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TRANSDUCER, GENERALIZED RELAYS AND THEIR CHARACTERIZATION

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Abstract. In the present paper, a definition of a transducer is given, and then a special type of a transducer - a generalized relay is investigated. It is proved that a generalized relay is continuous in LCC-topology, and a characterization of transducers which are generalized relays is given.

Key words: Transducer, relay, LCC-topology.

Classification: Primary 93A05, 93A10; Secondary 54C35

Introduction. Various special methods are used to overcome difficulties that arise in describing processes and systems containing elements with hysteresis. One of these methods (used by Krasnosel'skij et al. in [1,2] for a description of hysteresis effects appearing if plasticity and elasticity of bodies is studied) is based on a conception of a transducer.

Let sets X, Y be given. Roughly speaking, a transducer is a mapping attaching to a pair - a time-dependent variable $x(t)\in X$ and a point $y\in Y$ - a time-dependent variable $y(t)\in Y$. Here the definition of a transducer is given and the conceptions of a transducer and of a dynamical system are compared. Several examples of transducers are presented and two of them - a relay and a generalized relay are studied more in detail. We shall give two equivalent definitions of relays and generalized relays, one of them purely topological, and prove the continuity of a generalized relay in the LCC-topology. The main result of this paper is a complete characterization of generalized relays that is, according to the terminology of Krasnosel'skij [2] the identification problem of relays and generalized relays is solved.

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1. Notations and definition of a transducer

1.1. Let f be a mapping from a set X into a set Y. The domain of the map f will be denoted by D(f) and its image by Im f. We shall say that a map f is on X iff X is its domain and that it is onto Y iff Y is its range. A map f on X into Y will be denoted by $f: X \longrightarrow Y$.

Let X and Y be topological spaces and f be a map from X into Y. Then f is a continuous map if it is continuous with respect to the relative topology of its domain.

1.2. Let $(T, \leq)$ be a chain i.e. a linearly ordered set. A subset I of T will be called an interval if for any $a, b \in I$, $b \leq x \leq a$ implies $x \in I$. The set of all left-closed intervals (an interval I is a left-closed interval if inf I exists and inf $I \in I$) will be denoted by $\mathcal{J}$. For an interval I let $l(I)$ = inf I, $r(I)$ = sup I.

Let T be a chain and let X be an arbitrary set. A subset $\mathfrak{X}$ of the set of all mappings from T into X will be called a full set of trajectories into X, if it meets the following two conditions:

i) if $f \in \mathfrak{X}$, then $D(f) \in \mathcal{J}$,

ii) if $f \in \mathfrak{X}$, $I \in \mathcal{J}$, $I \subset D(f)$, then $f \upharpoonright I \in \mathfrak{X}$.

1.3. If $J = \{j\}$ is a set and $\{X_j\}$ is a family of sets indexed by J, their disjoint union is denoted by $\coprod_{j\in J} X_j = \bigcup_{j\in J} (j\times X_j)$.

The topological sum of an indexed collection of topological spaces $\{X_j\}_{j\in J}$ is the disjoint union $\coprod_{j\in J} X_j$, equipped with the finest topology in which each inclusion $i_j: X_j \longrightarrow \coprod_{j\in J} X_j$, $i_j: x \longmapsto (j,x)$, $x\in X_j$, is continuous.

Given a left-closed interval I, the set $\{f\in \mathfrak{X} \mid D(f) = I\}$ will be denoted by $\mathfrak{X}(I)$. The set $\mathfrak{X}$ may be considered either as a disjoint union of the family $\{\mathfrak{X}(I)\}_{I\in\mathcal{I}}$ or, if $\mathfrak{X}(I)$ are topological spaces, as a topological sum of this family.

1.4. Definiton. Let T be a chain and let X, Y be sets, possibly endowed with a structure and let $\mathfrak{X}$, $\mathfrak{Y}$ be full sets of trajectories into X and into Y, respectively. A mapping $W: S \longrightarrow \exp\mathfrak{Y}$ will be called a transducer if it satisfies the following conditions:

i) $S \subset \mathfrak{X}\times Y$, $(f,y)\in S \Longrightarrow W[f,y] \neq \emptyset$,

ii) if $g\in W[f,y]$ and $D(f) = I$, then $D(g) = I$ and $g(l(I)) = y$,

iii) causality: if $(f_1,y)\in S$, $(f_2,y)\in S$, $l(D(f_1)) = l(D(f_2))$ and if there exists an interval $I\subset D(f_1)\cap D(f_2)$ such that $l(I) = l(D(f_1))$ and $f_1 \upharpoonright I = f_2 \upharpoonright I$, then $\{g\upharpoonright I \mid g \in W[f_1,y]\} = \{g\upharpoonright I \mid g\in W[f_2,y]\}$.

