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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

24,1 (1983)

A NOTE ABOUT M_1-SPACES AND STRATIFIABLE SPACES

Ju. H. BREGMAN

Abstract: M_1-spaces and stratifiable spaces are characterized as preimages of metrizable spaces under one-to-one mappings, satisfying special conditions.

Key words: M_1-space, stratifiable, paracompact, closure-preserving, σ-space.

Classification: 54E20

In our paper, we deal with bijections (= one-to-one mappings) of M_1-spaces and stratifiable spaces onto metrizable ones. M_1- and stratifiable or M_3-spaces were defined in 1961 by J.G. Ceder [3]. It was also shown there that every M_1-space is stratifiable, but whether the converse is true is still unknown.

Since every stratifiable space is paracompact and has a G_δ-diagonal, it can be bijected onto a metrizable one [6],[2]. But it was not studied before, as far as we know, with what special properties can the bijections of stratifiable and M_1-spaces onto metrizable ones be chosen. Furthermore, it would be interesting to find characterizations of these two classes in terms of preimages of metrizable spaces under bijections. Recall that not long ago has A.V. Archangel'skii characterized the topology of a paracompact σ-space [7] as that one which has a metrizable Δ-approach [1], but this is obviously equivalent to the existence of a special bijection onto a metric space. Quite recently A.P. Šostak [9] has studied special properties of bijections which hold for some classes of paracompacta with G_δ-diagonals. The aim of the present paper is to obtain characterizations of M_1- and stratifiable spaces as preimages of metrizable spaces under bijections which satisfy special conditions.

Let us recall the basic definitions from [2],[3],[8] which we use in our paper. A regular space is called an M_1-space if it has a σ-closure-preserving base. A family of pairs of subsets $\mathcal{P} = \{(p'_\alpha, p''_\alpha): \alpha \in A\}$ of a space X, satisfying $p'_\alpha \subset p''_\alpha$ for every $\alpha \in A$ is called a pair-network in X if for every $x \in X$ and every its open neighbourhood U there exists $\alpha \in A$ such that $x \in p'_\alpha \subset U$. If, moreover, every p'_α is open, then $\mathcal{P}$ is called a pair-base in X. A family $\mathcal{P} = \{(p'_\alpha, p''_\alpha): \alpha \in A\}$ is called cushioned if $[\cup\{p'_\alpha : \alpha \in A_0\}]_X \subset \cup\{p''_\alpha : \alpha \in A_0\}$ for every $A_0 \subset A$. A family $\mathcal{P} = \{(p'_\alpha, p''_\alpha): \alpha \in A\}$ is called σ-cushioned if it is a countable union of cushioned families $\mathcal{P}_n = \{(p'_\alpha, p''_\alpha): \alpha \in A_n\}$. A T_1-space which has a σ-cushioned pair-base is called stratifiable. A topological space which has a σ-discrete network is called a σ-space. Every M_1-space is stratifiable [3] and every stratifiable space is a paracompact σ-space [5]. For the definitions of other conceptions used in the paper see, e.g. [4].
We denote the closure of a set A in a topological space X by $[A]_X$.

<u>Proposition 1</u>. For every closure-preserving family $\mathcal{K}$ of subsets of a paracompact σ-space X there exists a bijection φ

of X onto a metrizable space Y such that $\varphi([K]_X) = [\varphi(K)]_Y$ holds for every $K \in \mathcal{K}$.

