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ON THE MONOIDS OF HOMOMORPHISMS OF SEMIGROUPS WITH UNITY

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Abstract: It is proved that
- any semigroup with unity and zero element is isomorphic to a semigroup of endomorphisms of some monoid (i.e. semigroup with unity),
- any small category with zero morphisms is isomorphic to a small full subcategory of the category of monoids and their homomorphisms,
- any concrete category with zero morphisms is isomorphic to a full subcategory of the category of monoids and their homomorphisms, provided the non-existence of measurable cardinals is supposed.

Key words: Category theory, full embedding, homomorphisms of monoids, zero morphisms.

Classification: 18B15

The aim of the present paper is to characterize monoids which can be represented as the monoids of homomorphism of semigroups with unity.

Let M be a monoid of homomorphism of a semigroup S with the unity element 1. M necessarily contains the unity and zero elements corresponding to the identity mapping of S and to the constant mapping to the element 1 of S. We are going to show that there is no other restriction to monoids in question. More generally, we prove that every concrete category K with 0-morphisms is isomorphic to a full subcategory of the category of monoids (semigroups with unity) and their homomorphisms, provided (M) there exists a cardinal number α such that every α-additive two-valued measure is trivial.

In some cases (e.g. if K has a set of objects only or K is a category of universal algebras of a given type and their homomorphisms) the axiom (M) is not necessary, on the other hand the existence of a full embedding (i.e. a full and faithful functor) of e.g. the category of compact abelian groups into the category of monoids would imply (M) [7].

The proof is based on the fact that every concrete category K can be fully embedded into the category of oriented graphs and compatible mappings [1, 6] (see also [8]). Some special cases of this theorem are proved in [3, 4, 5]. Using this result we shall prove that a concrete category with O-morphism can be fully embedded into a special subcategory of the category of oriented graphs with one loop. (O-morphisms will correspond to constant mapping to the loops.)

The category of one-loop graphs will be fully embedded into the category of monoids by a modification of the method used in the paper [2].

0. Preliminary definitions. An oriented graph is a couple $G = (X,R)$, where X is a set and $R \subset X \times X$. X (R, resp.) is called the underlying set (the relation, resp.) of G. A loop of G is an element $x \in X$ such that $(x,x) \in R$. A mapping $f: X \longrightarrow Y$ is a compatible mapping from (X,R) into (Y,S) if $(x,y) \in R$ implies $(f(x),f(y)) \in S$. Note that a constant mapping to a loop is compatible.

GRA is a category of all oriented graphs and their compatible mappings.

GOL is a full subcategory of GRA determined by graphs $G = (X,R)$ such that G has exactly one loop x_0,

$(x_0,x),(x,x_0) \in R$ implies $x = x_0$,

if $x \neq x_0$ and either $(x,x_0) \in R$ or $(x_0,x) \in R$, then it is $(x,y) \in R$ iff $(y,x) \in R$.

GOL(I), where I is a set, is a category defined as follows:

objects are triples $(X,(R_i)_{i\in I},x_0)$, where X is a set, $R_i \subset X \times X$ for all $i \in I$, $x_0 \in X$, such that for every $i \in I$ it is $(x,x) \in R_i$ iff $x = x_0$,

morphisms from $(X,(R_i),x_0)$ into $(Y,(S_i),y_0)$ are mappings $f:X \longrightarrow Y$ such that for every $i \in I$, $(x,y) \in R_i$ implies $(f(x), f(y)) \in S_i$. (Note that in this case it is $f(x_0) = y_0$.)

A set $\prod_{i \in I} X_i$ is considered as the set of all mappings q from I into $\bigcup_{i \in I} X_i$ such that $q(i) \in X_i$.

MON is the category of monoids (semigroups with unity) and their homomorphisms. We shall say that a category K has O-morphisms if for any two objects A, B of K there is a morphism $Z_{A,B}:A \longrightarrow B$ such that for every morphism $f:A \longrightarrow B$, $g: :B \longrightarrow C$ it is $Z_{B,C} \circ f = g \circ Z_{A,B} = Z_{A,C}$.

1. Embedding into GOL

Theorem 1: If a category K has O-morphisms and if it can be fully embedded into GRA then there exists a full embedding of K into GOL (I) for some nonempty set I.

Proof: Without loss of generality we can suppose that K is a full category of GRA and that there exists an object O of K such that $Z_{o,o}$ is the identity morphism of O. The object O is uniquely determined as an image of any O-morphism in K.