The transducer W will be called deterministic if $W[f,y]$ is a one-element set for each pair $(f,y)\in S$. The set Y will be called as a state set of the transducer W, its elements - as states, trajectories from $\mathfrak{X}$ - as input functions or inputs; an input from a pair $(f,y)\in S$ will be called an admissible input with respect to the initial state y, and trajectories from $\mathfrak{Y}$ will be called outcomes or outcome functions of the tranducer W.

Let $W: S \to \exp \mathcal{Y}$ be a transducer. This transducer will be called full if for every pair $(f,y) \in S$, every left-closed interval $I \subset D(f)$ and every output $g \in W[f,y]$, the pair $(f \upharpoonright I, g(l(I)))$ is also an element of S.

For a full transducer, another useful notion will be introduced. We shall say that a full transducer W is compositive (or has the composition property), if for every pair $(f,y) \in S$, for every left-closed subinterval I of D(f), and for each $g \in W[f,y]$ the equality $W[f \upharpoonright I, g(l(I))] = \{h \upharpoonright I \mid h \in W[f,y],\ h(l(I)) = g(l(I))\}$ is valid.

1.5. The definition of a transducer is similar to that of a dynamical system. Differences between those concepts are small and a transducer could be considered to be a generalized dynamical system. The main reason for using this different term - a transducer - is to avoid confusion.

Let us concentrate on deterministic transducers only and compare their definition with the definition of dynamical systems as it is stated in definition 1.1, in Chapter I of the book of Kalman, Falb and Arbib [3].

The sets X, Y from the definition of a transducer may be identified with the sets U and X from the definition of a dynamical system, respectively. As for the time sets T, there appears a slight generalization in the definition of the transducer.

Another difference which should be mentioned is in defining the input sets Ω and $\mathfrak{X}$. The domains of mappings from Ω are supposed to be the whole set T and for two inputs from Ω, their concatenation is also an input from Ω. (See(c)2) from the definition of a dynamical system.) The domain of an input from $\mathfrak{X}$

is supposed to be a left-closed subinterval of T and there is no demand concerning concatenation of inputs.

In the definition of a transducer there are no sets corresponding to the sets Γ and Y. But this difference is not substantial - because, as it is well known, an equivalent definition of a dynamical system can be given, in which sets Y and Γ are not explicitly involved. These are implicitly included in the set of inner states of this dynamical system and in the set of time evolutions of inner state of this. The advantage of this equivalent definition is that only one mapping - sometimes a bit more complicated - may be studied. This was also the reason for leaving out an output value set and an readout map in the definition of a transducer.

The state-transition function approximately corresponds to the mapping W. The most important difference is that W is not generally defined on the whole product $\mathfrak{X} \times Y$. Another difference is that a transducer does not have be compositive.

1.6. Examples. It follows from the previous paragraphs that an automaton could be treated as a compositive deterministic transducer with T = N.

A differential equation with parameters, can serve, as another example of the transducer. Let $F: R \times R^n \times R^m \to R^n$ be a mapping such that the differential equation $dx/dt = F(t,x,a)$ with a parameter $a \in R^m$ has a unique solution. We will denote a solution of this equation with an initial condition (t',y) by $x(t',y,a;t)$. In this example, let T = R and let $\mathcal{R}^m$ be the set of all C^2-mappings from R into R^m defined on left-closed intervals. If $W: \mathcal{R}^m \times R^n \to \mathcal{R}^n$ is such a mapping that $D(W) = S =$ $= \{(a,y) \in \mathcal{R}^m \times R^n \mid D(a)$ is a domain of the maximal solution of the above equation for the initial condition $(l(D(a)), y)$ and the evolution of the parameter $a = a(t)\}$ and $W[a,y](t) =$ $= x(l(D(a)), y, a(t); t)$, then W is a deterministic transducer possessing the composition property.

More examples are presented in [2].