<u>Proof.</u> Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a σ-discrete network in X where every $\mathcal{P}_n$ is a discrete family of closed sets. For a set $P \subset X$ and a closure-preserving family $\mathcal{B}$ of subsets of X we define an open set $U(P,\mathcal{B}) = X \setminus \bigcup\{[V]_X : V \in \mathcal{B}, [V]_X \cap P = \emptyset\}$. We shall construct recursively a sequence $\{\mathcal{C}_k : k \in \mathbb{N}\}$ of discrete families $\mathcal{C}_k$ of open sets of a space X such that for every $k \in N$ the following three properties hold:

(1) $\mathcal{C}_k = \{V_k(P) : P \in \mathcal{P}_k, P \subset V_k(P)\}$;

(2) if $P \in \mathcal{P}_k$ and $P \subset V \in \bigcup\{\mathcal{C}_l : l < k\}$ then $[V_k(P)]_X \subset V$;

(3) $V_k(P) \subset (P, \mathcal{B}_k)$ where $\mathcal{B}_k = \bigcup\{\mathcal{C}_l : l < k\} \cup \mathcal{K}$.

Assume that the families $\mathcal{C}_k$ are already defined for all $k < n$ in such a way that they satisfy the properties (1) - (3). In order to define the family $\mathcal{C}_n$ consider the discrete family of closed sets $\mathcal{P}_n$ and applying the paracompactness of the space X find a discrete family $\mathcal{V} = \{V(P) : P \in \mathcal{P}_n\}$ of open neighbourhoods of the sets $P \in \mathcal{P}_n$. It is easy to notice that the family $\mathcal{B}_n = \bigcup\{\mathcal{C}_k : k < n\} \cup \mathcal{K}$ is closure-preserving and therefore $U(P, \mathcal{B}_n)$ is an open neighbourhood of P. Consider now for every $P \in \mathcal{P}_n$ the neighbourhood $V'(P) = V(P) \cap U(P, \mathcal{B}_n)$ then the family $\mathcal{V}' = \{V'(P) : P \in \mathcal{P}_n\}$ is discrete.

For every $P \in \mathcal{P}_n$ and every $k < n$ there exists at most one open set $V \in \mathcal{C}_k$ containing P. If there is such a set V then take $W_k(P)$ satisfying $P \subset W_k(P) \subset [W_k(P)]_X \quad V$. If there is no such a set $V \in \mathcal{C}_k$ then let $W_k(P) = X$. Define now for every $P \in \mathcal{P}_n$ a neighbourhood $V''(P) = \bigcap\{W_k(P) : k < n\}$ and let $V_n(P) = V'(P) \cap V''(P)$. It is easy to verify that the family

$\mathcal{C}_n = \{V_n(P) : P \in \mathcal{P}_n\}$ satisfies the conditions (1) - (3). Let now $\mathcal{C} = \bigcup \{\mathcal{C}_n : n \in N\}$

It is easy to notice that the σ-discrete family $\mathcal{C}$ is a base of a new Hausdorff topology on the set X. We denote the corresponding topological space by Y. It is obvious that $\mathcal{C}$ is a σ-discrete base of its topology.

Let φ denote the natural bijection of X onto Y. We shall prove now the equality $\varphi([V]_X) = [\varphi(V)]_Y$ for every $V \in \bigcup \{\mathcal{B}_n : n \in \mathbb{N}\}$. The inclusion $\varphi([V]_X) \subset [\varphi(V)]_Y$ is obvious since the mapping φ is continuous.
Let $x \notin \varphi([V]_X)$ for some $V \in \mathcal{B}_n$. The family $\mathcal{P}$ is a network in X and therefore there exists $P \in \mathcal{P}_m$, $m \geq n$ such that $x \in P \subset$ $\subset X \setminus [V]_X$. Then $V_m(P) \in \mathcal{C}_m$ and $V_m(P) \cap V = \emptyset$; hence $x \notin [\varphi(V)]_Y$. Thus the equality $\varphi([V]_X) = [\varphi(V)]_Y$ holds for every $V \in \bigcup\{\mathcal{B}_n : n \in N\}$ and, in particular, for every $V \in \mathcal{K}$. This property together with the condition (2) readily implies the regularity of the space Y. Therefore it is metrizable as a regular space with a σ-discrete base.

<u>Remark</u>. It is easy to notice that the topology $\mathcal{T}_Y$ of the space Y is an Δ-approach in the sense of A.V. Archangel'-skii [1] to the initial topology $\mathcal{T}_X$ of an M_1-space X. In terms of bijections this means that the image of the family $\mathcal{P}$ under the bijection φ is a σ-discrete network in Y.

