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NOTES ON DISTRIBUTIVE GROUPOIDS

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Abstract: It is proved that every distributive groupoid is strongly trimedial. Various other similar results on the structure of distributive groupoids are derived.

Key words: Distributive groupoid, quasigroup.

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1. Introduction. We have begun the investigation of distributive groupoids in the paper [2] (with which the reader is assumed to be acquainted). Chapter IV of [2] revealed some deep connections between the distributive and medial laws, but left the following two important questions unanswered: Is every distributive idempotent groupoid symmetric-by-medial? Is every free distributive idempotent groupoid cancellative? Recently ([1]), the authors succeeded in answering both these questions, and namely - in the affirmative. The aim of the present paper is to derive various (rather scattered) consequences of these two results and to continue in the structure theory of distributive groupoids.

2. Subdirectly irreducible distributive groupoids

2.1. Proposition. Let G be a subdirectly irreducible (or, more generally, subdirectly q-irreducible) cancellative distributive groupoid. Then G is a locally finite quasigroup.

Proof. By [7], there exists a distributive quasigroup Q such that G is a dense subgroupoid of Q. Using Proposition V.2.5 of [2], we see that Q is subdirectly q-irreducible. The variety of pointed distributive quasigroups is equivalent to the variety of special R-quasimodules for a commutative noetherian ring R (see [6] and [3]). Using this and Propositions 4.17 and 5.5 of [3], it is easy to show that every finitely q-generated subquasigroup of Q is finite. In particular, every subgroupoid of Q is a quasigroup.

2.2. Proposition. Let G be a subdirectly irreducible distributive idempotent groupoid containing no zero.

(1) If $\lambda_l(G)=id_G=\lambda_r(G)$ then G is a locally finite quasigroup.

(2) If either $\lambda_l(G)\neq id_G$ or $\lambda_r(G)\neq id_G$ then G is medial.

Proof. (1) By Lemma 3.3 of [1], G is cancellative and the result follows from 2.1.

(2) Let $\lambda_r(G)\neq id_G$. By Proposition V.5.10 of [2], $\eta(G)\neq id_G$. On the other hand, by Theorem 4.1 of [1], there exists a congruence r of G such that G/r is medial and every block of r is symmetric. Clearly, $r\cap\eta(G)=id_G$, so that $r=id_G$ and G is medial.

2.3. Proposition. Let G be a subdirectly irreducible distributive groupoid. Then at least one of the following three cases takes place:

(1) G is medial.

(2) G is a quasigroup.

(3) G contains a zero element 0, $K=G\setminus\{0\}$ is a subgroupoid of G and K is a quasigroup.

Proof. If G is not idempotent then G is medial by Corollary III.1.9 of [2]. If G is idempotent, the assertion follows from 2.2 and from Proposition V.5.4 of [2].

Denote by W the variety of distributive groupoids satisfying the identities xy=yx and x(x.xy)=xy.

2.4. Proposition. Let $G\in W$ be idempotent and subdirectly irreducible. Then either G is symmetric or G contains a zero element 0, $K=G\setminus\{0\}$ is a subgroupoid of G and K is symmetric.

Proof. We can assume that G contains no zero element. Since G is commutative, $\lambda_l(G)=id_G=\lambda_r(G)$ and G is a quasigroup by 2.2. Then G is symmetric.

3. Some consequences

3.1. Proposition. Every distributive groupoid satisfies the following identities:

((x.xy)y)(uv)=((x.xy)u)(yv),

((yx.x)y)(uv)=((yx.x)u)(yv),

((x.yx)y)(uv)=((x.yx)u)(yv),

(xy.yx)(uv)=(xy.u)(yx.v).

Proof. Any of these identities is satisfied in every cancellative distributive groupoid by Theorem IV.3.7 of [2]. However, free distributive idempotent groupoids are cancellative by Theorem 4.2. of [1]. For the non-idempotent case see Proposition IV.1.1 of [2].

3.2. Proposition. Let G be a distributive groupoid and let a,b,c,d,d G be such that ab.cd=ac.bd. Then the subgroupoid of G generated by a,b,c,d is medial.

Proof. We can assume that G is subdirectly irreducible. Now, the result is an easy consequence of 2.3, Proposition IV.2.7 of [2] and Theorem IV.2.8 of [2].

In the terminology of [2], this means that every distributive groupoid is strongly trimedial.

For a distributive groupoid G, define a relation $\mu(G)$ on G by $(a,b) \in \mu(G)$ iff $ab.xy = ax.by$ for all $x,y \in G$. By 3.2, we have $\mu(G) = \mu_G$, where μ_G is defined in Section IV.3 of [2].

3.3. Proposition. Let G be a distributive groupoid and $a,b \in G$. Then $(a.ab,b)$, $(ba.a,b)$, $(a.ba,b)$, (ab,ba) belong to $\mu(G)$.