Denote an underlying set of an object (i.e. a graph) G of K by X_G and its relation by R_G.
Given $x \in X_o$, denote $Z_{G,o}^{-1}(x) = X_{G,x}, Z_{o,G}(x) = a_{G,x}$. We have $X_{G,x} \cap X_{G,y} = \emptyset$ for $x \neq y$, $a_{G,x} \in X_{G,x}$, $a_{o,x} = x$, $X_{o,x} = \{x\}$, $X_G = \bigcup_{x \in X_o} X_{G,x}$. If $f:G \longrightarrow H$ is a morphism of K then f maps $X_{G,x}$ into $X_{H,x}$ and $f(a_{G,x}) = a_{H,x}$.

A full embedding F of K into GOL $(X_o \cup R_o)$ can be defined as follows:

$F(G) = (\prod_{x \in X_o} X_{G,x}, (R_G), Z_{o,G})$, where relations $R_{G,i}$ are defined in the following way:

$(q_1, q_2) \in R_{G,i}$ for $i = x \in X_o$, $q_1(x) = q_2(x)$,

$(q_1, q_2) \in R_G$ for $i = (x,y) \in R_o$, $(q_1(x), q_2(x)) \in R_G$,

$F(f)(q) = f \circ q$.

If $(x,y) \in R_o$ then $(Z_{o,G}(x), Z_{o,G}(y)) \in R_G$ which implies $(Z_{o,G}, Z_{o,G}) \in R_{G,(x,y)}$. Conversely, if $(q,q) \in R_{G,(x,y)}$ for every $(x,y) \in R_o$ then $q:X_o \longrightarrow X_G$ is a mapping such that $(x,y) \in R_o$ implies $(q(x), q(y)) \in R_G$. Hence $q:O \longrightarrow G$ is a morphism of K and $q = q \circ 1_o = q \circ Z_{o,o} = Z_{o,G}$.

Now, it is easy to see that F is a faithful factor. We shall prove that F is full:

Let $h:F(G) \longrightarrow F(H)$ be a compatible mapping of $GOL(X_o \cup R_o)$ and $a \in X_G$. There exists a unique $x \in X_o$ such that $a \in X_{G,x}$ and there is $q \in \prod_{x \in X_o} X_{G,x}$ such that $q(x) = a$. Put $f(a) = (h(q))(x)$. This does not depend on the choice of q, because $q_1 \in \prod_{x \in X_o} X_{G,x}$, $q_1(q) = a$ implies $(q, q_1) \in R_{G,x}$, $(h(q), h(q_1)) \in R_{H,x}$, $(h(q))(x) = (h(q_1))(x)$. We have obtained a mapping $f:X_G \longrightarrow X_H$ such that

$h(q) = f \circ q$. Let us suppose that $(a,b) \in R_G$. There exist $x,y \in X_0$ such that $a \in X_{G,x}, b \in X_{G,y}$ and $q_k, q_2 \in \prod_{x \in X_0} X_{G,x}$ such that $q_1(x) = a, q_2(y) = b$. We have $(x,y) = (Z_{G,o}(a), Z_{G,o}(b)) \in R_0$, $(q_1,q_2) \in R_{G,(x,y)}$, $(h(q_1), h(q_2)) \in R_{H,(x,y)}$, $(f(a),f(b)) = (fq_1(x), fq_2(y)) = (h(q_1)(x), (h(q_1)(y)) \in R_G$. Thus, $f: G \to H$ is a morphism of K and $F(f) = h$.

<u>Theorem 2</u>. If K is a category with O-morphisms and if there exists a full embedding of K into GRA then there exists a full embedding of K into GOL.

Proof: In view of Theorem 1 it suffices to construct a full embedding GOL (I) $\longrightarrow$ GOL for every set I. For the sake of simplicity we shall divide the construction into two parts:

1. A full embedding GOL (I) $\longrightarrow$ GOL (3)

According to [9], there exists an oriented graph $T = (I,U)$ which has the parameter set I as an underlying set such that the only compatible mapping of I into itself is the identity mapping.

Define F as follows:

$F((X,(R_i)_{i \in I}, x_0) = (((X - \{x_0\}) \times I) \cup \{x_0\}, (r_i)_{i=0,1,2}, x_0)$,

where $(a,b) \in r_i$ iff

either $i = 0$, $a = (x,p)$, $b = (x,y)$,

or $i = 1$, $a = (x,p)$, $b = (x,q), (p,q) \in U$,

or $i = 2$, $a = (x,p)$, $b = (y,p), (x,y) \in R_p$,

or $i = 2$, $a = (x,p)$, $b = x_0, (x,y_0) \in R_p$,

or $i = 2$, $a = x_0$, $b = (y,p), (x_0,x) \in R_p$,

or $i = 0,1,2$, $a = b = x_0$,

for some $x,y \in X - \{x_0\}$, $p,q \in I$,

$$F(f)((x,i)) = \begin{cases} (f(x),i) & \text{if } f(x) \text{ is not a loop,} \\ f(x) & \text{if } f(x) \text{ is a loop,} \end{cases}$$

$F(f)(x_o) = f(x_o)$.

