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SOLVABILITY OF NONLINEAR PROBLEMS AT RESONANCE

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Abstract: This paper deals with the solvability of nonlinear operator equations with finite-dimensional kernel of the linear part and with nonlinearity given by odd real function g with $\int_0^{\infty} g(z)dz \in \mathbb{R} \cup \{\pm \infty\}$ and with no restrictions on

$$\lim_{t \to \infty, \tau \in (a,t\rangle} t \min g(\tau).$$

Key words: Noncoercive problems at resonance, weakly nonlinear boundary value problems, vanishing nonlinearities.

Classification: 47H15, 35J40

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1. Assumptions. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $H = L^2(\Omega)$ be the real Hilbert space with usual inner product $\langle \cdot , \cdot \rangle$ and with the norm $\|u\| = \langle u,u \rangle^{1/2}$. Suppose that

$$L: D(L) \subset H \longrightarrow H$$

is a symmetric linear operator with dense domain D(L), with nontrivial finitedimensional nullspace N(L) and closed range R(L). Let

$$H = N(L) \oplus R(L)$$

and suppose that

$$K = (L|R(L))^{-1}: R(L) \longrightarrow R(L)$$

(so called the right inverse of L) is completely continuous.

We assume that N(L) has "unique continuation property" in the sense that the only function $w \in N(L)$ vanishing on a set of positive measure in Ω is $w \equiv 0$.

Let G be the Nemytskii operator associated with continuously differentiable odd bounded function $g: \mathbb{R} \longrightarrow \mathbb{R}$, $g \neq 0$,

$$G: u \longmapsto g \circ u.$$

Obviously G maps H into H and has bounded range.

Let us suppose that

(1) $\qquad c = \|K\| \sup_{z \in \mathbb{R}} |g'(z)| < 1,$

(2) there exists $\int_0^{+\infty} g(z)dz$.

Let us denote $I = \int_0^{+\infty} g(z)dz$ (we admit $I = \pm\infty$).

In distinction from papers [1] and [2] we assume nothing about the limit

$$\lim_{t \to +\infty} t \min_{\tau \in \langle a,t \rangle} g(\tau).$$

This paper also generalizes in some sense the results from [3], [4] and [6] because we may have $\dim N(L) > 1$.

2. Theorem. _Let $f \in R(L)$. Then the operator equation_

(3) $$Lu + G(u) = f$$

has at least one solution.

3. Proof of the theorem. We use the global Lyapunov-Schmidt method. For this purpose we denote P and Q the orthogonal projections from H onto N(L) and R(L), respectively. It is easy to see that the solvability of (3) is equivalent to the solvability of the bifurcation system

(3a) $$v + KQG(w + v) - Kf = 0,$$

(3b) $$PG(w + v) = 0,$$

$w \in N(L), v \in R(L), w = Pu, v = Qu.$

<u>Step 1</u>. <u>For each</u> $w \in N(L)$ <u>there exists exactly one</u> $v(w) \in$ $\in R(L)$ <u>such that</u>

(3a) $$v(w) + KQG(w + v(w)) - Kf = 0.$$

Define $F(w,.) : R(L) \longrightarrow R(L),$

$$F(w,.) : v \longmapsto Kf - KQG(w + v),$$

for each $w \in N(L)$. Then using Hölder inequality we obtain that

$$\|F(w,v_1) - F(w,v_2)\| \le \|K\| \, \|Q\| \sup_{\|u\|=1} \left| \int_\Omega [g(w + v_1) - g(w + v_2)] \, u \right| \le \|K\| \sup_{\|u\|=1} \int_\Omega |g(w + v_1) - g(w + v_2)| \, |u| \le$$
$$\le \|K\| \sup_{z \in \mathbb{R}} |g'(z)| \; \|v_1 - v_2\| = c \, \|v_1 - v_2\|$$

holds for each $w \in N(L), v_1, v_2 \in R(L)$. The Banach contraction theorem implies that for each $w \in N(L)$ there exists exactly one $v(w) \in R(L)$ that

$$v(w) = F(w, v(w)).$$

<u>Step 2</u>. <u>There exists</u> $r > 0$ <u>such that for each</u> $w \in N(L)$ <u>it is</u>

(4) $$\|v(w)\| \le r.$$

The proof follows immediately from the boundedness of G.

