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BASIC EQUIVALENCES IN THE ALTERNATIVE SET THEORY

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Abstract: In the paper we study a special case of equivalences of indiscernibility, so-called basic equivalences. The equivalences, whose definition has quite a set-logical character, play an important role in non-standard descriptions of topology and other areas of the alternative set theory. We proved here among others that there is not possible to include a proper set-theoretically definable class into a monad and that each set-theoretically definable function which has a fixpoint with respect to the basic equivalence in a monad is necessarily an identity on this monad.

Key words: Alternative set theory, basic equivalence, monad, set-theoretically definable class.

Classification: Primary 03E70
Secondary 54J05

This work is devoted to the studying of some properties of equivalence $\overset{\circ}{=}_{\{c\}}$. The equivalence $\overset{\circ}{=}$ is defined in [V], ch. V, § 1. Already from the results presented in the quoted book it follows that $\overset{\circ}{=}$ is of consequence in the alternative set theory. In the paper [V 1], the definition of the equivalence $\overset{\circ}{=}_X$ which is a generalization of $\overset{\circ}{=}$, is given, and several essential theorems are proved there. Other works in the alternative set theory, especially [Č - V], confirm the importance of these equivalences and, above all, the significance of $\overset{\circ}{=}_{\{c\}}$; we shall call it basic equivalence.

Now we remind (from [V] and [V 1]) several crucial definitions and assertions which we shall need later on.

We put $x \overset{\circ}{\underset{X}{=}} y$ iff the formula $\varphi(x) \equiv \varphi(y)$ holds for any set-formula $\varphi(z)$ of the language FL_X.

Even now we can see that the non-standard description of topology is much closer to set-logical considerations than the standard one.

If X is a finite or a countable class, then $\overset{\circ}{\underset{X}{=}}$ is an equivalence of indiscernibility (cf. [V], ch. III) which is totally disconnected. The clopen figures in $\overset{\circ}{\underset{X}{=}}$ are just the classes which belong to Sd_X.

Moreover, it was proved in [V 1] that for each equivalence of indiscernibility $\doteq$ there is an equivalence $\overset{\circ}{\underset{\{c\}}{=}}$ which is finer. This fact actually led to the name - basic equivalence - for $\overset{\circ}{\underset{\{c\}}{=}}$.

Monads in $\overset{\circ}{\underset{\{c\}}{=}}$, i.e. classes of decomposition of V according to $\overset{\circ}{\underset{\{c\}}{=}}$, correspond (by a one-one correspondence) with ultrafilters on the ring of classes $Sd_{\{c\}}$ (recall that $Sd_{\{c\}}$ denotes the system of all classes of the form $\{x; \varphi(x)\}$ where φ is a set-formula of the language $FL_{\{c\}}$; cf. [S - Ve 1]). The correspondence is described as follows: for $\mu \in V/\overset{\circ}{\underset{\{c\}}{=}}$ and $\mathcal{F}$ being an ultrafilter on $Sd_{\{c\}}$, we have $X \in \mathcal{F} \equiv X \supseteq \mu$ for each $X \in Sd_{\{c\}}$.

We shall define an ordering $\underset{\{c\}}{\preceq}$ on monads (note that it is similar to Rudin-Keesler's ordering on ultrafilters) and investigate its properties.

Perhaps, the most interesting result of this paper is the theorem, analogous to the classical theorem of the set theory, which asserts that two monads have the same strength (in ordering by $\underset{\{c\}}{\dashv}$), iff there exists a one-one mapping between them.

Note that through the whole paper we do not use the axiom of extensional coding (the axiom of choice) and the axiom of cardinalities. When we speak about ordering on V, we bear in mind the natural ordering on the class (see [V], ch. II, § 1).

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§ 1. At first we prove that the following statement holds for each function $F \in Sd_{\{c\}}$: if F has a fixpoint with respect to $\underset{\{c\}}{\overset{\circ}{=}}$, then F is an identity. We also show that the condition cannot be generalized in the sense that if $F, G \in Sd_{\{c\}}$ and $F(x) \underset{\{c\}}{\overset{\circ}{=}} G(x)$, then $F(x) = G(x)$ is valid; see Example 1.

<u>Theorem 1</u>. Let $F \in Sd_{\{c\}}$, F be a function. Then $(\forall x)[F(x) \underset{\{c\}}{\overset{\circ}{=}} x \Longrightarrow (\exists X \in Sd_{\{c\}})(F \restriction X = Id \restriction X \ \& \ \mu_{\{c\}}(x) \subseteq X)]$.