2. Relay and generalized relay

2.1. In the rest of this paper we shall study deterministic transducers with real time only, i.e. with $T = R$, and with the two state set $Y = \{0,1\}$. Furthermore, we shall suppose that $X = R$. Inputs will be the continuous maps. Thus, in the rest of this paper we shall denote by $\mathcal{R}$ the set of all continuous maps from R into R, the domain of which is a left-closed interval. From now on, the set of all trajectories from R into $\{0,1\}$ will be denoted by $\mathcal{Y}$.

The simplest non-trivial example of such a transducer is a relay - $r(\alpha,\beta)$ (see also [1]), for $\alpha,\beta \in R$, $\alpha \neq \beta$. The set S of admissible inputs of $r(\alpha,\beta)$ is $\{(f,i) \in \mathcal{R} \times \{0,1\} \mid f(l(D(f))) = \alpha \Rightarrow i \neq 0$ and $f(l(D(f))) = \beta \Rightarrow i \neq 1\}$. Let us denote $t_\alpha = \inf\{t \in D(f) \mid f(t) = \alpha\}$, $t_\beta = \inf\{t \in D(f) \mid f(t) = \beta\}$ and $\tau_t = \sup\{\tau \leq t \mid f(\tau) \in \{\alpha,\beta\}\}$ for $t \in D(f)$ and $t \geq \min\{t_\alpha, t_\beta\}$.

We define
$$r(\alpha,\beta)[f,0](t) = 0 \text{ if } t < t_\alpha,$$
$$r(\alpha,\beta)[f,1](t) = 1 \text{ if } t < t_\beta,$$
$$r(\alpha,\beta)[f,i](t) = 0 \text{ if } t \geq \min\{t_\alpha,t_\beta\} \text{ and } f(\tau_t) = \alpha,$$
$$r(\alpha,\beta)[f,i](t) = 1 \text{ if } t \geq \min\{t_\alpha,t_\beta\} \text{ and } f(\tau_t) = \beta.$$

It is easy to check that a relay is a transducer with the composition property.

2.2. For each relay $r(\alpha,\beta)$, there exists the minimal topology τ on the set $R \times \{0,1\}$ such that the following statement holds for the space $R(\alpha,\beta) = (R\times\{0,1\},\tau)$.

Statement(St): For each pair $(f,i)\in S$ there is exactly one continuous map $F: I \to R(\alpha,\beta)$, $I = D(f) \subset R$, such that

i) $F(l(D(f))) = (f(l(D(f))), i)$,

ii) $\pi_1 \circ F = f$, ($\pi_1 : R\times\{0,1\} \to R$ is a projection,

iii) $\pi_2 \circ F$ is a right-side continuous mapping, ($\pi_2 : R\times\{0,1\} \to \{0,1\}$ is a projection).

Moreover, $\pi_2 \circ F = r(\alpha,\beta)[f,i]$.

Thus, the correspondence $[f,i] \to \pi_2 \circ F$ induced by the space $R(\alpha,\beta)$, is the mapping $r(\alpha,\beta): S \to \mathcal{F}$.

Let us describe the topology τ. The points $(\alpha,0)$ and $(\beta,1)$ are isolated. The local base of the point $(\alpha,1)$ is $\{U\cup V \mid U$ is a euclidean neighborhood of $(\alpha,1)$ in $R\times\{1\}$ and V is a euclidean nieghborhood of $(\alpha,0)$ in $R\times\{0\}$ without this point$\}$. The local base of $(\beta,0)$ is defined analogically and local basis of the other points, $(x,i)\in R\times\{0,1\}$, are the sets $\{U \mid U$ is a euclidean neighborhood of (x,i) in $R\times\{i\}\}$. It is simple to verify that this is the topology which the statement (St) does hold for.

2.3. The idea of relay can be generalized in a natural way. Let A, B be two disjoint, closed subsets of R. Let the local basis of the points (a,i), $a\in A$, resp. (b,i), $b\in B$, resp. (x,i), $x\notin A\cup B$, be the same as the local basis of the points (α,i), resp. (β,i), resp. (x,i). The space with this topology will be denoted by $R(A,B)$. If the set $S = \{(f,i)\in \mathcal{R}\times\{0,1\} \mid f(l(D(f)))\in A \Rightarrow i \neq 0$ and $f(l(D(f)))\in B \Rightarrow i \neq 1\}$, then the statement (St) holds also for the space $R(A, B)$ and thus the correspondence $[f,i] \to \pi_2 \circ F$ induces the mapping $r(A, B): S \to \mathcal{Y}$. This mapping will be called a generalized relay. A generalized relay is a deterministic transducer with the composition property.