It is easy to see that F is a faithful functor. Let $h: F((X,(R_i),x_o)) \longrightarrow F((Y,(S_i),y_o))$ be a compatible mapping. We have $h(x_o) = y_o$. r_o is an equivalence with the equivalence classes $\{x\} \in I$, $x \in X$, $x \neq x_o$ and $\{x_o\}$; similarly for s_o. The mapping h preserves these partitions. According to the definition of r_1, s_1 and the properties of $T = (I,U)$, there exists a mapping $f: X \longrightarrow Y$ such that

$$h((x,i)) = \begin{cases} (f(x),i) & \text{if } f(x) \neq y_o, \\ y_o & \text{if } f(x) = y_o, \end{cases}$$

$h(x_o) = y_o$.

In view of the definition of r_2, s_2 and the properties of the mapping f we know that $(x,y) \in R_p$ implies $(f(x),f(y)) \in S_p$.

Therefore $f: (X,(R_i),x_o) \longrightarrow (Y(S_i),y_o)$ is a morphism of GOL (I) such that $F(f) = h$.

A full embedding GOL (3) $\longrightarrow$ GOL

$F((X,(v_i)\ i = o,1,2,x_o)) = (((X - \{x_o\}) \times \{1,2,3,4\} \times \{1,2,3,4\}) \cup \cup \{x_o\}, R)$, where $(a,b) \in R$ if there exist $x,y \in X - \{x_o\}$ such that either $a = (x,i,p), b = (x,j,p), p = 1,3$ and

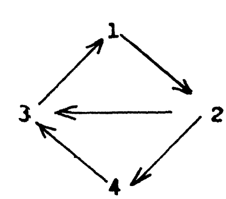
either $i = 1,\ j = 2$,

or $i = 2,\ j = 3$,

or $i = 3,\ j = 1$,

or $i = 2,\ j = 4$,

or $i = 4,\ j = 3$,



or $a = (x,i,p), b = (x,i,q)$, $i = 1,2,3,4$ and

either $p = 1,3$, $q = p + 1$,

or $\quad p = 2,4$, $q = p - 1$,

or $a = (x,1,1)$, $b = (x,1,3)$,

or $a = (x,i + 2,2)$, $b = (y,i + 3,4)$, $(x,y) \in r_i \quad i = 0,1,2$,

or $a = (x,i + 2,2)$, $b = x_0$, $\qquad (x,x_0) \in r_i \quad i = 0,1,2$,

or $a = x_0$, $\qquad b = (y,i + 2,4) \ (x_0,y) \in r_i \quad i = 0,1,2$,

or $a = b = x_0$.

$$F(f)((x,i,p)) = \begin{cases} (f(x),i,p) & \text{if } f(x) \text{ is not a loop,} \\ f(x) & \text{if } f(x) \text{ is a loop,} \end{cases}$$

$F(f)(x_0) = f(x_0)$.

It is easy to see that F is a faithful functor from GOL (3) into GOL. We shall prove that F is full:

Let $h:F((X,(r_i),x_0)) \longrightarrow F((Y,(S_i),q_0))$ be a compatible mapping.

Given $x \in X$, $p = 1,3$, the points $(x,1,p)$, $(x,2,p)$, $(x,3,p)$ form a cycle and therefore there is $y \in Y$, $q = 1,3$, $u = 0,1,2$ such that either

$h((x,i,p)) = y_0$ for $i = 1,2,3$,

$$\text{or } h((x,i,p)) = \begin{cases} (y,i + u,q) & \text{if } i + u \leq 3, \\ (y,i + u - 3,x) & \text{if } i + u > 3 \text{ for } i \quad 1,2,3. \end{cases}$$

Considering the arrows $((x,2,p),(x,4,p))$ and $((x,4,p),(x,3,p))$, we can show that

$h((x,i,p)) = y_0$ for $i = 1,2,3,4$ in the first case,

$h((x,i,p)) = (y,i,q)$ for $i = 1,2,3,4$ in the second case.