<u>Proof</u>. Let $F(x) \underset{\{c\}}{\overset{\circ}{=}} x$ for $F \in Sd_{\{c\}}$. Let us denote $X = \{t; F(t) = t\}$. Because $X \in Sd_{\{c\}}$ and hence X is a clopen figure, it suffices to prove that $x \in X$ since this implies $\mu_{\{c\}}(x) \subseteq X$.

Suppose $x \notin X$ and put $Y = dom(F) - X$. Obviously $Y \in Sd_{\{c\}}$. Moreover, $x \in Y$ and hence $\mu_{\{c\}}(x) \subseteq Y$.

Let us construct the graph G of F; its chromatic number being less than or equal to 3. Therefore, the field of G is the union $Y_1' \cup Y_2' \cup Y_3'$ where Y_i' $(i = 1,2,3)$ are mutually disjoint (Y_i' contains just the elements of G which are coloured with the same colour). Hence $(F''Y_i') \cap Y_i' = \emptyset$. Firstly, we prove that we can choose Y_i' in such a way that $Y_i' \in Sd_{\{c\}}$; we simply colour the graph G.

Let $\{K_\ell\}_{\ell \in N}$ be the system of components of G. Firstly, let K_ℓ terminated by a cycle. Then we colour the smallest element of the cycle by the colour 1 and going back around it the vertices will be alternately coloured by colours 1 and 2, eventually 3 (when the cycle has an odd number of vertices). If K_ℓ ends by a vertex which does not belong to dom(F), we colour it by the colour 1 and when going backward we alternate colours 1 and 2. If K_ℓ is confinal with N (K_ℓ is now a proper class) we find its least element and colour it by the colour 1. Then, starting from the point to both the opposite sides, we alternate colours 1 and 2. Thus, $Y_i' \in Sd_{\{c\}}$ $(i = 1,2,3)$.

Put $Y_i = Y_i' \cap \mathrm{dom}(F)$ for $i = 1,2,3$. Obviously $Y_i \in Sd_{\{c\}}$. Since $x \in Y$, there exists $j \in \{1,2,3\}$ such that $x \in Y_j$; then $\mu_{\{c\}}(x) \subseteq$ $\subseteq Y_j$. Moreover, for each i there is $F''Y_i \cap Y_i = \emptyset$ and therefore also $F''\mu_{\{c\}}(x) \cap \mu_{\{c\}}(x) = \emptyset$; this is in contradiction to $F(x) \overset{\circ}{=}_{\{c\}} x$.

<u>Remark</u>. It is possible to reformulate Theorem 1 into the following equivalent version:

$$(\forall x,c)\ \mathrm{Def}_{\{x,c\}} \cap \mu_{\{c\}}(x) = \{x\},$$

since the formula $y \in \mathrm{Def}_{\{x,c\}}$ is equivalent to the formula $y =$ $= F(x)$ for a suitably chosen function $F \in Sd_{\{c\}}$.

<u>Example 1</u>. There are functions $F,G \in Sd$ such that $(\exists x)(F(x) \overset{\circ}{=} G(x)\ \&\ F(x) \neq G(x))$.

We shall define functions F, G, as follows: for each $\langle t,u\rangle$ we put $F(\langle t,u\rangle) = t$ and $G(\langle t,u\rangle) = u$. Let v, w be such that $v \neq w$

and $v \overset{\circ}{=} w$. Then it suffices to put $x = \langle v,w \rangle$.

Now we shall take an interest in a question how monads are mapped by set-theoretically definable relations and how functions of $Sd_{\{c\}}$ behave on monads in $\overset{\circ}{=}_{\{c\}}$.

<u>Theorem 2</u>. $(\forall x,y,c)\ \mathrm{dom}(\mu_{\{c\}}(\langle y,x\rangle)) = \mu_{\{c\}}(x)\ \&$
$\&\ \mathrm{rng}(\mu_{\{c\}}(\langle y,x\rangle)) = \mu_{\{c\}}(y)$.

