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SUM THEOREMS FOR THE TIGHTNESS AND π-CHARACTER IN THE CLASS OF COMPACT SPACES

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Abstract: Let a compact non-empty space X be the union of some family γ of its subspaces and τ be an infinite cardinal. We prove that if $|\gamma| \le \tau$ and $t(M) < \tau$ for any $M \in \gamma$ then there exists a point $p \in X$ such that $\pi\chi(p,X) < \tau$. With the aid of this result "the increasing chain" version (the term due to Juhász) of Shapirovskii inequality $w(X) \le t(X)^{\delta(X)}$ is proved for any compact space X.

Key words and phrases: Compact space, irreducible map, chain of subspaces, π-character, tightness, Souslin number, weight.

Classification: Primary 54B25, 54A25
Secondary 54C05

Introduction. The paper is the natural continuation and development of the ideas and methods exposed by the author in [3]. Let τ be an infinite cardinal and a compact X be a union of a family γ such that M is a subspace of X and the tightness of M is less than τ for each $M \in \gamma$. How the tightness of X depends on the power of the family γ ? What restrictions should be required on the power of γ in order that the space X would have points of a "small" π-character? How these questions can be solved in the case when γ is a chain (i.e. $M \subset N$ or $N \subset M$ for each $M,N \in \gamma$)?

These and similar problems have been investigated in [3], [4]. So in the sequel we shall use a number of results from

[3] and [4] essentially.

Notations and terminology. The symbols $w(X)$, $nw(X)$, $c(X)$ and $t(X)$ stand for the weight, net weight, Souslin number and tightness of X accordingly; $\chi(x,X)$ and $\pi\chi(x,X)$ for the character and π-character of a point $x \in X$ in the space X. Further, $|A|$ is the power of a set A and $[A]$ is the closure of A in the space X, $A \subset X$. Let $x \in X \supset A$ and $x \in [A]$. We put $t(x,A,X) = \min\{|M| : M \subset A, x \in [M]\}$, $t(x,X) = \sup\{t(x,A,X) : A \subset X,\ x \in [A]\}$ and $\nabla t(x,X) = \min\{\tau$: if $A \subset X$ and $x \in [A]$ then $t(x,A,X) < \tau\}$.

Each ordinal is considered as the set of all preceding ordinals. Cardinals are identified with the corresponding ordinals. The values of all cardinal functions are assumed to be infinite.

Some necessary fact For the sake of the reader's comfort the formulations of the most important results used in the sequel are given here.

Proposition 1. Let k, μ be cardinals and $\aleph_0 \leq k < cf\,\mu$ Then $\mu^k = \sum_{\rho < \mu} \rho^k$.

Assertion 1 (see [1]). Let τ be a cardinal, X be a space and γ be a family of subspaces of X where $X = \cup\gamma$ and $nw(M) \leq \tau$ for each $M \in \gamma$. Then $nw(X) \leq \tau \cdot |\gamma|$.

Assertion 2 (see [1]). If X is a compact, then $w(X) = nw(X)$.

Let $\varkappa$ be an infinite regular cardinal and $\{M_\alpha : \alpha < \varkappa\}$ be a family of subspaces of X such that $M_\alpha \subset M_\beta$ whenever $\alpha < \beta < \varkappa$ and $X = \cup\{M_\alpha : \alpha < \varkappa\}$. Then the following results hold.

Assertion 3 (see [4], Th. 8). Let X be a compact, μ be a cardinal and $t(M_\alpha) < \mu$ for each $\alpha < \varkappa$. Then

(a) $t(X) \leq \mu$;

(b) $t(X) < \mu$, if $\mu < \varkappa$;

(c) $t(X) < \mu$, if $\varkappa < cf\,\mu$.