Proof. This is an immediate consequence of 3.1.

3.4. Proposition. Let G be a distributive groupoid. Then there exists a congruence r of G such that $r \subseteq \mu(G)$ and $G/r \in W$.

Proof. We can assume that G is subdirectly irreducible, idempotent and not medial. The result then follows from 2.4 and Theorem IV.3.7 of [2].

3.5. Proposition. Let G be a distributive groupoid and let $a,b,c,d \in G$ be such that $ab.cd \neq ac.bd$. Denote by K the subgroupoid generated by these elements. Then there exists a congruence r of K such that K/r is a finite non-medial distributive quasigroup and K/r is subdirectly irreducible.

Proof. There exists a congruence r of K such that $H = K/r$ is subdirectly irreducible and not medial. If H contains no zero element then the result follows from 2.3 and 2.1. Suppose that H contains a zero O and put $A=H\setminus\{0\}$. Then A is a distributive quasigroup and A is not medial. On the other hand, H is generated by four elements and it is easy to see that A is generated by three elements. Hence A is medial, a contradiction.

3.6. <u>Corollary</u>. Let V be a class of distributive groupoids closed under subgroupoids and homomorphic images. Suppose that no groupoid from V is a finite non-medial quasigroup. Then every groupoid from V is medial.

3.7. <u>Proposition</u>. Let V be a class of groupoids closed under isomorphic images and subgroupoids and not containing a non-trivial symmetric groupoid. Let G be a distributive idempotent groupoid and r be a congruence of G such that G/r is medial and every block of r belongs to V. Then G is medial.

Proof. By Theorem 4.1 of [1], there is a congruence of G such that G/s is medial and every block of s is symmetric. Clearly, $r\cap s=id_G$, and hence G is medial.

4. <u>Ideals</u>

4.1. <u>Proposition</u>. Let I be an ideal of a distributive idempotent groupoid G. Then G is isomorphic to a subgroupoid of $I^{2\times I}\times(G/I)$.

Proof. Denote by r the congruence $(I\times I)\cup id_G$. For every $a\in I$, both L_a and R_a can be viewed as homomorphisms of G into I. Clearly, $r\cap\bigcap\{Ker(L_a)\cap Ker(R_a);\ a\in I\}=id_G$.

4.2. <u>Corollary</u>. Let I be an ideal of a non-medial distributive idempotent groupoid G. Then either I or G/I is not medial.

4.3. <u>Lemma</u>. Let I and K be two left ideals of a distributive groupoid G. Suppose that both I and K are medial groupoids. Then the left ideal $I \cup K$ is a medial groupoid.

Proof. Put $A = I \cup K$. It suffices to show that f(A) is medial whenever f is a homomorphism of G onto a subdirectly irreducible distributive groupoid H. To this purpose, we can assume that H is not medial. If H is a quasigroup, then f(I)=H, since f(I) is a left ideal of H, and hence H is medial, a contradiction. Now, by 2.3, H has a zero O and $H \setminus \{O\}$ is a quasigroup. Again, since H is not medial and both f(I) and f(K) are left ideals of H, we must have $f(I) = \{O\} = f(K$. Consequently, $f(A) = \{O\}$ is medial.

4.4. <u>Lemma</u>. Let G be a distributive groupoid and I be a left ideal of G such that I is a medial groupoid. Then the ideal K of G generated by I is a medial groupoid.

Proof. It suffices to show that f(K) is a medial groupoid whenever f is a homomorphism of G onto a subdirectly irreducible groupoid H. Proceeding similarly as in the proof of 4.3, we can assume that H contains a zero element O and $H \setminus \{O\}$ is a non-medial quasigroup. Since f(I) is a left ideal of H and $f(I) \neq H$, we have $f(I) = \{O\}$. However, then $f(K) = \{O\}$ is medial.

For every distributive groupoid G denote by M(G) the union of all ideals of G which are medial groupoids.

4.5. <u>Proposition</u>. Let G be a distributive groupoid such that M(G) is non-empty. Then:

(1) M(G) is an ideal of G and it is a medial groupoid.

(2) Every left (or right) ideal of G which is a medial groupoid is contained in M(G).

Proof. Apply 4.3 and 4.4.

5. Perfect distributive groupoids. A distributive groupoid G is called perfect if it satisfies the following quasiidentities:

$$(xu.vz=xv.uz \;\&\; (xy.u)(vz)=(xy.v)(uz)) \rightarrow yu.vz=yv.uz,$$
$$(xu.vz=xv.uz \;\&\; (yx.u)(vz)=(yx.v)(uz)) \rightarrow yu.vz=yv.uz,$$
$$(ux.vz=uv.xz \;\&\; (u.xy)(vz)=(uv)(xy.z)) \rightarrow uy.vz=uv.yz,$$
$$(ux.vz=uv.xz \;\&\; (u.yx)(vz)=(uv)(yx.z)) \rightarrow uy.vz=uv.yz,$$
$$(uv.zx=uz.vx \;\&\; (uv)(z.xy)=(uz)(v.xy)) \rightarrow uv.zy=uz.vy,$$
$$(uv.zx=uz.vx \;\&\; (uv)(z.yx)=(uz)(v.yx)) \rightarrow uv.zy=uz.vy.$$

The class of perfect distributive groupoids is thus a quasivariety.