<u>Step 3</u>. <u>It is</u>

(5) $$\lim_{\ell \to +\infty} \operatorname{meas} \{x \in \Omega \, ; |v(w)(x)| \ge \ell\} = 0,$$

<u>uniformly with respect to</u> $w \in N(L)$.

The equality (5) follows from (4).

<u>Step 4</u>. <u>For each</u> $k \in \mathbb{N}$ <u>we have</u>

$$\lim_{\substack{\|w\| \to \infty \\ w \in N(L)}} \operatorname{meas} \{x \in \Omega \, ; |w(x)| \le k\} = 0.$$

Suppose on the contrary that there exists $k_0 \in \mathbb{N}$, $w_n \in N(L)$, $\|w_n\| \longrightarrow +\infty$ such that

$$\text{meas}\{x \in \Omega ; |w_n(x)| \leq k_0\} \geq \varepsilon_0 > 0.$$

Put $\hat{w}_n = w_n / \|w_n\|$. Then we have

$$\text{meas}\{x \in \Omega ; |\hat{w}_n(x)| \leq k_0/\|w_n\|\} \geq \varepsilon_0. \tag{6}$$

Since $\dim N(L) < +\infty$ we can suppose that $\hat{w}_n \longrightarrow w_0$ in $L^2(\Omega)$, i.e. by Jegorov's theorem for each $\eta > 0$ there exists $\Omega' \subset \Omega$, meas $\Omega' < \eta$ and $\hat{w}_n \rightrightarrows w_0$ (uniformly) on $\Omega \setminus \Omega'$. If we put $\eta = \varepsilon_0/2$ and take the limit for $n \longrightarrow +\infty$ in (6), we obtain

$$\text{meas}\{x \in \Omega ; |w_0(x)| = 0\} \geq \varepsilon_0/2 > 0,$$

which is a contradiction with $w_0 \in N(L)$ and the unique continuation property of $N(L)$.

<u>Step 5.</u> If $I \in \mathbb{R}$ then it is

$$\lim_{\|w\| \to +\infty} v(w) = Kf \quad \text{and} \quad \lim_{\|w\| \to +\infty} Lv(w) = g.$$

Using Hölder inequality we obtain

$$\|v(w) - Kf\|^2 \leq \|K\|^2 \left(\sup_{\|u\| \leq 1} \int_\Omega |g(w + v(w))u|\right)^2 \leq$$
$$\leq \|K\|^2 \left(\int_\Omega |g(w + v(w)|^2\right);$$

analogously $\|Lv(w) - f\|^2 \leq \left(\int_\Omega |g(w + v(w))|^2\right)$.

Choose $\varepsilon > 0$. Then there exists $k > 0$ such that

$$\left(\sup_{|z| \geq k} |g(z)|^2 \text{ meas } \Omega\right) < \varepsilon/2. \tag{7}$$

According to <u>Steps 3</u> and <u>4</u> we obtain the existence of such $\varkappa > 0$ that for $\|w\| \geq \varkappa$ it is

$$\text{meas } \Omega_k = \text{meas}\{x \in \Omega ; |w(x) + v(w)(x)| \leq k\} < \tag{8}$$
$$< \varepsilon / (2 \sup_{z \in \mathbb{R}} |g(z)|^2).$$

Using (7) and (8) we obtain

$$\|v(w) - Kf\|^2 \le \|K\|^2 \{ (\int_{\Omega_k} |g(w + v(w))|^2) +$$
$$+ (\int_{\Omega \setminus \Omega_k} |g(w + v(w))|^2) \} \le \|K\|^2 \{ (\sup_{z \in \mathbb{R}} |g(z)|^2 \text{ meas } \Omega_k) +$$
$$+ (\sup_{|z| \ge k} |g(z)|^2 \text{ meas } \Omega) \} < \|K\|^2 \varepsilon \quad ;$$

analogously we obtain $\|Lv(w) - f\|^2 < \varepsilon$.