Assertion 4 (see [4], Th. 3), Let λ be a cardinal and $c(M_\alpha) < \lambda$ for each $\alpha < \varkappa$. Then

(a) $c(X) \leq \lambda$;

(b) $c(X) < \lambda$, if $\lambda < \varkappa$;

(c) $c(X) < \lambda$, if $\varkappa < cf\,\lambda$.

One should pay a special attention to the following remarkable result.

Assertion 5 (see [5], Th. 3). Let $\wp$ be a cardinal, X be a regular space, $S = \{x \in X: \pi\chi(x,X) \leq \wp\}$ and $F = [S]$. Then $w(F) \leq \wp^{c(X)}$.

The formulations of the prior assertions 3 and 4 slightly differ from their primary form (compare with Theorems 3 and 8 from [4]).

Main Results. Our first result is a supplement to Theorem 1.1 from [3], but its proof is based on another idea.

Theorem 1. Let $\tau > \aleph_0$ be a cardinal and a non-empty compact X be the union of a family $\mathcal{M}$ of subspaces such that $|\mathcal{M}| \leq \tau$ and $t(M) < \tau$ for every $M \in \mathcal{M}$. Then there exists a point $p \in X$ such that $\pi\chi(p,X) < \tau$.

Proof. Let us assume that $\pi\chi(p,X) \geq \tau$ for each point $p \in X$. Then the Theorem of Shapirovskii ([6],Th.3.18) implies that there exists a continuous map $g: X \to I^\tau$ of a compact X onto Tichonoff cube I^τ. Let $2 = \{0,1\} \subset I$. Then $2^\tau \subset I^\tau$ and we put $Y = g^{-1}2^\tau$ and $f = g|_Y$. Let us represent the set τ in the form of the union of some disjoint family $\{P_\alpha : \alpha < \tau\}$ such that $|P_\alpha| = \tau$ for each $\alpha < \tau$. Put $A_\alpha = \bigcup\{P_\beta : \beta \leq \alpha\}$, $\alpha < \tau$. For every $\alpha < \tau$ let $\mathcal{A}_\alpha$ be the family $\{A_\alpha \cup P : P \subset \tau, |P| \leq |\alpha| \cdot \aleph_0\}$. Let also $\mathcal{F}_\alpha$ be the family of all closed subsets K of Y such that the set f(K) has a form $\{a\} \times 2^{\tau \setminus A}$ where $A \in \mathcal{A}_\alpha$ and $a \in 2^A$. Primarily we must prove the following assertion:

(∗) If $A \cup B \cup C = \tau$, A, B, C are disjoint, $|B| = \tau$, $M \subset K \subset Y$, $t(M) < \tau$, $f(K) = \{a\} \times 2^B \times 2^C$ and $a \in 2^A$ then there exist a finite set $L \subset C$, a point $b \in 2^B \times 2^L$, and a closed subset $F \subset K$ such that $F \cap M = \emptyset$ and $f(F) = \{a\} \times \{b\} \times 2^{C \setminus L}$.

Let us assume the contrary. Let $\mathcal{F}$ be the family of all closed subsets $K' \subset K$ such that $f(K') = \{a\} \times \{b\} \times 2^{C \setminus L}$ for some finite set $L \subset C$ and a point $b \in 2^B \times 2^C$. Then $K' \cap M \neq \emptyset$ for every $K' \in \mathcal{F}$. Since the map f is continuous and K is closed in Y, there exists a closed subset $\widetilde{K} \subseteq K$ such that $f(\widetilde{K}) = f(K)$ and the restriction $f|_{\widetilde{K}}$ is irreducible. Let $\overline{0}$ and $\overline{1}$ be the points of 2^B with zero and unit coordinates, respectively. Let also σ be the set of all points of 2^B which differ from the point $\overline{1}$ on finitely many coordinates only. We claim that the set $S = \pi_B^{-1}(\sigma) \cap f(\widetilde{K} \cap M)$ is dense in $\{a\} \times 2^B \times 2^C$, where π_B stands for the natural projection of 2^τ onto 2^B. Indeed, let T be any finite subset of $B \cup C$ and x be an arbitrary point of 2^T. It is sufficient to show that $S \cap \pi_T^{-1}(x) \neq \emptyset$ because $S \subset \{a\} \times 2^B \times 2^C$. We define the point $b \in 2^{B \cup T}$ by the rule: $b|_T = x$ and $b(\alpha) = 1$ for every $\alpha \in B \setminus T$. Now we put $\Phi = \{a\} \times \{b\} \times 2^{C \setminus T}$ and $K' = \widetilde{K} \cap f^{-1}(\Phi)$. Then $f(K') = \Phi$, so $K' \in \mathcal{F}$ and $K' \cap M \neq \emptyset$. Let y be any point from $K' \cap M$. Then $f(y) \in \Phi$, i.e. $f(y) \in S \cap \pi_T^{-1}(x) \neq \emptyset$. Therefore S is dense in $\{a\} \times 2^B \times 2^C$.