In view of the existence of an arrow $((x,1,1),(x,1,3))$ there is $y \in Y$ such that $h((x,i,p)) = (y,i,p)$ for $i = 1,2,3,4$, $p = 1,3$. Since we have $((x,i,p),(x,i,p + 1))$, $((x,i,p + 1)$,

$(x,i,p)) \in R$ for $i = 1,2,3,4$, $p = 1,3$, necessarily $h((x,i,q)) =$ $= (y,i,q)$, for $i = 1,2,3,4$, $q = 2,4$.

Therefore there is a mapping $f: X \longrightarrow Y$ such that

$$h((x,i,p)) = \begin{cases} (f(x),i,p) & \text{if } f(x) \neq y_0, \\ y_0 & \text{if } f(x) = y_0, \end{cases}$$

$h(x_0) = y_0$.

Now, it can be easily seen that f is a compatible mapping from $(X,(r_i),x_0)$ into $(Y,(s_i),y_0)$ and that $h = F(f)$.

2. Embedding into MON. The next three theorems constitute the main results of the paper:

Theorem 3. Assuming (M), a category K is isomorphic to a full subcategory of the category of monoids and their homomorphisms if and only if it is a concrete category with O-morphisms.

Theorem 4. If K is either a small category or a category of universal algebras of a given type and their homomorphisms then K is isomorphic to a full subcategory of the category of monoids and their homomorphisms if and only if K has O-morphisms.

Theorem 5. Every multiplicative semigroup with the unity and zero elements is isomorphic to a semigroup of endomorphisms of some monoid.

Proof of Theorems 3 - 5. The theorem 5 is an immediate consequence of the theorem 4. The "only if" part of the theorems 3, 4 follows from the fact that any full subcategory of MON is a concrete category with O-morphisms.

Now, we are going to prove the "if" part of Theorems 3, 4. It follows from the assumption of the theorems and from [3, 4, 6] (see also [8]) that K can be fully embedded into GRA.

Since K has 0-morphisms, the theorem 2 gives a full embedding $K \longrightarrow GOL$. Therefore it is sufficient to construct a full embedding of GOL into MON. It can be defined as follows.

Given a graph $G = (X,R)$, which is an object of GOL, let $M'(G)$ be a free monoid over $X' = X - \{x_0\}$, where x_0 is the loop of G, i.e. $M'(G)$ be a set of all finite (possibly empty) sequences of elements of X', the composition in $M'(G)$ is given by concatenation and the unity is the empty sequence.

Let $\equiv$ be the smallest congruence on $M'(G)$ such that

(1a) $x\, z^2\, y\, x^2\, z \equiv x\, z^2\, y^2\, x^2\, z$ whenever $x,y,z \in X'$ and $(x,y),(y,z) \in R$ (note that it is $x \neq y$ and $z \neq x$),

(1b) $x\, y\, x^2 \equiv x\, y^2\, x^2$, whenever $x,y \in X'$ and (x,y), $(y,x_0) \in R$ (note that $x \neq y$),

(1c) $z^2\, y\, z \equiv z^2\, y^2\, z$ whenever $y,z \in X'$, and (x_0,y), $(y,z) \in R$ (note that $y \neq z$).

Put $F(G) = M'(G)/\equiv$.

A) It is evident that $x^p \not\equiv x^q$ for $x \in X'$, $p \neq q$ (especially $x \not\equiv 1$) and that $x,y \in X'$, $x \equiv y$ implies $x = y$.

B) Let $a = x_1 \ldots . x_k$ be a word over X'. Define $C(a)$ to be the number of indices $i = 1,2,\ldots,k-1$ such that $x_i \neq x_{i+1}$. It is easy to see that $a \equiv b$ implies $C(a) = C(b)$. Moreover, $C(a\, b\, c) \leqq C(a\, b^2\, c)$ and the equality holds iff $b = x^k$, $x \in X'$, with a nonnegative integer k. Especially, if $a\, c^2\, b\, a^2\, c \equiv$ $\equiv a\, c^2 b^2 a^2 c$ then $b = x^k$, $x \in X'$, $k \geqq 0$.

C) Let $u,v,w \in X'$, p,q,r be natural numbers and one of the following equalities hold:

(2a) $u^p w^{2r} v^q u^{2p} w^r \equiv u^p w^{2r} v^{2q} u^{2p} w^r$,

(2b) $u^p v^q u^{2p} \equiv u^p v^{2q} u^{2p}$,

(2c) $w^{2r} v^q w^r \equiv w^{2r} v^{2q} w^r$.

We have to transform the right side of (2) by subsequent applications of the equations (1 a,b,c) into the left side of (2). During the application of (1) which changes the exponent of v for the first time necessarily $v = y$, $2q \leqq 2$, which implies $q = 1$.