<u>Proof</u>. We prove only the first assertion; the second one can be proved analogously. Firstly, note that if $X \in Sd_{\{c\}}$ and $\langle y,x\rangle \in X$, then $x \in \mathrm{dom}(X)$ and $\mathrm{dom}(X) \in Sd_{\{c\}}$. Let $\{X_n; n \in FN\}$ be a descending sequence of classes from $Sd_{\{c\}}$ such that $\mu_{\{c\}}(x) = \bigcap\{X_n; n \in FN\}$ and let $\{Y_n; n \in FN\}$ be such a descending sequence of classes from $Sd_{\{c\}}$ for which $\mu_{\{c\}}(\langle y,x\rangle) = \bigcap\{Y_n; n \in FN\}$ and $\mathrm{dom}(Y_n) \subseteq X_n$. Then according to [V], ch. II, § 5, we have $\mathrm{dom}(\mu_{\{c\}}(\langle y,x\rangle)) = \mathrm{dom}(\bigcap\{Y_n; n \in FN\}) = \bigcap\{\mathrm{dom}\ Y_n; n \in FN\} = \mu_{\{c\}}(x)$.

<u>Theorem 3</u>. Let $R \in Sd_{\{c\}}$, R be a relation. Then for each x, the class $R''\mu_{\{c\}}(x)$ is a closed figure in $\overset{\circ}{=}_{\{c\}}$.

<u>Proof</u>. The fact that $R''\mu_{\{c\}}(x)$ is a figure follows immediately from the previous theorem when applying it to $\mu_{\{c\}}(\langle y,x\rangle)$ for $\langle y,x\rangle \in R$. It remains to prove that $R''\mu_{\{c\}}(x)$ is a π-class (cf. § 2 ch. III [V]). Since $R''\mu_{\{c\}}(x) = \mathrm{dom}((R \cap (V \times \mu_{\{c\}}(x)))^{-1})$ and since the classes R, V, $\mu_{\{c\}}(x)$ are π-classes, the class $R''\mu_{\{c\}}(x)$ is also a π-class (see § 5 ch. II [V]).

From Theorem 3 it follows immediately:

<u>Theorem 4</u>. Let $R \in Sd_{\{c\}}$, R be a relation. Let $\langle y,x\rangle \in R$ and $\bar{x} \overset{\circ}{=}_{\{c\}} x$. Then there is a set $\bar{y}$ such that $\bar{y} \overset{\circ}{=}_{\{c\}} y$ and

$\langle \bar{y}, \bar{x} \rangle \in R$.

The next theorem asserts that functions of $Sd_{\{c\}}$ are both continuous and open with respect to $\overset{\circ}{=}_{\{c\}}$.

Theorem 5. Let $F \in Sd_{\{c\}}$, F be a function. Let μ be a monad in $\overset{\circ}{=}_{\{c\}}$. Then $F''\mu$ is either empty or $F''\mu$ is a monad in $\overset{\circ}{=}_{\{c\}}$.

Proof. The assertion follows directly from Theorem 4.

Remark. Realize that Theorem 3 results in: The inverse image of a monad in $\overset{\circ}{=}_{\{c\}}$ is a closed figure.

Lemma 1. $(\forall x,y,t,c)\ x \overset{\circ}{=}_{\{c,t\}} y$ iff $\langle t,x \rangle \overset{\circ}{=}_{\{c\}} \langle t,y \rangle$.

Proof. At first, let $x \overset{\circ}{=}_{\{c,t\}} y$. We know that $x \overset{\circ}{=}_{\{c,t\}} y$ iff for each formula φ the condition $\varphi(x,c,t) \equiv \varphi(y,c,t)$ holds. We have to prove that for each formula ψ, it is provable: $\psi(\langle t,x \rangle,c) \equiv \psi(\langle t,y \rangle,c)$. Thus, let ψ be given, then we put $\varphi(x,c,t) \equiv (\exists z)(z = \langle t,x \rangle \,\&\, \psi(z,c))$. Conversely, assume that $\langle t,x \rangle \overset{\circ}{=}_{\{c\}} \langle t,y \rangle$ is valid. Now the formula φ is given and we find a corresponding formula ψ : We take $\psi(z,c) \equiv (\exists t,x)(z = \langle t,x \rangle \,\&\, \varphi(x,c,t))$.

Theorem 6. Let $F \in Sd_{\{c\}}$, F be a function. Let $x_1 \overset{\circ}{=}_{\{c\}} x_2$ and $F(x_1) = F(x_2) = y$. Then $x_1 \overset{\circ}{=}_{\{c,y\}} x_2$.