2.4. For the sake of completeness we shall present a definition of the generalized relay formulated analogously as the former definition of a relay. Similarly to the previous, we shall denote $t_A = \inf\{t \in D(f) \mid f(t) \in A\}$, $t_B = \inf\{t \in D(f) \mid f(t) \in B\}$ and $\tau_t = \sup\{\tau \leq t \mid f(\tau) \in A \cup B\}$ for $t \in D(f)$ and $t \geq \min\{t_A, t_B\}$ and define $r(A, B)[f,0](t) = 0$ if $t < t_A$,

$r(A, B)[f,1](t) = 1$ if $t < t_B$,

$r(A, B)[f,i](t) = 0$ if $t \geq \min\{t_A, t_B\}$ and $f(\tau_t) \in A$,

$r(A, B)[f,i](t) = 1$ if $t \geq \min\{t_A, t_B\}$ and $f(\tau_t) \in B$.

It is quite easy to see that the outcomes of a generalized relay defined in this paragraph are right-side continuous, too. It is simple to verify that both the definitions are equivalent, therefore we need not present it.

2.5. It is natural to consider two outcomes with the same domain to be close if the total time of their different states (in which they are in different states) is short. Let $f \in \mathcal{Y}(I)$. The local base of an outcome f is defined to be the family $\{U(\varepsilon,K)(f) \mid \varepsilon > 0,\ K \subset I,\ K$ is a compact set$\}$, $U(\varepsilon,K)(f)$ being the set $\{g \in \mathcal{Y}(I) \mid \int_K |f(t) - g(t)|\, dt < \varepsilon$ and $r(I) \in I \Rightarrow f(r(I)) = g(r(I))\}$. The topology on $\mathcal{Y}$ is defined, as was said in the paragraph 1, to be the topological sum of $\mathcal{Y}(I)$ over the family $\mathcal{I}$.

Convention.

Speaking about a topological space $\mathcal{S}$, we shall mean the set $\mathcal{S}$ endowed with the topology defined above.

2.6. Proposition. Let P be the set of all right-side continuous maps from $\mathcal{S}$. Then P with the relative topology is a Hausdorff space.

Proof. Denoting by P(I) the set of all right-side continuous mappings from $\mathcal{S}(I)$, we see that $P = \coprod_{I \in \mathcal{J}} P(I)$ and so it is sufficient to prove that P(I) is a Hausdorff space.

Let us choose $u, v \in P(I)$, $u \neq v$. It may be supposed that there exists $t \in I$ for which $u(t) = 0$ and $v(t) = 1$. The proof is complete if $t = r(I)$. Let us suppose this is not the case. Then we can find a positive real number such that the closure of the interval $J = [t, t+\delta)$ is included in I and $u \upharpoonright J \equiv 0$ and $v \upharpoonright J \equiv 1$. Let $K \subset I$ be a compact set and $J \subset K$. We choose neighborhoods $U_1 = U(\delta/3, K)(u)$ and $U_2 = U(\delta/3, K)(v)$ and suppose that there is a mapping w belonging to them both. It implies the Lebesgue measure of each of the sets $\{\tau \in J \mid w(\tau) = u(\tau)\}$, $\{\tau \in J \mid w(\tau) = v(\tau)\}$ is at least $2/3\,\delta$; this is an obvious contradiction with the size of the interval J. Thus the neighborhoods U_1, U_2 must be disjoint and the proof is completed.

2.7. A generalized relay r(A,B) is a time-covariant transducer, i.e. $r(A,B)[f \circ \varphi, i](t) = r(A,B)[f,i](\varphi(t))$ for any which maps $D(f \circ \varphi)$ onto $D(f)$ continuously and preserves the orientation of the real-line. In fact, we shall need a weaker property of the transducer in the following part.

A deterministic transducer $W: S \to \mathcal{Y}$, $S \subset \mathcal{R} \times Y$ is said to be linearly time-covariant if for each $(f,y) \in S$, $f(t) = at + b$, $a \neq 0$ the following conditions hold:

i) there exists a pair (g,y) in S such that $D(g) = \mathrm{Im}\, f$ and $g(t) = t$ if $a > 0$ and $g(t) = -t$ if $a < 0$,

ii) for $t \in D(f)$ the equality $W[f,y](t) = W[g,y](at + b)$ holds.