D) Suppose that $u,v,w \in X'$ and one of the following equalities holds:

(3a) $uw^2v\ u^2w \equiv u\ w^2v^2u^2w$,

(3b) $uvu^2 \equiv uv^2u^2$,

(3c) $w^2v\ w \equiv w^2v^2w$.

We have to transform the left hand side of (3) into its right hand side by means of the equations (1 a,b,c). (1 b) is the only equation which can be applied to (3 b). Thus, $u = x$, $v = y$ and hence $(u,v),(v,x_0) \in R$. Similarly in the case (3 c) we have $(x_0,v)(v,w) \in R$. If (1 q) is applied to (3 a) then $u\ \ x$, $v = y$, $w = z$ and $(u,v),(v,w) \in R$.

If (1 b) is applied to the left hand side of (3 a), then

either $u = v + w$, $u\ w^2v\ u^2w = u\ w^2u^3w$, which could be equivalent to $u\ w\ u^3w$ if $(u,w),(w,x_0) \in R$, but no other word is equivalent to $u\ w^2v\ u^2w$ which is a contradiction, or $u = w = x$, $v = y$, $(u,v),(v,x_0) \in R$ and according to the properties of G we have $(v,w) = (v,u) \in R$.

Analogously, if (1 c) is applied to (3 a) then $u = w = z$,

$v = y$, $(x_0, v), (v,w) \in R$ which implies $(u,v) = (w,v) \in R$.

We have proved that

(3 a) implies $(u,v),(v,w) \in R$,

(3 b) implies $(u,v), (v,x_0) \in R$,

(3 c) implies $(x\ \ ,v), (v,w) \in R$.

F can be defined equivalently as a factorization of a free monoid M(G) over X by the smallest equivalence $\sim$ defined by

(4a) $x\ z^2y^2z \sim xz^2y^2x^2z$ whenever $x,y,z \in X$, $(x,y)\ (y,z) \in$ $\in R$,

(4b) $x_0 \sim 1$.

We can reformulate the above results as follows:

A') given $x,y \in X$, $x \sim y$ implies $x = y$,

B') given words a,b,c over X, $a\ c^2b\ a^2c \sim ac^2b^2a^2c$ implies that there exists $x \in X$ and a natural number p such that $b = x^p$,

C') given $u,w \in X$, $v \in X'$, p,q,r natural, $u^pw^{2r}v^qu^{2p}w^r \sim$ $\sim u^pw^{2r}v^{2q}u^{2p}w^r$, then $q = 1$,

D') given $u,w \in X$, $v \in X'$, $u\ w^2v\ u^2w \sim u\ w^2v^2u^2w$, then $(u,v),(v,w) \in R$.

A compatible mapping $f:G \longrightarrow H$ can be uniquely extended to a homomorphism from M(G) into M(H). The extended homomorphism preserves congruence and therefore gives rise to a homomorphism $F(f):F(G) \longrightarrow F(H)$. It is easy to see that F is a functor from GPL into MON. F is faithful in view of A'.

To prove that F is full, let us consider a homomorphism $h:F(G) \longrightarrow F(H)$.

Given $y \in X$, there are $x,z \in X$ such that $(x,y),(y,z) \in R$, which implies $h(x)(h(z))^2\ h(y)(h(x))^2\ h(z) \sim h(x)(h(z))^2\ (h(y))^2$

$(h(x))^2\ h(z)$. In view of B', there exists $v \in X$ and a natural number q such that $h(y) = v^2$. Similarly, we can show that there exists $u, v \in X$ and natural numbers p, r such that $b(x) = u^p$, $h(y) = v^r$. Thus it is either $v = y_0$ and $h(y) = y_0^q \sim y_0$, or $v \neq y_0$ and $q = 1$.

Therefore there exists a mapping $f: X \longrightarrow Y$ such that $h(x) \sim f(x)$ for $x \in X$.

Given $(a,b) \in R$, then either $f(a) = f(b) = y_0$ and $(f(a), f(b)) \in S$, or there are $u, v, w \in X$ such that $(u,v), (v,w) \in R$ and either $u = a$, $v = b$, $f(b) \neq y_0$ or $v = a$, $w = b$, $f(a) \neq y_0$. Because $u\, w^2 v\, u^2 w \sim u\, w^2 v^2 u^2 w$, we have

$f(u)(f(w))^2\ f(v)(f(u))^2\ f(w) \sim f(u)(f(w))^2 (f(v))^2 (f(u))^2 f(w)$

and it follows from D' that $(f(a), f(b)) \in S$. Thus, $f: G \longrightarrow H$ is a morphism and $h = F(f)$.

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