<u>Step 6</u>. <u>Put</u>

$$\varphi(w) = 1/2 \langle Lv(w), v(w) \rangle + \int_\Omega dx \int_0^{w+v(w)} g(z)dz - \int_\Omega fv(w).$$

<u>Then</u>

$\lim_{\|w\| \to \infty} \varphi(w) = I\text{meas } \Omega - 1/2 \langle f, Kf \rangle$, in the case $I \in \mathbb{R}$ and $\lim_{\|w\| \to \infty} \varphi(w) = \pm \infty$, if $I = \pm \infty$.

We shall prove the assertion for $I \in \mathbb{R}$ and $I = +\infty$ (the case $I = -\infty$ is analogous). Let $I \in \mathbb{R}$. According to Step 5 it is $\lim_{\|w\| \to \infty} [1/2 \langle Lv(w), v(w) \rangle - \int_\Omega fv(w)] = - 1/2 \langle f, Kf \rangle$.

Choose $\varepsilon > 0$. There exists $k > 0$ such that

(9) $$| \int_0^{\pm k} g(z)dz - I | < \varepsilon .$$

Let $\varkappa > 0$ be such that (see Steps 3, 4)

(10) $$\text{meas } \Omega_k < \varepsilon ,$$

for all $w \in N(L)$, $\|w\| \ge \varkappa$. Then for $\|w\| \ge \varkappa$ we obtain using (9) and (10)

$$| \int_\Omega dx \int_0^{w+v(w)} g(z)dz - I\text{meas } \Omega | \le | \int_{\Omega \setminus \Omega_k} dx \int_0^{w+v(w)} g(z)dz -$$
$$- I\text{meas } (\Omega \setminus \Omega_k) | + | \int_{\Omega_k} dx \int_0^{w+v(w)} g(z)dz | + I\text{meas } \Omega_k <$$
$$< \varepsilon (\text{meas } \Omega + \int_0^\infty |g(z)| dz + I),$$ which implies

$$\lim_{\|w\| \to \infty} \int_\Omega dx \int_0^{w+v(w)} g(z)dz = I\text{meas } \Omega .$$

Let $I = +\infty$. Then for arbitrary $\ell > 0$ there exists $k>0$ such that

$$\int_0^{\pm k} g(z)dz > \ell .$$

Let $\varkappa > 0$ be such that meas $\Omega_k < \min\ (1/\ell \int_0^{k} |g(z)|dz,$ $1/2$ meas Ω), for all $w \in N(L)$, $\|w\| \geq \varkappa$. Thus for $\|w\| \geq \varkappa$ it is

$$\int_\Omega dx \int_0^{w+v(w)} g(z)dz \geq \int_{\Omega\setminus\Omega_k} dx \int_0^{w+v(w)} g(z)dz -$$

$$- \left| \int_{\Omega_k} dx \int_0^{w+v(w)} g(z)dz \right| \geq \ell \text{meas } (\Omega \setminus \Omega_k) -$$

$$- \text{meas } \Omega_k \int_0^{k} |g(z)|dz \geq 1/2\ \ell \text{ meas } \Omega - 1/\ell$$, which implies

$$\lim_{\|w\| \to \infty} \int_\Omega dx \int_0^{w+v(w)} g(z)dz = +\infty .$$

This together with Step 2 proves the assertion for $I = +\infty$.

Step 7. The function $v(\cdot): w \longmapsto v(w)$ is Fréchet differentiable on $N(L)$. Since $c<1$ (see (1)), the Fréchet derivative of

$$(v,w) \longmapsto v - F(v,w)$$

with respect to the first variable is invertible (lemma of Minty) and the assertion then follows from the implicit function theorem.