2.8. Definitions. We shall call a continuous function $f \in \mathcal{R}$ piecewise linear if there exists a finite set S_f such that for each $x \in D(f) \setminus S_f$ a neighborhood of x can be found on which the function f is linear.

A function $f \in \mathcal{R}$ will be called standard if either $f = (\pm \mathrm{id} + \mathrm{const}_a) \restriction D(f)$, $a \in R$, or $f = \mathrm{const}_a \restriction D(f)$, $a \in R$.

Proposition. Let $W: S \to \mathcal{Y}$, $S \subset \mathcal{R} \times \{0,1\}$ be a deterministic transducer, whose outcomes are righ-side continuous. Let the following conditions be satisfied:

i) for any $x \in R$, there exist $(f_1, k_1), (f_2, k_2) \in S$ such that $f_1 = (\mathrm{id} + \mathrm{const}_a) \restriction D(f_1)$, $f_2 = (-\mathrm{id} + \mathrm{const}_b) \restriction D(f_2)$, $D(f_1)$, $D(f_2)$, are nondegenerated intervals and $f_1(l(D(f_1))) = f_2(l(D(f_2))) = x$,

ii) if $(f,i), (g,k) \in S$, f and g are standard, $t \in D(f)$, $\tau \in D(g)$, $f(t) = g(\tau)$, and $W[f,i]$ is discontinuous at t, then $W[g,k](\tau) = W[f,i](t)$.

Then there exist disjoint closed sets $A \subset R$, $B \subset R$ such that the restrictions of W and $r(A, B)$ to $S_1 = \{(f,i) \in S \mid f$ is standard$\}$ coincide.

If, in addition, W is a compositive linearly time covariant transducer and $S_2 = \{(f,i) \in S \mid f$ is a piecewise linear map$\}$, then W and $r(A, B)$ coincide on S_2.

Proof. Let us define the sets A and B:

$A = \{x \in R \mid$ if $(u,i) \in S_1$, $t \in D(u)$ and $u(t) = x$ then $W[u,i](t) = 1$,

$B = \{x \in R \mid$ if $(u,i) \in S_1$, $t \in D(u)$ and $u(t) = x$ then $W[u,i](t) = 0$.

The sets A, B are closed. Let us suppose for example that this is not true for the set A. Then there exists $x \in \bar{A} \setminus A$ and a sequence $\{x_n\}_{n=1}^{\infty} \subset A$ converging to x. The first condition and x not being in A imply that there exist $(f_1 0), (f_2, 0) \in S$, $f_1 = (id + const_a) \restriction D(f_1)$, $f_2 = (-id + const_b) \restriction D(f_2)$, $a, b \in R$, the domains of f_1 and f_2 are nontrivial intervals and $f_1(l(D(f_1))) = f_2(l(D(f_2)))$ $= x$. At least for one of these maps, say for f_1, a sequence $\{t_k\}_{k=1}^{\infty} \subset D(f_1)$, $t_k \searrow l(D(f_1))$, can be chosen such that $\{f_1(t_k)\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$. But this contradicts the right-side continuity of $W[f_1,0]$ as $t_k \searrow l(D(f_1))$ and $W[f_1,0](t_k) = 1$.

The disjointness of the sets A and B is evident. The sets A and B determine the relay $r(A, B)$. It is an easy observation that S_1 is included in the domain of $r(A, B)$. It follows directly from the definition of the sets A, B.

Now let us show that the restrictions of W and of $r(A, B)$ to S_1 coincide. Let F be the set $\{t \in D(f) \mid W[f,i](t) \neq r(A, B)[f,i](t)\}$ for a pair $(f,i) \in S_1$ and let us suppose that F is a non-void set. Put $t_0 = \inf F$. The right-side continuity of outcomes implies $t_0 \in F$. From that it follows that just one of the outcomes $r(A, B)[f,i]$, $W[f,i]$ is discontinuous at t_0 in the euclidean topology. On the other hand, the definition of the sets A, B, the assumptions of the proposition and the fact that t_o = inf F guarantee that the outcomes W[f,i] and r(A, B) [f,i] are at t_o either both continuous, or both discontinuous. Thus F = 0 and $W \restriction S_1 = r(A, B) \restriction S_1$.