Put $F = \{a\} \times \{\overline{0}\} \times 2^C$. We have:

$S \subset \pi_B^{-1}(\mathcal{G}) \cap (\{a\} \times 2^B \times 2^C)$ and $|B| = \tau$,
hence the definition of the set $\mathcal{G}$ implies that $[Q] \cap F = \emptyset$ for any set $Q \subset S$ with $|Q| < \tau$. Consequently $[R] \cap K^* = \emptyset$ for any $R \subset \widetilde{M}$ with $|R| < \tau$, where $K^* = \widetilde{K} \cap f^{-1}(F)$ and $\widetilde{M} = M \cap \widetilde{K} \cap f^{-1}(S)$. It is clear that $f(F^*) = F$, $F^* \in \mathcal{F}$ and $f(\widetilde{M}) = S$. As S is dense in $\{a\} \times 2^B \times 2^C$ and the map $f|_{\widetilde{K}}$ is irreducible, so $\widetilde{M}$ is dense in $\widetilde{K}$. The set $K^* \cap M$ is non-empty, because $K^* \in \mathcal{F}$. Choose a point $y \in K^* \cap M$. Then $y \in [\widetilde{M}]$, but $y \notin [R]$ for any subset $R \subseteq \widetilde{M}$ with $|R| < \tau$. It contradicts the fact that $t(M) < \tau$. So the assertion $(\ast)$ is proved.

Let us enumerate the family $\mathcal{M} = \{M_\alpha : \alpha < \tau\}$. There exist a closed subset $K_0 \subset Y$, a set $A_0 \in \mathcal{A}_0$ and a point $a_0 \in 2^{A_0}$ such that $K_0 \cap M_0 = \emptyset$ and $f(K_0) = \{a_0\} \times 2^{C_0}$ where $C_0 = \tau \setminus A_0$ (assertion $(\ast)$). It is clear that $K_0 \in \mathcal{F}_0$. Let $0 < \alpha < \tau$ and for every $\beta < \alpha$ we have defined a set $A_\beta \in \mathcal{A}_\beta$, a point $a_\beta \in 2^{A_\beta}$ and a closed set $K_\beta \in \mathcal{F}_\beta$ such that $K_{\beta''} \subset K_{\beta'}$, $A_{\beta'} \subset A_{\beta''}$ and $a_{\beta''}|_{A_{\beta'}} = a_{\beta'}$ whenever $\beta' < \beta'' < \alpha$. Put $A = \bigcup \{A_\beta : \beta < \alpha\}$, $a = \bigcup \{a_\beta : \beta < \alpha\}$ and $K = \bigcap \{K_\beta : \beta < \alpha\}$. Then $f(K) = \{a\} \times 2^{\tau \setminus A}$. Put also $B_\alpha = P_\alpha \setminus A$ and $C_\alpha = \tau \setminus (A \cup P_\alpha)$. Then $|B_\alpha| = \tau$ and the assertion $(\ast)$ implies that there exist a finite subset $L_\alpha \subset C_\alpha$, a point $b \in 2^{B_\alpha} \times 2^{L_\alpha}$ and a closed subset $K_\alpha \subset K$ such that $K_\alpha \cap M_\alpha = \emptyset$ and $f(K_\alpha) = \{a\} \times \{b\} \times 2^{C_\alpha \setminus L_\alpha}$. We put $A_\alpha = A \cup L_\alpha$ and define a point $a_\alpha \in 2^{A_\alpha}$ by the rules $a_\alpha|_A = a$ and $a_\alpha|_{L_\alpha} = b$. Obviously, $A_\alpha \in \mathcal{A}_\alpha$ and $K_\alpha \in \mathcal{F}_\alpha$. This completes our recursive construction.