5.1. Proposition. A distributive groupoid G is perfect, provided it satisfies at least one of the following conditions:

(1) G is cancellative.

(2) G is regular.

(3) G is medial.

Proof. See Proposition IV.2.7 of [2] and Theorem 4.1 of [1].

5.2. Proposition. Every ideal-free distributive groupoid is perfect.

Proof. Let G be an ideal-free distributive groupoid. Without loss of generality,we can assume that G is subdirectly irreducible and not medial. Then G contains no zero and G is a quasigroup by 2.3. Hence G is perfect by 5.1.

5.3. <u>Proposition</u>. Let G be a left (or right) cancellative distributive groupoid. Then G is perfect.

Proof. By Lemma 2.5 of [1], G is a subgroupoid of a distributive groupoid H such that H is a left quasigroup. Then H is ideal-free and 5.2 can be applied.

5.4. <u>Proposition</u>. Let G be a distributive groupoid which can be generated by four elements. Then G is perfect.

Proof. Proceeding similarly as in the proof of 3.5, we can show that every subdirectly irreducible factor of G is perfect.

5.5. <u>Proposition</u>. Let G be a perfect distributive groupoid. Then $\mu(G)$ is a congruence of G and $G/\mu(G)$ is symmetric.

Proof. Apply 3.3 and Proposition IV.3.3 of [2].

5.6. <u>Corollary</u>. Every perfect distributive groupoid is medial-by-symmetric.

5.7. <u>Proposition</u>. Let H be a dense subgroupoid of a perfect distributive groupoid G. If H is medial then G is medial.

Proof. Suppose that H is medial and denote by K a subgroupoid of G such that $H \subseteq K$, K is medial and K is maximal with respect to these properties. It is enough to show that K is closed in G. For, let $a \in G$, $b \in K$ and $ab \in K$. Denote by A the subgroupoid generated by the set $B = K \cup \{a\}$. Since G is perfect and K is medial, $xy.uv = xu.yv$ for all $x,y,u,v \in B$. Now, A is medial by Proposition IV.2.2 of [2], $A = K$ and $a \in K$.

5.8. Corollary. **Let G be a perfect, non-medial distributive groupoid and I be a left (or right) ideal of G. Then I is not medial. Consequently, M(G)=∅.**

5.9. Proposition. **Let V be a class of groupoids closed under isomorphic images and subgroupoids and containing no non-trivial symmetric groupoid. Let G be a distributive idempotent groupoid and r be a congruence of G such that G/r is perfect and every block of r belongs to V. Then G is perfect.**

Proof. Similar to that of 3.7.

5.10. Proposition. Let G be a finite, left- and right-ideal-free distributive groupoid. Then G is a quasigroup.

Proof. We shall proceed by induction on the number of elements of G. By Theorem V.6.6(i) of [2], G is regular and idempotent. It follows that if $\eta(G)=id_G=\wp(G)$ then G is cancellative, and hence a quasigroup, since it is finite. Now, we can assume that $\eta(G)\neq id_G$. Then, according to the induction hypothesis, the groupoid $H=G/\eta(G)$ is a quasigroup. Since G is regular, H is isomorphic to the subgroupoid Ga of G for every $a\in G$. Define a relation r on G by $(a,b)\in r$ iff Ga=Gb. Then r is an equivalence. Further, let $(a,b)\in r$ and $c\in G$. We have b=da for some $d\in G$, bc=dc.ac, cb=cd.ca, $bc\in G.ac$, $cb\in G.ca$, (G.ac)(bc)=G.ac and (G.ca)(cb)=G.ca, since both G.ac and G.ca are quasigroups. Hence $G.ac\subseteq G.bc$ and $G.ca\subseteq G.cb$. The converse inclusions can be proved similarly and we see that r is a congruence of G. As $ab\in Gb$ and thus Gb=G.ab for all $a,b\in G$, G/r is a semigroup of right zeros. On the other hand, if $(a,b)\in r\cap\eta(G)$ then aa=cb for some $c\in G$ and $(a,c)\in\eta(G)$, since H is a quasigroup. Then a=aa=cb=ab=bb=b and we get

$r \cap \eta(G)=id_G$. Finally, G/r is a left- and right-ideal-free semigroup of right zeros, so that $r=G\times G$ and consequently $\eta(G)=id_G$, a contradiction.