Proof. Define a function G as follows: $G(t) = \langle F(t),t \rangle$. Then $G \in Sd_{\{c\}}$ and thus G is continuous in $\overset{\circ}{=}_{\{c\}}$. Therefore $\langle y,x_1 \rangle = G(x_1) \overset{\circ}{=}_{\{c\}} G(x_2) = \langle y,x_2 \rangle$. According to Lemma 1 we have $x_1 \overset{\circ}{=}_{\{c,y\}} x_2$.

Remark. It follows immediately from Theorem 6 that for $F \in Sd_{\{c\}}$, F a function, the inverse image of each element y

restricted to a monad in $\overset{\circ}{=}_{\{c\}}$ is a monad in $\overset{\circ}{=}_{\{c,y\}}$.

Our next remarks are concerned with the question whether it is possible to converse Theorem 5; i.e. if for each c, $F \in \in Sd_V$, F a function, the assertion

$$(\forall \mu_{\{c\}})(\exists \bar{\mu}_{\{c\}})(F'' \mu_{\{c\}} = \bar{\mu}_{\{c\}} \Rightarrow F \in Sd_{\{c\}})$$

is valid. We shall show that the answer is negative. Let us reformulate the problem in this way: Let $F \in Sd_{\{d\}}$, F a function, and let, for each $\mu_{\{c\}}$, $\bar{\mu}_{\{c\}}$ such that $F'' \mu_{\{c\}} = \bar{\mu}_{\{c\}}$ exist. What kind of definability holds then between c and d ?

At first, we introduce a new notion.

Definition. The sets c, d are called incomparable iff $c \notin Def_{\{d\}}$ and $d \notin Def_{\{c\}}$.

The following theorem 7 gives the example of such a function which belongs to $Sd_{\{d\}} - Sd_{\{c\}}$ (c, d are incomparable) and transforms monads in $\overset{\circ}{=}_{\{c\}}$ onto monads in $\overset{\circ}{=}_{\{c\}}$.

For proving the theorem we need two lemmas. Remember now that in [S - Ve 1], there is proved that there exists at least one class of indiscernibles which is a proper π-class and which is an intersection of countably many classes from Sd_0. We shall choose one of them and denote it Ind.

Lemma 2. Ind is a monad in $\overset{\circ}{=}$

Proof. Ind is a figure in $\overset{\circ}{=}$ (see [V]). For Ind being a monad in $\overset{\circ}{=}$ it is sufficient to prove that $x \overset{\circ}{=} y$ for each $x, y \in \in$ Ind; in other words, we must prove that for an arbitrary formula $\varphi \in FL_0$, $\varphi(x) \equiv \varphi(y)$ holds. According to the definition of indiscernibles we know that for each ordered n-tuple it is true $\varphi(x_1,\dots,x_n) \equiv \varphi(y_1,\dots,y_n)$ and hence also

$\varphi(x) \equiv \varphi(y)$.

Lemma 3. $(\forall c > \mathrm{Def})\ (\forall d_1, d_2 \in \mathrm{Ind})\ [(d_1, d_2 > \mathrm{Def}_{\{c\}}\ \&$ $\&\ d_1 \overset{\circ}{=}_{\{c\}} d_2 \ \&\ d_1 \neq d_2) \Rightarrow c \notin \mathrm{Def}_{\{\langle d_1, d_2\rangle\}}]$.

Proof. Let $c \in \mathrm{Def}_{\{\langle d_1, d_2\rangle\}}$. Then there is $d \in \mathrm{Ind}$ such that $d < c < d_1 < d_2$ (note that $c > \mathrm{Def}$ and monads tend confinally to Def). Since $c \in \mathrm{Def}_{\{\langle d_1, d_2\rangle\}}$ we have $F(d_1, d_2) = c$ for a suitable function $F \in \mathrm{Sd}_0$. Construct $\mu_{\{c\}}(d_1)$; the monad is a class of indiscernibles, for $\mu_{\{c\}}(d_1) \subseteq \mathrm{Ind}$ holds. But the there is d_3 such that $d_3 < d_1$, $d_3 < d_2$ and $d_3 > \mathrm{Def}_{\{c\}}$. Let $\psi(t, t_1, t_2) \equiv F(t_1, t_2) > t$. Obviously ψ is true for d, d_1, d_2. These elements are, however, indiscernibles and hence it is also true $\psi(d_3, d_1, d_2)$. Thus $c > d_3$, which is in contradiction to $d_3 > \mathrm{Def}_{\{c\}}$.