Put $K^* = \bigcap \{K_\alpha : \alpha < \tau\}$. Then $K^* \neq \emptyset$. However, $K^* \cap M_\alpha \subset K_\alpha \cap M_\alpha = \emptyset$ for every $\alpha < \tau$. That is a contradiction. The Theorem is proved.

Remark 1. Theorem 1 is valid for $\tau = \aleph_0$. In this case each $M \in \mathfrak{M}$ is discrete, hence X is a scattered sequential compact ([8], Th. 3). Therefore $t(X) \leq \aleph_0$.

Remark 2. The author does not know the answer to the following question:

Let $\tau > \aleph_0$. Does there exist any zero-dimensional space M with $w(M) \leq \tau > t(M)$, which can be continuously mapped onto 2^{τ} ?

The equivalent question:
Is there any subspace $M \subset 2^A \times 2^B$ such that $\pi_A(M) = 2^A$ and $t(M) < \tau$? Here $A = B = \tau$ and π_A is the projection of the product $2^A \times 2^B$ onto the first factor.

Now we shall consider the situation when a compact is represented as the union of some chain of its subspaces.

Lemma 1. Let a compact X be the union of a chain $\mathscr{C}$ of its subspaces and $\mu = \sup\{t(M) : M \in \mathscr{C}\}$, where $cf\,\mu < \mu$ and $t(M) < \mu$ for each $M \in \mathscr{C}$. Then $\pi\chi(x,X) < \mu$ and $\nabla t(x,X) \leq \mu$ for each point $x \in X$.

Proof. Primarily we show that $\pi\chi(x,Y) < \mu$ for every closed subset $Y \subset X$ and any point $x \in Y$. Let Y be a closed subset of X. Put $\tilde{\mathscr{C}} = \{M \cap Y : M \in \mathscr{C}\}$ and $\tilde{\mu} = \sup\{t(N) : N \in \tilde{\mathscr{C}}\}$. If $\tilde{\mu} < \mu$ then $t(Y) \leq \tilde{\mu}^+$ (assertion 3(a)). However, μ is a singular cardinal, hence $\tilde{\mu}^+ < \mu$ and $\pi\chi(x,Y) \leq t(Y) < \mu$ for any point $x \in Y$ ([7], Th. 1).

Let us consider now the case $\tilde{\mu} = \mu$. We have: $t(N) < \mu$ for any $N \in \tilde{\mathscr{C}}$. Put $\varkappa = cf\,\mu$. Then $\varkappa < \mu$ and there exists a subfamily $\mathscr{D} \subset \tilde{\mathscr{C}}$ such that $|\mathscr{D}| = \varkappa$ and $\bigcup\mathscr{D} = \bigcup\tilde{\mathscr{C}} = Y$. For every $\wp < \mu$ we put $Y_{\wp} = \{y \in Y : \pi\chi(y,Y) \leq \wp\}$. We claim that $Y = \bigcup\{Y_\rho : \rho < \mu\}$.

