indecomposable, and thus is contained in one summand, say K. Then $E/A \cong K/B \oplus L \oplus \bar{C}/C$. Since E/A is algebraically compact [Corollary 42.2 in 2], so is L. But E has no non-trivial algebraically compact subgroups, so $L = 0$, as desired.

Out next theorem is based on Fuch's generalization of Corner's result [Theorem 91.2 in 2].

Theorem 2. There exists a torsion-free group $\bar{A}$ of the form $\bar{A} = \prod_1^\infty B_n \oplus \bar{C} = \bar{X} \oplus \bar{Y}$ where $\bar{C}$, $\bar{X}$, $\bar{Y}$ are indecomposable and each B_n has rank one.

Proof. (a) first we construct a torsion-free group A $= \bigoplus_1^\infty B_n \oplus C = X \oplus Y$ such that C, X, Y are indecomposable and each B_n has rank one. The proof of Theorem 91.2 in [2] provides just such an example (with other lettering). Let $\{p, q, p_n\}$, n in N, be a set of distinct primes and let $A = B \oplus C$ where, for independent b_n and c_n, we define $B = \bigoplus_N \langle p_n^{-\infty} b_n \rangle$ and $C = \langle p_n^{-\infty} c_n,\ p^{-1}q^{-1}(c_n - c_{n+1})$ for all n in N$\rangle$.

For s and t such that $ps - qt = 1$ let $x_n = pb_n + tc_n$ and $y_n = qb_n + sc_n$ and set $X = \langle p_n^{-\infty} x_n,\ p^{-1}(x_n - x_{n+1})$ for all n$\rangle$ and $Y = \langle p_n^{-\infty} y_n,\ q^{-1}(y_n - y_{n+1})$ for all n$\rangle$. We also define $E_n = \langle p_n^{-\infty} b_n,\ p_n^{-\infty} p^{-1} q^{-1} c_n \rangle$ for each n and $E = \prod_N E_n$. Now we have $A = B \oplus C = X \oplus Y \subseteq \prod E_n = E$ with C, X, Y indecomposable and B a direct sum of rank one groups.

(b) Secondly we show that $\bar{A} = \bar{B} \oplus \bar{C} = \bar{X} \oplus \bar{Y}$ in E. From the structure of E it is clear that $\overline{B \oplus C}$ (or $\bar{A}$) equals $\bar{B} \oplus \bar{C}$. To show $\overline{X \oplus Y}$ (or $\bar{A}$) equals $\bar{X} \oplus \bar{Y}$ we apply criterion (*). Suppose the elements $a_1, a_2, \ldots$ in A converge to 0. We may suppose each a_n is in E^n. From the definitions of x_n, y_n, and E^n we see that $f_X(a_n)$ and $f_Y(a_n)$ are in E^n for each n where f_X and f_Y are the projections to X and Y. Therefore $f_X(a_n)$ and $f_Y(a_n)$ both converge to 0 and $\bar{A} = \bar{X} \oplus \bar{Y}$.

(c) Finally we show that $\bar{C}$ is indecomposable (the proofs for $\bar{X}$ and $\bar{Y}$ are similar). Suppose $\bar{C} = K \oplus L$. Since

C is fully invariant in $\overline{C}$ and indecomposable, we may suppose $C \subseteq K$. Let $P = \prod_N \langle p_n^{-\infty} c_n \rangle$. Since $P/\bigoplus_N \langle p_n^{-\infty} c_n \rangle$ is algebraically compact, P is in K. Since $pq\overline{C}$ is in P and L is torsion-free, $\overline{C}$ must be in K and $L = 0$. The proof is complete.

In the last two theorems the E_n's all had rank greater than one. This was no mere coincidence as our next theorem will show. The theorem is a natural consequence of some well-known facts. First we need a lemma.

Lemma 3. If f is an endomorphism of a group of the form $V = \prod_I Re_i$ with $R \subseteq Q$, then the pure subgroup generated by $f(e_i)$ is a direct summand of V for each i.