Theorem 7. There is a set-formula $\varphi \in \mathrm{FL}$ such that for each $c > \mathrm{Def}$ there exists d incomparable with c and φ defines a function $F \in \mathrm{Sd}_{\{d\}} - \mathrm{Sd}_{\{c\}}$ for which the condition

$$(\forall \mu_{\{c\}})(\exists \bar{\mu}_{\{c\}})\ F'' \mu_{\{c\}} = \bar{\mu}_{\{c\}}$$

holds.

Proof. Let $c > \mathrm{Def}$ and let $d_1, d_2 \in \mathrm{Ind}$ be sets satisfying the assumptions of Lemma 3. It is easy to verify that $\langle d_1, d_2\rangle$ and c are incomparable: $c \notin \mathrm{Def}_{\{\langle d_1, d_2\rangle\}}$ follows directly from Lemma 3 and for $d_1, d_2 > \mathrm{Def}_{\{c\}}$ we have $\langle d_1, d_2\rangle \notin \mathrm{Def}_{\{c\}}$. Denote $d = \langle d_1, d_2\rangle$. Furthermore, define a function F by: $F(d_1) = d_2$, $F(d_2) = d_1$ and $F(t) = t$ for each t different from d_1, d_2. Then obviously $F \in \mathrm{Sd}_{\{d\}}$ and F transforms each monad in $\overset{\circ}{=}_{\{c\}}$ onto a monad in $\overset{\circ}{=}_{\{c\}}$.

§ 2. In the beginning of this paragraph we shall investigate the "strength" of monads from the standpoint of definability. Further let μ_1, μ_2 denote monads in $\{\overset{\circ}{=}_c\}$.

<u>Definition</u>. We say that μ_2 is stronger than μ_1 (notation: $\mu_1 \{\overset{\prec}{=}_c\} \mu_2$) iff there is a function $F \in Sd_{\{c\}}$ such that $F'' \mu_2 = \mu_1$. If $\mu_1 \{\overset{\prec}{=}_c\} \mu_2$ and $\mu_2 \{\overset{\prec}{=}_c\} \mu_1$ at the same time, we say that μ_1 and μ_2 have the same strength (notation $\mu_1 \{\overset{\approx}{=}_c\} \mu_2$). We write $\mu_1 \{\overset{\prec}{c}\} \mu_2$ if simultaneously $\mu_1 \{\overset{\prec}{=}_c\} \mu_2$ and $\mu_1 \{\overset{\not\approx}{=}_c\} \mu_2$ and we say that μ_2 is strictly stronger than μ_1.

<u>Remark</u>. Notice that from the results of § 1 it follows:

$$\mu_1 \{\overset{\prec}{=}_c\} \mu_2 \equiv (\exists F \in Sd_{\{c\}})(F'' \mu_2 \subseteq \mu_1) \equiv (\exists F \in Sd_{\{c\}})\ F'' \mu_2 \cap \\ \cap \mu_1 \neq \emptyset.$$

<u>Lemma 4</u>. $(\forall \mu_1, \mu_2)\ [\mu_1 \{\overset{\approx}{=}_c\} \mu_2 \equiv (\exists H \in Sd_{\{c\}})$
$H: \mu_1 \longleftrightarrow \mu_2]$.

<u>Proof</u>. Let $\mu_1 \{\overset{\approx}{=}_c\} \mu_2$. Then there are functions $F, G \in Sd_{\{c\}}$ such that $F(x) \in \mu_1$ for each $x \in \mu_2$ and $G(y) \in \mu_2$ for each $y \in \mu_1$. Construct a composite of F and G. Obviously $F \circ G \in Sd_{\{c\}}$, $\mathrm{dom}(F \circ G) \supseteq \mu_1$ and $(F \circ G)'' \mu_1 = \mu_1$. Thus $y \{\overset{\circ}{=}_c\} (F \circ G)(y)$ for each $y \in \mu_1$. In accordance with Theorem 1 there is a class $X \in Sd_{\{c\}}$ such that $F \circ G$ is the identity function on X and $\mu_{\{c\}}(y) \subseteq X$. Since $\mu_{\{c\}}(y) \subseteq \mu_1$, we have $F \circ G = \mathrm{Id} \restriction \mu_1$ and therefore $G \restriction (F''X) = (F \restriction X)^{-1}$. Hence it suffices to put $H = G \restriction (F''X)$. The converse implication is obvious.

<u>Lemma 5</u>. Let $\mu_1 \{\overset{\approx}{=}_c\} \mu_2$ and let $F, G \in Sd_{\{c\}}$, be such functions for which $F: \mu_1 \longleftrightarrow \mu_2$ and $G: \mu_1 \longleftrightarrow \mu_2$ hold.