Assume the contrary. Then there exists a point $p \in Y \setminus \bigcup\{Y_\rho : \rho < \mu\}$. Let $\{\rho_\alpha : \alpha < \varkappa\}$ be a strictly increasing sequence of cardinals such that $\rho_\alpha < \mu$ for each $\alpha < \varkappa$ and $\mu = \sup\{\rho_\alpha : \alpha < \varkappa\}$. For every $\alpha < \varkappa$ let us put $\mathfrak{D}_\alpha = \{Y_{\rho_\alpha} \cap N : N \in \mathfrak{D}\}$. Then $Y_{\rho_\alpha} = \bigcup \mathfrak{D}_\alpha$ and $|\mathfrak{D}_\alpha| \leq |\mathfrak{D}| = \varkappa$.

We should note that for any $\alpha < \varkappa$ and $K \in \mathfrak{D}_\alpha$ the point p does not belong to the closure of a set K. Otherwise there exist $\alpha < \varkappa$ and $K \in \mathfrak{D}_\alpha$ such that $p \in [K]$. Obviously there exists $N \in \mathfrak{D}$ such that $K = Y_{\rho_\alpha} \cap N$. As $\mathfrak{D}$ is a chain and $Y = \bigcup \mathfrak{D}$, we can find an element $\widetilde{N} \in \mathfrak{D}$ such that $p \in \widetilde{N}$ and $N \subset \widetilde{N}$. Put $\widetilde{K} = Y_{\rho_\alpha} \cap \widetilde{N}$. Then $K \subset \widetilde{K}$ hence $p \in [\widetilde{K}]$. However, $t(\widetilde{N}) < \mu$ which implies the existence of a subset $S \subset \widetilde{K}$ such that $|S| < \mu$ and $p \in [S]$. We have: $S \subset \widetilde{K} \subset Y_{\rho_\alpha}$, hence for every point $x \in S$ there exists a π-base $\mathcal{B}_x$ at x in a compact Y such that $|\mathcal{B}_x| \leq \rho_\alpha$. Put $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in S\}$. Then the family $\mathcal{B}$ is a π-base at the point p in Y and $|\mathcal{B}| \leq \rho_\alpha \cdot |S| < \mu$, i.e. $\pi\chi(p,Y) < \mu$. It contradicts the choice of the point p.

So for each $\alpha < \varkappa$ there exists a $G_\varkappa$ x)-set $\mathcal{O}_\alpha$ in Y such that $p \in \mathcal{O}_\alpha$ and $\mathcal{O}_\alpha \cap Y_{\rho_\alpha} = \emptyset$. Then $\mathcal{O} = \bigcap\{\mathcal{O}_\alpha : \alpha < \varkappa\}$ is a $G_\varkappa$-set in Y, $p \in \mathcal{O}$ and $\mathcal{O} \cap \bigcup\{Y_\rho : \rho < \mu\} = \emptyset$. Now we fix a closed subset $\Phi \subset Y$ such that $p \in \Phi \subset \mathcal{O}$ and $\chi(\Phi, Y) \leq \varkappa$. Then first, $\pi\chi(y,\Phi) \geq \varkappa$ for each point $y \in \Phi$. For if there exists a point $y \in \Phi$ with $\pi\chi(y,\Phi) = \lambda < \mu$ then $\pi\chi(y,Y) \leq \pi\chi(y,\Phi) \cdot \chi(\Phi,Y) \leq \lambda \cdot \varkappa < \mu$ ([2], § 3, Lemma 1), which contradicts the fact that $\Phi \cap \bigcup\{Y_\rho : \rho < \mu\} = \emptyset$. Second,

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x) A subset $\mathcal{O} \subset Y$ is said to be a $G_\varkappa$-set in Y if $\mathcal{O}$ is an intersection of some family γ of open subsets of Y with $|\gamma| \leq \varkappa$.

Theorem 1 implies that there exists a point $y \in \Phi$ with $\pi\chi(y, \Phi) < \mu$. This contradiction shows that $Y = \bigcup\{Y_\wp : \wp < \mu\}$, i.e. $\pi\chi(y,Y) < \mu$ for any $y \in Y$.