<u>Lemma 1</u>. Let $\{\mathcal{K}_n : n \in \mathbb{N}\}$ be a sequence of families of subsets of a topological space X. Suppose that for every $n \in \mathbb{N}$ there exists a bijection φ_n of X onto a metrizable space M_n such that $\varphi_n([K]_X) = [\varphi_n(K)]_{M_n}$ holds for all $K \in \mathcal{K}_n$. Then the diagonal mapping $\varphi = \Delta \varphi_n : X \longrightarrow M = \varphi(X) \subset \Pi\{M_n : n \in \mathbb{N}\}$

satisfies the equality $\varphi([K]_X) = [\varphi(K)]_M$ for every $K \in \mathcal{K} = \bigcup\{\mathcal{K}_n : n \in \mathbb{N}\}$.

<u>Proof</u>. Let $\varphi_n(x) = x' \in M_n$ (thus we shall not distinguish in notations the corresponding points in spaces M_n for different n); therefore $\varphi(x) = y = (x', \ldots, x' \ldots)$. Take a set $K \in \mathcal{K}_m$. Since the diagonal mapping φ is continuous, the inclusion $\varphi([K]_X) \subset [\varphi(K)]_M$ is true. Let $y \in [\varphi(K)]_M$; to complete the proof we must show that $y \in \varphi([K]_X)$. If U_y is a neighbourhood of a point $y = (x', \ldots, x' \ldots)$ in M then $U_y \cap \varphi(K) \neq \emptyset$. Take $U_y = (M_1 \times \ldots \times M_{m-1} \times U_m \times M_{m+1} \times \ldots) \cap M$, where U_m is an arbitrary neighbourhood of the point x' in the space M_m. Denoting by p_m the projection of the product $\prod\{M_n : n \in \mathbb{N}\}$ onto the m-th coordinate space M_m we may conclude that $U_m \cap (p_m(\varphi(K)) \neq \emptyset$. Since $p_m \circ \varphi = \varphi_m$, this means that $U_m \cap \varphi_m(K) \neq \emptyset$ and hence $x' \in [\varphi_m(K)]_{M_m} = \varphi_m([K]_X)$. Therefore $x \in [K]_X$ and $y \in \varphi([K]_X)$.

Proposition 1 and Lemma 1 easily imply the following result.

<u>Proposition 2</u>. Let X be a paracompact σ-space and $\mathcal{K}$ a σ-closure-preserving family of its subset. Then there exists a bijection φ of X onto a metrizable space Y such that $\varphi([K]_X) = [\varphi(K)]_Y$ for every $K \in \mathcal{K}$.

This proposition allows us to establish the following characteristic property of M_1-spaces.

<u>Theorem 1</u>. Let X be an M_1-space and $\mathcal{B}$ a σ-closure-preserving base of it. Then there exists a bijection φ of X onto a metrizable space Y such that $\varphi([V]_X) = [\varphi(V)]_Y$ for every $V \in \mathcal{B}$.

The next characterization of M_1-spaces easily follows from Proposition 2, too.

<u>Theorem 2</u>. A regular space X is an M_1-space iff there exists a bijection φ of it onto a metrizable space Y and a σ-closure-preserving network $\mathcal{K}$ in Y such that $\mathcal{B} = \{\varphi^{-1}(K) : K \in \mathcal{K}\}$ is a σ-closure-preserving base in X.

We proceed now to the study of the bijections of stratifiable spaces onto metric ones.

<u>Proposition 3</u>. Let X be a paracompact σ-space and $\mathcal{K} = \{(K',K'')\}$ a cushioned family of pairs of its subsets. Then there exists a bijection φ of X onto a metrizable space Y such that $[\varphi(K')]_Y \subset \varphi(K'')$ holds for every $(K',K'') \in \mathcal{K}$.