6. <u>The radical</u> ε . In this section, the reader is supposed to be acquainted with the theory of semipreradicals, as developed in [4] and [5].

Let V be a variety of groupoids. Consider the following two conditions for a semipreradical r on V:

(D) If $G,H\in V$ and f is a homomorphism of G onto H then $f(r(G))\subseteq r(H)$.

(L) If $G,H\in V$, H is a closed subgroupoid of G and if $(a,b)\in$ $\in r(G)$ where $a\in H$, then $b\in H$ and $(a,b)\in r(H)$.

6.1. <u>Lemma</u>. Let r, s be two semipreradicals on V satisfying (D) and (L). Then r:s satisfies (D) and (L).

Proof. r:s satisfies (D) by Proposition 2.1 of [4]. Let H be a closed subgroupoid of a groupoid $G\in V$; let $(a,b)\in$ $\in (r:s)(G)$ and $a\in H$. Denote by f the natural projection of G onto $G/s(G)$ and put $K=f(H)$. Then K is isomorphic to H/t where $t=(H\times H)\cap s(G)$. By (L) we have $t\subseteq s(H)$. If $f(cd)=f(e)$ where $c,d,e\in G$ and $c,e\in H$ then $(cd,e)\in s(G)$, so that $cd\in H$ and $d\in H$. Similarly, if $f(cd)=e$ where $c,d,e\in G$ and $d,e\in H$, then $c\in H$. This shows that K is a closed subgroupoid of $G/s(G)$. We have $(f(a),f(b))\in r(G/s(G))$ and $f(a)\in K$. By (L) we get $f(b)\in K$ and $(f(a),f(b))\in r(K)$. There is a $c\in H$ such that $f(b)=f(c)$, i.e. $(b,c)\in s(G)$. However, then $b\in H$. Denote by g the natural projection of K onto $H/s(H)$. By (D) we have $(gf(a),gf(b))\in$ $\in r(H/s(H))$. Thus $(a,b)\in (r:s)(H)$.

Now consider the idempotent preradicals λ_l and λ_r on the variety of distributive idempotent groupoids. Define a chain $r_0, r_1, \dots$ of preradicals as follows: $r_0 = id$; if $i \geq 1$ is odd then $r_i = \lambda_l : r_{i-1}$; if $i \geq 2$ is even then $r_i = \lambda_r : r_{i-1}$. The join of this (countable) chain of preradicals will be denoted by ε.

6.2. <u>Proposition</u>. ε is an idempotent radical on the variety of distributive idempotent groupoids and ε satisfies (L). If G is a distributive idempotent groupoid then $G/\varepsilon(G)$ is both λ_l- and λ_r-torsionfree.

Proof. Evidently, both λ_l and λ_r satisfy (L). Now it follows easily from 6.1 that ε satisfies (L). The rest is easy.

6.3. <u>Proposition</u>. Let G be an ε-torsion distributive idempotent groupoid. Then G is medial.

Proof. Suppose that G is not medial. By 3.5, G contains a subgroupoid H such that a factorgroupoid of H is a non-medial quasigroup. Since (by 6.2) ε satisfies (L), G is just the least closed subgroupoid of G containing H. Hence by Proposition V.2.5 of [2] every normal congruence of H can be extended to a normal congruence of G. Consequently, a factorgroupoid K of G is a non-medial quasigroup. Now, K must be an ε-torsion groupoid; on the other hand, K is cancellative and so both λ_l- and λ_r-torsionfree, a contradiction.

6.4. <u>Lemma</u>. Let G be an ε-torsion distributive idempotent groupoid. Then every cancellative subgroupoid of G is trivial.

Proof. Let H be a cancellative subgroupoid of G. Since ε satisfies (L), we can assume that H is dense in G. Then the identity congruence of H can be extended to a cancellative congruence r of G. Let f denote the natural homomorphism of G onto K=G/r. If $a,b \in H$ then $(a,b) \in \varepsilon(G)$, $(f(a),f(b)) \in \varepsilon(K)=id_K$, $f(a)=f(b)$ and a=b, since $f|H$ is injective. We have proved that H is trivial.

6.5. Lemma. Let G be a distributive idempotent groupoid and let r be a congruence of G such that every block of r is cancellative. Then $r \cap \varepsilon(G)=id_G$.

Proof. Apply 6.4.

6.6. Proposition. Let G be a distributive idempotent groupoid such that $G/\varepsilon(G)$ is medial. Then G is medial.

Proof. By 6.5, $\varepsilon(G) \cap r=id_G$ where r is a congruence of G such that G/r is medial and every block of r is symmetric.

6.7. Proposition. Let G be a distributive idempotent groupoid such that $G/\varepsilon(G)$ is perfect. Then G is perfect.

Proof. Similar to that of 6.6.

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