In particular, in the case $Y = X$ we get the first assertion of the theorem. Now we show that $\nabla t(x,X) \leq \mu$ for each $x \in X$. Let $x \in X \supset S$ and $x \in [S]$. Put $Y = [S]$. Then $\pi\chi(x,Y) < \mu$, hence there exists a π-base $\mathcal{B}$ at x in Y with $|\mathcal{B}| < \mu$. For every $V \in \mathcal{B}$ we fix a point $x_V \in S \cap V$ and put $A = \{x_V : V \in \mathcal{B}\}$. Then $A \subset S$, $|A| \leq |\mathcal{B}| < \mu$ and $x \in [A]$. Thus the lemma is proved.

In the following theorem we straighten the Shapirovskii's inequality $w(X) \leq t(X)^{c(X)}$ which holds for any compact X, to a "chain case".

<u>Theorem 2</u>. Let τ be an infinite cardinal and a compact X be the union of a chain $\mathcal{C}$ of its subspaces, where $t(M)^{c(M)} \leq \tau$ for every $M \in \mathcal{C}$. Then $w(X) \leq \tau$.

<u>Proof</u>. One can assume that a chain $\mathcal{C}$ has no maximal element - otherwise all is trivial. We shall say that a subfamily $\mathcal{D} \subset \mathcal{C}$ is cofinal in X if $X = \bigcup\mathcal{D}$. For every cofinal in X subfamily $\mathcal{D} \subset \mathcal{C}$ put $\lambda_{\mathcal{D}} = \sup\{c(M) : M \in \mathcal{D}\}$. Put also $\lambda = \min\{\lambda_{\mathcal{D}} : \mathcal{D} \subset \mathcal{C}, \mathcal{D}$ is cofinal in $X\}$. There exists a cofinal subfamily $\mathcal{D} \subset \mathcal{C}$ in X such that $\lambda = \lambda_{\mathcal{D}}$. Obviously there exist a regular cardinal $\varkappa \geq \aleph_0$ and a subfamily $\mathcal{E} = \{M_\alpha : \alpha < \varkappa\} \subset \mathcal{D}$ such that 1) $X = \bigcup\mathcal{E}$ and 2) if $\alpha < \beta < \varkappa$ then $M_\alpha \subset M_\beta$ and $c(M_\alpha) \leq c(M_\beta)$. The definition of the cardinal λ implies that $\lambda = \sup\{c(M_\alpha) : \alpha < \varkappa\}$. Put $\mu = \sup\{t(M_\alpha) : \alpha < \varkappa\}$. Unlike the Souslin number the tightness is a monotonous cardinal function, hence $t(M_\alpha) \not\leq t(M_\beta)$ whenever $\alpha < \beta < \varkappa$.

There are two possibilities for the Souslin number:

(1 c) $c(M_\alpha) < \lambda$ for every $\alpha < \varkappa$;

(2 c) $c(M_\alpha) = \lambda$ for some $\alpha < \varkappa$.

Analogously there are two possibilities for the tightness:

(1 t) $t(M_\alpha) < \mu$ for every $\alpha < \varkappa$;

(2 t) $t(M_\alpha) = \mu$ for some $\alpha < \varkappa$.

For every $\alpha < \varkappa$ we put $F_\alpha = [M_\alpha]$ and show that $w(F_\alpha) \leq \tau$. The further proof is in consideration of four possible combinations of the cases formulated above.

I. (1 c) & (1 t).

Then $\mathrm{cf}\,\lambda = \varkappa = \mathrm{cf}\,\mu$ and the conditions of the theorem imply that $\rho^k \leq \tau$ for any $\rho < \mu$ and $k < \lambda$. In particular, $\varkappa \leq \mu \leq \tau$ and $\varkappa \leq \lambda \leq \tau$.

