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VARIETIES OF ASSOCIATIVE RINGS CLOSED UNDER IDEAL SUMS

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Abstract: All varieties of associative rings closed under ideal sums are described.

Key words: Variety, associative ring.

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A ring variety X is said to be closed under ideal sums if for any (associative) ring R and for any its ideals I and J belonging to X, the sum I+J belongs to X, too. B.J. Gardner has described in [1] varieties of algebras over any field which are closed under ideal sums and suggested describing varieties of rings with this property. The aim of the present paper is to solve this problem.

Let us recall that the (Malcev-)product of two varieties $\mathcal{M}$ and $\mathcal{N}$ is a class $\mathcal{M}\mathcal{N}$ of all rings R containing an ideal $I \in \mathcal{M}$ such that $R/I \in \mathcal{N}$. The product of two ring varieties is also a variety containing both $\mathcal{M}$ and $\mathcal{N}$. The variety of all rings satisfying the identity $nx=0$ will be denoted by $\mathfrak{B}_n$.

Theorem. A variety of associative rings is closed under ideal sums iff it coincides either with the variety of all associative rings or with a variety of the kind $\mathfrak{B}_n\mathcal{S}$ where the variety $\mathcal{S}$ is generated by a finite (possibly, empty) set of finite fields.

Proof. Necessity. Let X be a variety closed under ideal sums, By [1], Corollary 2.3, X either consists of all associative rings or satisfies an identity of the kind mx=0 for some m. In the latter case there exists the greatest number n such that $\mathcal{B}_n \subseteq X$. By [2], Corollary 2, we have $X = \mathcal{B}_n \mathcal{S}$ for some variety $\mathcal{S}$. Denote by $\mathcal{A}_p$ the ring variety given by the identities px=xy=0 where p is a prime number. If the variety $\mathcal{S}$ contains $\mathcal{A}_p$ for some p, we consider the zero-ring A over a cyclic group with pn elements. We obtain $pA \in \mathcal{B}_n$ and $A/pA \in \mathcal{A}_p$, hence, $A \in \mathcal{B}_n \mathcal{A}_p \subseteq \mathcal{B}_n \mathcal{S} = X$. By [1], Proposition 1.1 it follows $\mathcal{B}_{pn} \in X$, and we have now a contradiction with the choice of n. Thus, $\mathcal{A}_p \nsubseteq \mathcal{S}$ for any p. By [3], Corollary 3.8, $\mathcal{S}$ may be generated by a finite ring S. It is easy to see that S does not contain non-zero nilpotent elements, and hence S is either the direct product of a finite number of finite fields, or the trivial ring.

Sufficiency. We prove that any variety of the kind $\mathcal{B}_n \mathcal{S}$ where $\mathcal{S}$ is generated by a finite (possibly, empty) set of finite fields has the following property which is stronger than the property to be closed under ideal sums: for any ring R and for any ideal I and subring S which both belong to $\mathcal{B}_n \mathcal{S}$ the subring S+I belongs to $\mathcal{B}_n \mathcal{S}$, too. Consider the sets $I_n = \{i \in I \mid ni=0\}$ and $S_n = \{s \in S \mid ns=0\}$. It is easy to verify that $I_n + S_n$ and $I+S_n$ are ideals in I+S. The ring $(I+S)/(I+S_n) \simeq S/(S_n + S \cap I)$ is a homomorphic image of the ring S/S_n. Since $S \in \mathcal{B}_n \mathcal{S}$, and S_n is the greatest ideal of S belonging to $\mathcal{B}_n$, it follows that $(I+S)/(I+S_n) \in \mathcal{S}$. Analogously, the ring

$(I+S_n)/(I_n+S_n) \simeq I/(I_n+S_n \cap I)$ belongs to $\mathcal{S}$. Thus, the ring $(S+I)/(S_n+I_n)$ lies in $\mathcal{S}\mathcal{S}$, but by [4], Lemma 10, $\mathcal{S}\mathcal{S} = \mathcal{S}$. Since the ideal I_n+S_n belongs to the variety $\mathcal{B}_n$, we obtain $S+I \in \mathcal{B}_n \mathcal{S}$.

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