Then $F \upharpoonright \mu_1 = G \upharpoonright \mu_1$.

Proof. The assertion is an evident corollary of Theorem 1.

Remark. Note that the assumption $\mu_1 \overset{\approx}{\{c\}} \mu_2$ in the previous lemma is essential. Namely, it follows from Example 1 that:

$(\exists x)(\exists \mu_1, \mu_2)(\exists F,G \in Sd_{\{c\}})\ \mu_1 \doteq \mu_2 \,\&\, F: \mu_1 \longrightarrow \mu_2 \,\&$
$\&\, G: \mu_1 \longrightarrow \mu_2 \,\&\, F(x) \neq G(x)$.

Lemma 6. Let $\mu_1 \overset{\approx}{\{c\}} \mu_2$. Then

$(\forall F)(F$ is a figure in $\overset{\circ}{\{c\}}$ $)\ F \cap (\mu_1 \times \mu_2) \neq \emptyset \Longrightarrow F: \mu_1 \longleftrightarrow \mu_2$.

Proof. Let $x \in F \cap (\mu_1 \times \mu_2)$, then $\langle x, F(x)\rangle \in \mu_1 \times \mu_2$. Denote $\nu = \mu_{\{c\}}(\langle x, F(x)\rangle)$. Since ν is a monad, there is a descending sequence of classes $X_n \in Sd_{\{c\}}$ such that $\nu = \bigcap\{X_n; n \in \in FN\}$.

We prove that there is $k \in FN$ such that X_k is a function. Assume that for each $n \in FN$ there is $x_n \in \mathrm{dom}(X_n)$ such that $X_n'' \{x_n\}$ has at least two elements. We prolong the sequence $\{x_n; n \in FN\}$ by the axiom of prolongation. Let α_i be the greatest element such that for each β, $i \leq \beta \leq \alpha_i$, the class $X_i'' \{x_\beta\}$ has at least two elements. Evidently $\alpha_i \supset FN$ for each $i \in FN$. The sequence $\{\alpha_i\}$ is a descending one. Therefore there exists γ such that for each $i \in FN$ we have $i \in \gamma \in \alpha_i$. Construct $\bigcap\{X_i \upharpoonright \{x_\gamma\}; i \in FN\}$; by a consequence of the axiom of prolongation, the class has at least two elements, too. At the same time, however, $\bigcap\{X_i \upharpoonright \{x_\gamma\}; i \in FN\} \subseteq \nu$ and ν is a function - a contradiction.

Thus let $k \in FN$ be such an element for which X_k is a function. Since $\nu \subseteq X_k$ we obtain that $X_k \in Sd_{\{c\}}$ is a function which is a one-one mapping of μ_1 onto μ_2. It is true now that $F \upharpoonright \mu_1 = I_k \upharpoonright \mu_1 = \nu$ (recall that the domain of a monad is a monad); this completes the proof.

Further we shall formulate several criteria which enable us to verify whether $\mu_1 \prec \mu_2$.

Lemma 7. Let $F \in Sd_{\{c\}}$, F be a function. If $F'' \mu_2 = \mu_1$, then the following are equivalent:

(1) $\mu_1 \prec \mu_2$

(2) $(\forall y)[y \in \mu_1 \Longrightarrow (F^{-1''}\{y\} \cap \mu_2$ has at least two elements$)]$

(3) $(\forall y)[y \in \mu_1 \Longrightarrow (F^{-1''}\{y\} \cap \mu_2$ is infinite$)]$

(4) $(\forall y)[y \in \mu_1 \Longrightarrow (F^{-1''}\{y\} \cap \mu_2$ is a nontrivial monad in $\overset{\circ}{=}_{\{c,y\}})]$.

Proof. For $(1) \Longrightarrow (4)$ see Theorem 6. The implications $(4) \Longrightarrow (3)$ and $(3) \Longrightarrow (2)$ are trivial. For $(2) \Longrightarrow (1)$ realize that $\neg(1)$ says actually that $\mu_1 \underset{\{c\}}{\approx} \mu_2$ and hence, in accordance with Lemma 6, the function F is a one-one function and therefore $\neg(2)$ is valid.

Remark. It is possible to rewrite (equivalently) the statements (2),(3),(4) using only the quantifier $\exists$.