I A. Suppose that $\varkappa < \mu$.

We have: $c(F_\alpha) = c(M_\alpha) < \lambda$ for every $\alpha < \varkappa$. Put $\mathscr{E}_\alpha = \{F_\alpha \cap M : M \in \mathscr{E}\}$ and $\mu_\alpha = \sup\{t(N) : N \in \mathscr{E}_\alpha\}$. If $\mu_\alpha < \mu$ then $t(F_\alpha) \leq \mu_\alpha^+$ (assertion 3(a)) and $\mu_\alpha^+ < \mu$ because μ is a singular cardinal. Therefore $w(F_\alpha) \leq t(F_\alpha)^{c(F_\alpha)} \leq \tau$. Now let $\mu_\alpha = \mu$. For every cardinal $\rho < \mu$ put $Z_\alpha(\rho) = \{x \in F_\alpha : \pi\chi(x, F_\alpha) \leq \rho\}$. Then Lemma 1 implies that $F_\alpha = \bigcup\{Z_\alpha(\rho) : \rho < \mu\}$. Put $F_\alpha(\rho) = [Z_\alpha(\rho)]$. Then $w(F_\alpha(\rho)) \leq \rho^{c(F_\alpha)} \leq \tau$ for every $\rho < \mu$ (assertion 5). Consequently $nw(Z_\alpha(\rho)) \leq w(Z_\alpha(\rho)) \leq w(F_\alpha(\rho)) \leq \tau$. The equality $F_\alpha = \bigcup\{F_\alpha(\rho) : \rho < \mu\}$ implies that $w(F_\alpha) = nw(F_\alpha) \leq \tau \cdot \mu = \tau$ (assertions 1 and 2).

I B. Suppose that $\varkappa = \mu$.

Then $\mu \leq \lambda$. Assertion 3(a) implies that $t(X) \leq \mu$, hence $t(F_\alpha) \leq \mu$ for every $\alpha < \varkappa$. We consider two subcases.

1) $\mu = \lambda$. The cardinal μ is regular, hence

$\mu^k = \sum_{\rho<\mu} \rho^k \leq \tau \cdot \mu = \tau$ for every $k < \mu$ (Proposition 1). So $w(F_\alpha) \leq t(F_\alpha)^{c(F_\alpha)} \leq \tau$ for every $\alpha < \varkappa$ because $t(F_\alpha) \leq \leq t(X) \leq \mu$ and $c(F_\alpha) = c(M_\alpha) < \lambda = \mu$.

2) $\mu < \lambda$. We have: $\rho^k \leq \tau$ for any $\rho < \mu$ and $k < \lambda$. In particular, $2^k \leq \tau$ for any $k < \lambda$. Consequently $w(F_\alpha) \leq t(F_\alpha)^{c(F_\alpha)} \leq \mu^{c(F_\alpha)} \leq (2^\mu)^{c(F_\alpha)} = 2^{\mu \cdot c(F_\alpha)} \leq \tau$ for every $\alpha < \varkappa$, because $\mu < \lambda$ and $c(F_\alpha) = c(M_\alpha) < \lambda$.

II. (1 c) & (2 t).

Then $\varkappa = \mathrm{cf}\,\lambda$ and our theorem's conditions imply that $\mu^k \leq \tau$ for every $k < \lambda$. In particular, $2^k \leq \tau$ for every $k < \lambda$ hence $\lambda \leq \tau$. Consider two subcases.

1) $\mu^+ = \varkappa$. As $t(M_\alpha) \leq \mu < \varkappa$ for every $\alpha < \varkappa$, so $t(X) \leq \varkappa$ (assertion 3(a)). Obviously $\mu^+ = \varkappa \leq 2^\mu$. Consequently

$$w(F_\alpha) \leq t(F_\alpha)^{c(F_\alpha)} \leq \varkappa^{c(F_\alpha)} \leq (2^\mu)^{c(F_\alpha)} = 2^{\mu \cdot c(F_\alpha)} \leq \tau$$

for every $\alpha < \varkappa$, because $t(F_\alpha) \leq t(X) \leq \varkappa$, $\mu < \varkappa \leq \lambda$ and $c(F_\alpha) = c(M_\alpha) < \lambda$.