Proof. We may assume R is reduced and that $f(e_m) = x \neq 0$ for some m in I. Since the characteristic of e_m is $\leq$ that of each component of x, we may write $x = \Sigma(a_i/b_i)e_i$ with a_i, b_i in Z and $b_iR = R$ for each i. If d is the g.c.d. of the a_i's, then x/d is in V, so we can assume $d = 1$. Also $V = \prod_I Ru_i$ where $u_i = (1/b_i)e_i$. For some finite subset $J = \{1, 2, \ldots, n\}$ of I, we have $(a_1, a_2, \ldots, a_n) = 1$. There is a $n \times n$ matrix $A = (a_{jk})$ over Z such that $a_{1k} = a_k$ for each k and $|A| = 1$. Write $x_j = \sum_k a_{jk}u_k$ for $j = 2, 3, \ldots, n$. Then $V = Rx \oplus (\bigoplus_{j=2}^{n} Rx_j^*) \oplus \prod_{I \setminus J} Ru_i$ and Rx is the pure subgroup generated by x.

Theorem 4. An infinite direct product of rank one torsion-free reduced groups cannot equal the direct sum of indecomposable subgroup

<u>Proof</u>. Suppose a group V equals $\prod_I R_i = \bigoplus_J A_j$ where I is infinite, each R_i is torsion-free reduced of rank one, and each A_j is indecomposable. Let B and C be the direct sum of the A_j's of rank 1 and rank >1 respectively. Then B is slender [Theorem 95.3 in 2] and some R_i, say R_1, must be contained in C. Let each R_i have type t_i and set $t = t_1$. Write $V_t = \prod_{t_i = t} R_i$ and $V^t = \prod_{t_i > t} R_i$. Both $V_t \oplus V^t$ and V^t are fully invariant subgroups of V [Theorem 96.1 in 2] and each is then a direct sum of A_j's since the A_j's are indecomposable. Hence, by cancelling V^t we may assume $V_t = \bigoplus_{J'} A_j$ for some subset J' of J. For some j in J' and projection $f: V_t \to A_j$ we have $f(R_1) \neq 0$. Since R_1 is in C, this A_j has a proper rank one direct summand by Lemma 3 and is not indecomposable. This contradiction proves the theorem.

In a final theorem we illustrate the fact that many unusual decompositions of direct products can be derived immediately from corresponding direct sum decompositions. For verification of the theorem we will cite another theorem of Corner on direct sums and then indicate why the transfer from direct sum to direct product is permissible.

<u>Theorem 5</u>. There exists a group G such that, for every sequence of positive integers $r_1, r_2, \ldots$, infinitely many of which exceed 1, there exist indecomposable subgroups A_n of rank r_n in G such that $G = \prod_N A_n$.

Proof. Let $\{p, p_n\, q_n\}$, n in N, be a set of distinct primes and for independent u_n and x_n define $B_n = \langle p^{-\infty} u_n, p_n^{-\infty} x_n, q_n^{-1}(u_n + x_n)\rangle$. Suppose the sequence $r_1, r_2, \ldots$ is given. Let $A = \underset{N}{\oplus} B_n$ and $G = \underset{N}{\Pi} B_n$. We now make an observation. Suppose N has partitions $\{N_i\}$ and $\{M_i\}$ for $i = 1, 2, \ldots$ with each N_i and M_i finite; and suppose, for each i, that $\underset{n \in N_i}{\oplus} B_n = \underset{m \in M_i}{\oplus} C_m$ for some subgroups C_m. Then $A = \underset{N}{\oplus} C_m$ and $G = \underset{N}{\Pi} C_m$. Now, by a finite number of such operations, Corner showed [Theorem 2 in I of 1; also Theorem 91.3 in 2] that we can obtain a decomposition $A = \underset{N}{\oplus} A_n$ where each A_n is indecomposable of rank r_n. Hence $G = \underset{N}{\Pi} A_n$, as desired.

References

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University of Detroit

Detroit, Michigan 48221

U.S.A.

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