Finally, we prove that for each monad in $\overset{\circ}{=}_{\{c\}}$ there is no proper class $X \in Sd_V$ which is a part of the monad. The assertion is interesting with respect to the prolongation theorem which implies that in each infinite σ-class (and therefore also in each semiset) there exists an infinite set which is a part of it. Thus, if we want to use the direct analogy to the prolongation axiom for classes, we have to turn to the technique of Sd^*_V classes (see [S - V 2]).

Theorem 8. Let $X \in Sd_o$. Then

$(\forall x,y)\,[X''\{x\} \subseteq \mu(y) \,\&\, X''\{x\}$ has at least two elements$] \Rightarrow$

$\Rightarrow \mu(y) \prec \mu(x)$.

Proof. If $z \stackrel{\circ}{=} x$, then obviously $X''\{z\} \subseteq \mu(y)$ and $X''\{z\}$ has at least two elements. Define functions F_1, F_2 as follows: for each t let $F_1(t)$ be the first element of $X''\{t\}$ and let $F_2(t)$ be the second element of the same class. Then $F_1, F_2 \in Sd_o$ and $F_i'' \mu(x) = \mu(y)$ for $i = 1,2$. This implies $\mu(y) \preceq \mu(x)$. If $\mu(y) \approx \mu(x)$ then $F_1 = F_2$, which is a contradiction. Thus $\mu(y) \prec \mu(x)$.

Remark. Theorem 8 and also the following Theorem 9 hold obviously also for the relation $\underset{\{c\}}{\stackrel{\circ}{=}}$.

Theorem 9. Let μ be a monad in $\stackrel{\circ}{=}$. Then

$\neg[(\exists X \in Sd_V)$ (X a proper class $\& X \subseteq \mu$) .

Proof. Let $y \in \mu$ and let there be a proper class $X \in Sd_V$ such that $X \subseteq \mu(y)$. Then there are x, $\bar{X}$ such that $\bar{X}''\{x\} = X$ and $\bar{X} \in Sd_o$. Since $X \subseteq \mu(y)$ we have $\bar{X}''\{x\} \subseteq \mu(y)$ and by Theorem 8 the assertion $\mu(y) \prec \mu(x)$ holds. Define (by induction) a function G by the rule: for each t let G(t) be the smallest element of $X''\{t\} - rng(G \upharpoonright \{z; z < t\})$. Evidently $G \in Sd_o$ and G is a one-one function. Furthermore, $G(x) \in \bar{X}''\{x\} \subseteq \mu(y)$. Thus G is a one-one mapping of x into $\mu(y)$ and hence $x \approx y$; this is in contradiction to $\mu(y) \prec \mu(x)$.

§ 3. In the last part of this paper we shall formulate several interesting statements concerning algebraical properties of the relation $\underset{\{c\}}{\preceq}$.

Theorem 10. There is no maximal monad (in ordering by

$\overset{\rightarrow}{\{c\}}$).

Proof. Let μ be a maximal monad. Let $x,y \in \mu$ and $x \neq y$. We claim that $\mu(\{x,y\})_{\{c\}}^{\leftarrow} \mu$. Obviously $\mu(\{x,y\})_{\{c\}}^{\leftarrow} \mu$. Since $\epsilon''\{\{x,y\}\} = \{x,y\} \subseteq \mu$ we have $\mu(\{x,y\}) \leftarrow \mu$ due to Theorem 8.

Theorem 11. There are uncountably many minimal monads (in $\overset{\rightarrow}{\{c\}}$).

Firstly we prove the following assertion:

Lemma 8. μ is a minimal monad (in $\overset{\rightarrow}{\{c\}}$) iff each function $F \in Sd_{\{c\}}$ is either constant or one-one mapping on μ .

Proof. Suppose μ is not a minimal monad. Then there is a monad ν such that $\nu \overset{\rightarrow}{\{c\}} \mu$ and ν is not trivial. This implies the existence of a function $G \in Sd_{\{c\}}$ for which $\nu = G''\mu$ and G is not one-one mapping. Thus G is a constant function, which is a contradiction (ν is not a singleton).

Conversely, let μ be minimal. Let $F \in Sd_{\{c\}}$, F be a function which is not one-one on μ . We shall prove that then F is a constant function on μ . By Theorem 5 we know that $F''\mu$ is a monad. Moreover, $F''\mu \overset{\rightarrow}{\{c\}} \mu$. According to the definition of minimal monads we have, however, that $F''\mu$ is a singleton and therefore F is constant on μ .