<u>Proof</u>. Consider a σ-discrete network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in N\}$ in which every $\mathcal{P}_n$ is a discrete family of closed subsets of X. For a set $P \subset X$ and a cushioned family $\mathcal{B}$ of pairs of subsets of X let $\tilde{U}(P,\mathcal{B}) = X \setminus [\bigcup\{V' : (V',V'') \in \mathcal{B}, V'' \cap P = \emptyset\}]_X$. Quite analogously as in the proof of Proposition 1 we recursively construct a sequence $\{\mathcal{C}_k : k \in N\}$ of discrete families $\mathcal{C}_k$ of open sets of the space X such that for every $k \in N$ the following three properties hold:

(1) $\mathcal{C}_k = \{V_k(P) : P \in \mathcal{P}_k, P \subset V_k(P)\}$;

(2) if $P \in \mathcal{P}_k$ and $P \subset V \in \bigcup\{\mathcal{C}_l : l < k\}$ then $[V_k(P)]_X \subset V$;

(3) $V_k(P) \subset \tilde{U}(P, \mathcal{B}_k)$ where $\mathcal{B}_k = \bigcup\{(V,[V]_X) : V \in \mathcal{C}_l, l < k\} \cup \mathcal{K}$.

As in the proof of Proposiion 1 one can easily verify that the σ-discrete family $\mathcal{C}$ of subsets of X is a base of a new metrizable topology on the set X. The corresponding metrizable space will be denoted by Y and let $\varphi : X \longrightarrow Y$ be the natural bijection. There is no difficulty to show also that the inclusion $[\varphi(K')]_Y \subset \varphi(K'')$ holds for every pair $(K',K'') \in \mathcal{K}$.

Furthermore, patterned after the proof of Lemma 1 one can easily prove the following.

<u>Lemma 2</u>. Let X be a topological space and $\{\mathcal{K}_n : n \in \mathbb{N}\}$ a sequence of cushioned families of pairs of its subsets. Suppose that for every $n \in \mathbb{N}$ there exists a bijection φ_n of X onto a metrizable space M_n such that $[\varphi_n(K')]_{M_n} \subset \varphi_n(K'')$ for all $(K',K'') \in \mathcal{K}$. Then the diagonal mapping $\varphi = \Delta \varphi_n : X \rightarrow M = \varphi(X) \subset \Pi\{M_n : n \in \mathbb{N}\}$ satisfies the inclusion $[\varphi(K')]_M \subset \varphi(K'')$ for all pairs $(K',K'') \in \mathcal{K} = \cup\{\mathcal{K}_n : n \in \mathbb{N}\}$.

Proposition 3 and Lemma 2 immediately imply the following.

<u>Proposition 4</u>. Let X be a paracompact σ-space and $\mathcal{K}$ a σ-cushioned family of pairs of its subsets. Then there exists a metrizable space Y and a bijection $\varphi : X \rightarrow Y$ such that $[\varphi(K')]_Y \subset \varphi(K'')$ for every pair $(K',K'') \in \mathcal{K}$.

Next theorem which establishes a characteristic property of stratifiable spaces is a direct corollary of the previous result.

<u>Theorem 3</u>. Let X be a stratifiable space and $\mathcal{B}$ a σ-cushioned pair-base of it. Then there exists a bijection φ of X onto a metrizable space Y such that $[\varphi(V')]_Y \subset \varphi(V'')$ for every pair $(V',V'') \in \mathcal{B}$.

Next characterization of stratifiable spaces in terms of bijections onto metrizable spaces also follows from Proposition 4.

<u>Theorem 4</u>. A $\mathcal{T}_1$-space X is stratifiable iff there exists

a bijection φ of X onto a metrisable space Y and a σ-cushioned pair-network $\mathcal{K}$ in Y such that the family $\mathcal{B} = \{(\varphi^{-1}(K'), \varphi^{-1}(K'')) : (K', K'') \in \mathcal{K}\}$ is a σ-cushioned pair-base in X.

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