Let us prove the second part of the proposition. In the first step we choose a pair $(f,i) \in S_2$, the input of which is a linear function. W being a linearly time-covariant transducer implies the existence of $(g,i) \in S_1$ such that there holds: $r(A, B)[f,i](t) = r(A, B)[g,i](f(t)) = W[g,i](f(t)) = W[f,i](t)$. Because of this equality and because r(A, B) and W are compositive transducers, $r(A, B) \restriction S_2 = W \restriction S_2$.

2.9. The proposition 2.7 gives us an information about necessary conditions under which the transducer W is equal to a generalized relay on the set of piecewise linear inputs. This rises the question, under which conditions,this equality can be extended to a larger set.

Let us suppose a topology for the set $\mathcal{R} \times \{0,1\}$ is given, in which generalized relays are continuous and the set $L = \{(f,i) \mid f$ is a piecewise linear function$\}$ is a dense subset of $\mathcal{R} \times \{0,1\}$. Let $r(A, B): S \rightarrow \mathcal{Y}$ be a generalized relay and $W: S \rightarrow \mathcal{Y}$ be a continuous transducer identical with r(A, B) on the set $S \cap L$. Using the well-known theorem about extending identities we can deduce the identity of W and r(A, B) on the whole set S. Thus, providing the required topology does exist, we have proved the following theorem.

Theorem. Let $r(A, B): S \rightarrow \mathcal{Y}$ be a generalized relay and $W: S \rightarrow \mathcal{Y}$ be a transducer continuous on S. The necessary and sufficient condition for r(A, B) and W to be equal is their equality on the set $S \cap L$.

2.10. Summarizing the paragraphs 2.8 and 2.9 we obtain a characterization of a generalized relay.

Theorem. A transducer $W: S \to \mathcal{F}$ is a generalized relay iff

i) $S = \{(f,i) \in \mathcal{R} \times \{0,1\} \mid$ if there exist $(g,k) \in S$ and $\tau \in D(g)$ such that g is standard, $g(\tau) = f(l(D(f)))$ and $W[g,k](\tau) = i$

ii) W is deterministic, compositive and linearly time-invariant,

iii) $W[f,i]$ is righ-side continuous for each $(f,i) \in S$,

iv) for any $x \in R$, there exist $(f_1,k_1),(f_2,k_2)$, $f_1 = (id + const_a) \upharpoonright D(f_1)$, $f_2 = (-id + const_b) \upharpoonright D(f_2)$, $a,b \in R$, $D(f_1)$, $D(f_2)$ are nondegenerated intervals and $f_1(l(D(f_1))) = f_2(l(D(f_2))) = x$,

v) if $(f,i),(g,k) \in S$, f and g are standard, $t \in D(f)$, $\tau \in D(g)$, $f(t) = g(\tau)$ and $W[f,i]$ is discontinuous at t, then $W[g,k](\tau) = W[f,i](t)$,

iv) W is LCC-continuous.

3. Continuity of a generalized relay

3.1. To be able to consider continuity of a generalized relay we have to topologized the set $\mathcal{R} \times \{0,1\}$. If we take the topologies usual for function spaces, a generalized relay is not continuous. Because of that, a new suitable topology for these purposes was defined. It is called a level topology of compact convergence and another paper [4] is devoted to the study of its properties. In this paper, only a brief definition of this topology will be given and the continuity of the generalized relay will be demonstrated, provided the domain of a generalized relay is thus topologized.

3.2. Let X and Y be topological space and Y^X be the set of all continuous mappings on X into Y. We shall define the level topology on Y^X specifying the local base at each point of Y^X. Let us denote $U(\{V(x_i)\}_{i=1}^n)(f) = \{g \in Y^X \mid \forall i = 1,\dots,n \exists x_i' \in V(x_i)$ such that $f(x_i) = g(x_i')\}$. The local base at f is formed by the sets $U(\{V(x_i)\}_{i=1}^n)(f)$ for finite sequences $\{x_i\}_{i=1}^n$ and for all neighborhoods of their elements.

The level topology of compact convergence is the topology projectively generated by the topology of compact convergence and the level topology. We shall abbreviate its name to LCC-topology.

3.3. Let the sets $\mathcal{R}(I)$ be topologized by LCC-topology. In accordance with the paragraph 1.3, we shall consider $\mathcal{R}$ to be the topological sum of these spaces. The set $\{0,1\}$ will be supposed to be a discrete space and $\mathcal{R}\times\{0,1\}$ will be considered with the product topology.