Proof of Theorem 11. We shall prove that for each countable system of monads $\{\mu_i\}$ there is a minimal monad μ which is a proper class and which is disjoint with all μ_i.

Let us enumerate all functions of $Sd_{\{c\}}$; denote them F_i. We shall construct a descending sequence of proper classes

$X_i \in Sd_{\{c\}}$ for which two conditions hold: F_i is either constant or one-one on X_i and $X_i \cap \mu_i = \emptyset$. The classes X_i will be constructed by induction. Let X_i be formed, we produce X_{i+1} in such a way: Divide X_i into two disjoint proper classes $\bar{X}_i$, $X_i - \bar{X}_i$. Then the monad μ_{i+1} is a part of one and only one of them. Further we consider just the class from the couple $\bar{X}_i$, $X_i - \bar{X}_i$ which is disjoint with μ_{i+1} - denote it Y_i.

Now we investigate $F_{i+1} \upharpoonright Y_i$. If $F_{i+1} \upharpoonright Y_i$ is a set, then $F_{i+1} \upharpoonright Y_i \in Sd_{\{c\}}$. Denote $u = Y_i \cap \mathrm{dom}(F_{i+1})$. In this case, we put $X_{i+1} = Y_i - u$. Let further $F_{i+1} \upharpoonright Y_i$ be a proper class. Then either F_{i+1}" Y_i is a set or F_{i+1}" $Y_i \notin V$. In the first situation we have F_{i+1}" $Y_i \in Sd_{\{c\}}$. Let t be the smallest element of F_{i+1}" Y_i such that (F^{-1}_{i+1})"$\{t\} \notin V$; such a t exists since Y_i is a proper class. We shall put now $X_{i+1} = (F^{-1}_{i+1})$"$\{t\}$. If F_{i+1}" Y_i is a proper class, then F^{-1}_{i+1} generates a decomposition of Y_i according to the equivalence $x \approx y \equiv F(x) = F(y)$; denote $\{Z_t\}_{t \in F"Y_i}$ the system of classes of the decomposition. In this second case we shall put $X_{i+1} = \{z;$ z is the smallest element of $Z_t \,\&\, t \in F"Y_i\}$.

Let us construct $\bigcap X_i$. The intersection is a proper σ-class and therefore a figure in $\{\frac{\circ}{c}\}$. We claim that for each $F \in Sd_{\{c\}}$ the function F is either constant or one-one on $\bigcap X_i$ and that $\bigcap X_i \cap \mu_i = \emptyset$. The assertion $\bigcap X_i \cap \mu_i = \emptyset$ is trivial since for each μ_j we have $X_j \cap \mu_j = \emptyset$. Let further be $F_i \in Sd_{\{c\}}$; then - according to our construction - the function F_i is either constant or one-one on X_i. The same is therefore true also for $\bigcap X_i$.

Because $\bigcap X_i$ is a proper class, there is a monad in $\bigcap X_i$

which is a proper class, too. Thus we have constructed (see Lemma 8) at least one proper minimal monad.

It is easy to verify that we can produce an uncountable amount of such monads. If there is only a countable number of minimal monads then we create - in accordance with the above mentioned procedure - a next minimal monad which is different from all preceding ones. This completes the proof.

Remark. It follows from the results of J.B. Paris concerning non-standard models of PA that there is a monad which has no minimal monad "under itself".
Note further that using the familiar construction of the transfinite induction one can prove (by means of the axiom of choice and the axiom of cardinalities) that there is a chain $\mathcal{L}$ of monads in $\overset{\circ}{=}_{\{c\}}$, with ordering $\preceq_{\{c\}}$ of type Ω_1 such that each monad in $\overset{\circ}{=}_{\{c\}}$ "lies under" a monad of the chain $\mathcal{L}$.

Theorem 12. $(\exists \mu_1, \mu_2)(\mu_1 \not\preceq_{\{c\}} \mu_2 \ \& \ \mu_2 \not\preceq_{\{c\}} \mu_1)$.

Proof. We know from the previous theorem that there is an uncountable amount of minimal monads. We prove now that there are among them two monads which are not comparable with respect to ordering $\preceq_{\{c\}}$. Thus, if μ, ν are minimal monads and either $\mu \preceq_{\{c\}} \nu$ or $\nu \preceq_{\{c\}} \mu$ holds, then we have $\mu \cong_{\{c\}} \nu$. But there is only a countable number of monads like these, since there is just a countable amount of functions from $Sd_{\{c\}}$ which are one-one functions.

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