2) $\mu^+ \neq \varkappa$. Then $t(X) \leq \mu$ (assertion 3(a),(b),(c)). Consequently $t(F_\alpha) \leq t(X) \leq \mu$ and $w(F_\alpha) \leq \mu^{c(F_\alpha)} \leq \tau$ for every $\alpha < \varkappa$ because $c(F_\alpha) < \lambda$.

III. (2 c) & (1 t).

Then $\varkappa = \mathrm{cf}\,\mu$ and our theorem's conditions imply that $\rho^\lambda \leq \tau$ for every $\rho < \mu$. In particular, $\varkappa \leq \mu \leq \tau$. Consider two cases.

III A. $\varkappa < \mu$. In this case the following proof is completely analogous to the proof of the case I A.

III B. $\varkappa = \mu$. Then $t(X) \leq \mu$ (assertion 3(c)). There are two possibilites.

1) $\lambda < \mu$. The cardinal μ is regular, hence

$\mu^{\lambda} = \sum_{\rho < \mu} \rho^{\lambda} \leq \tau \cdot \mu = \tau$ (Proposition 1). Consequently $w(F_{\alpha}) \leq t(F_{\alpha})^{c(F_{\alpha})} \leq \mu^{\lambda} \leq \tau$ because $t(F_{\alpha}) \leq t(X) \leq \mu$ and $c(F_{\alpha}) = c(M_{\alpha}) \leq \lambda$ for every $\alpha < \varkappa$.

2) $\mu \leq \lambda$. We have: $\rho^{\lambda} \leq \tau$ for every $\rho < \mu$. In particular, $2^{\lambda} \leq \tau$. Consequently $w(F_{\alpha}) \leq t(F_{\alpha})^{c(F_{\alpha})} \leq \mu^{c(F_{\alpha})} \leq$ $\leq (2^{\mu})^{c(F_{\alpha})} = 2^{\mu c(F_{\alpha})} \leq \tau$ because $\mu \leq \lambda$ and $c(F_{\alpha}) \leq \lambda$.

Thus $w(F_{\alpha}) \leq \tau$ in each of the cases I, II, III and $\varkappa \leq \tau$. So $w(X) \leq \tau$ (assertion 1).

IV. (2 c) & (2 t).

The theorem's conditions imply that $\mu^{\lambda} \leq \tau$. In particular, $\mu \leq \tau$. Without loss of generality we can assume that $2^{\lambda} \leq \mu$. Consider three cases.

1) $\varkappa \leq \mu$. Then $t(X) \leq \mu$ (assertion 3(c)). Consequently $w(F_{\alpha}) \leq t(F_{\alpha})^{c(F_{\alpha})} \leq \mu^{\lambda} \leq \tau$ for every $\alpha < \varkappa$. This inequality and assertion 1 imply that $w(X) \leq \tau$ because $\varkappa \leq \mu \leq \tau$.

2) $\mu^{+} = \varkappa$. Then $c(X) \leq \lambda$ because $\lambda < \mu < \varkappa$ (assertion 4(c)). Theorem 1 of the paper implies that the set $S = \{x \in X: \pi\chi(x,X) \leq \mu\}$ is dense in X. Applying assertion 5 we conclude that $w(X) = w([\tilde{S}]) \leq \mu^{\lambda} \leq \tau$.

3) $\mu^{+} < \varkappa$. Then $t(X) \leq \mu$ (assertion 3(c)) and $c(X) \leq \lambda$ (assertion 4(c)). Consequently $w(X) \leq \mu^{\lambda} \leq \tau$.

The theorem is completely proved.

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