Theorem. Let $r(A,B)\colon S \to \mathcal{S}$ be a generalized relay. If we suppose the topology on the set S is the topology induced by the inclusion of S into $\mathcal{R}\times\{0,1\}$, then $r(A,B)$ is a continuous map.

Proof. Let us denote $S(I,i) = S \cap (\mathcal{R}(I)\times\{i\})$, $i = 0,1$, $I \in \mathcal{J}$. Then $S = (\coprod_{I\in\mathcal{J}} S(I,0)) \amalg (\coprod_{I\in\mathcal{J}} S(I,1))$. To prove the continuity of $r(A,B)$ it is sufficient to prove the continuity of $r(A,B) \restriction S(I,i)\colon S(.,i) \to \mathcal{S}(I)$.

Let us introduce the following symbols: for $u \in S(I,i)$ we define $t_u = \inf \{t | u(t) \in A \cup B\}$; for an arbitrary $t \in I$, $t > t_u$ we define $\tau_t = \sup\{\tau \in [t_u, t) | u(\tau) \in A \cup B\}$. Finally, we define $M_u = \{t \in [t_u, +\infty) \cap I | t = t_u$ or $u(t) \in A$ and $u(\tau_t) \in B$ or $u(t) \in B$ and $u(\tau_t) \in A\}$.

We are going to prove the following Lemma for the set M_u.

Lemma. Let $K \subset R$ be a connected, compact set. Then $M_u \cap K$ is a finite set.

Proof. We shall suppose that $M_u \cap K$ is not a finite set and deduce a contradiction. Let there exist a one-to-one sequence $\{t_n\}_{n=1}^{\infty} \subset M_u \cap K$. We may suppose that it converges to a point $\hat{t}$ and that $\{u(t_n)\}_{n=1}^{\infty} \subset A$. It follows that there must be another sequence $\{t'_m\}_{m=1}^{\infty} \subset M_u \cap K$ such that $\{u(t'_m)\}_{m=1}^{\infty} \subset B$ and $\{t'_m\}_{m=1}^{\infty}$ converges to $\hat{t}$, too. In more detail: for every $\tau, \tau' \in M_u, \tau < \tau'$ such that $u(\tau), u(\tau') \in A$, there exists $\tilde{\tau} \in M_u$ such that $u(\tilde{\tau}) \in B$ and $\tau < \tilde{\tau} < \tau'$ Thus, at least one point $t'_m \in M_u \cap K$ lies in each $1/m$-neighborhood of the point $\hat{t}$. It implies $u(t) \in A \cap B$ and this is the contradiction.

Let us come back to the proof of continuity of $r(A,B) \upharpoonright S(I,i) = r$. Let $u \in S(I,i)$ and let $U = U(\eta, K')(r(u))$ be a neighborhood of the evolution $r(u)$. Let us choose a compact set $K = [l(I), \tilde{t}] \subset I$ such that $K' \subset K$. Let us add the point $l(I)$ to the set $M_u \cap K$ and number the elements of the set created in this way, keeping the natural order given by their positions on the real line - $\{\tau_0, \dots, \tau_k\}$. Let us take $0 < \delta < \eta/2(k+1)$ and for each $i = 1, \dots, k$ define ε_i in the following way: if $u(\tau_i)$ is an element of A, then $\varepsilon_i = \inf \{|u(t) - B| | t \in [\tau_i, \tau_{i+1} - \delta]\}$

and analogously for $u(\tau_1)$ being an element of B. For $i = 0$ let us set $\varepsilon_0 = \inf \{|u(t) - (A \cup B)| \mid t \in [\tau_0, \tau_1 - \delta]\}$. Let $\varepsilon = \min \{\varepsilon_0, \dots, \varepsilon_k\}$. The parameters ε, K and the sequence of δ- neighborhoods of the points $\tau_1, \dots, \tau_k$ determine the neighborhood of the input u in the space $S(I,i)$ such that for every input v from this neighborhood there is $r(v) \in U$.

3.4. Having proved the LCC-topology is a suitable topology as concerns the continuity of a generalized relay, we may turn our attention to the question of the density of the set L. In the paper on LCC-topology [4] we have proved that the set $L(I) = \{f \in \mathcal{R}(I) \mid f$ is a piecewise linear function$\}$ is a dense subset of $\mathcal{R}(I)$. An obvious implication of that is the following proposition.

Proposition. The set L is a dense subset of $\mathcal{R} \times \{0,1\}$.

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