According to Step 6 the function $\varphi : N(L) \longrightarrow \mathbb{R}$ must attain its maximum or minimum in some point $w_0 \in N(L)$, if $I \in \mathbb{R}$, φ attains its maximum for $I = -\infty$ and minimum for $I = +\infty$

Then

(11) $$\langle \varphi'(w_0), h \rangle = 0$$

for each $h \in N(L)$. On the other hand, it is

$$\langle \varphi'(w_0), h \rangle = 1/2 \langle Lv'(w_0)h, v(w_0) \rangle + 1/2 \langle Lv(w_0), v'(w_0)h \rangle +$$

$+ \int_\Omega g(w_0 + v(w_0))h + \int_\Omega g(w_0 + v(w_0))v'(w_0)h - \int_\Omega fv'(w_0)h.$

Since L is symmetric, it is

$1/2 \langle Lv'(w_0)h, v(w_0)\rangle + 1/2 \langle Lv(w_0), v'(w_0)h\rangle = \langle Lv(w_0), v'(w_0)h\rangle$

and (because of $v'(w_0)h \in R(L)$ and (3a) holds)

$\langle Lv(w_0), v'(w_0)h\rangle + \int_\Omega g(w_0 + v(w_0))v'(w_0)h = \int_\Omega fv'(w_0)h$

for each $h \in N(L)$. From (11) we obtain that

$$\int_\Omega g(w_0 + v(w_0))h = 0,$$

for each $h \in N(L)$, which is nothing else than (3b).

The function $u = w_0 + v(w_0)$ is then the solution of (3).

4. Applications. The results of this paper may be applied, for instance, to the following types of semilinear elliptic boundary value problems:

$$(12) \quad \begin{cases} -\Delta u - \lambda_k u + \beta u e^{-u^2} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

$$(13) \quad \begin{cases} -\Delta u - \lambda_k u + \beta e^{-u^2} \sin u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

$$(14) \quad \begin{cases} \Delta^2 u - \lambda_k u + \dfrac{\beta u}{1+u^8} = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega; \end{cases}$$

$$(15) \quad \begin{cases} \Delta^2 u - \lambda_k u + g(u) = f & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where g is bounded, odd, continuously differentiable function with compact support in $\mathbb{R}$.

We put $D(L) = W_0^{1,2}(\Omega)$, resp. $D(L) = W_0^{2,2}(\Omega)$, in the cases (12),(13), resp. (14),(15). The operator L is defined by

$$\langle Lu, v\rangle = \int_\Omega \nabla u \nabla v - \lambda_k \int_\Omega uv,$$

in cases (12) and (13);

$$\langle Lu, v\rangle = \int_\Omega \Delta u \Delta v - \lambda_k \int_\Omega uv,$$

in the cases (14) and (15). We suppose that λ_k is any eigenvalue of the Laplace operator Δ , resp. the biharmonic operator Δ^2, with Dirichlet boundary conditions. Then the operator L satisfies all the assumptions from Section 1. Let us note that the assumption of "unique continuation property" is satisfied according to the result of Sitnikova [7]. The constant $\beta > 0$ depends on Ω and it must be such that the assumption (1) is fulfilled.

5. Remarks. As it was pointed out in Section 1, we assume nothing about the limit

$$\lim_{t \to \infty,\, \tau \in \langle \alpha, t\rangle} t \min g(\tau). \tag{16}$$

It means that this paper generalizes the results of Fučík, Krbec [1] and Hess [2]. The price we must pay for this generalization is the assumption (1) which is not very eligible.

This paper generalizes the results of de Figueiredo, Ni [3] and Concalves [6] because we may have dim $N(L) > 1$ and it need not be necessarily $g(t)t \geq 0$, $t \in \mathbb{R}$.

Following the proof of the theorem it is obvious that the assumption that g is odd can be replaced by the assumption

$$\int_{-\infty}^{0} g(z)dz = - \int_0^\infty g(z)dz.$$

Studying the function $\varphi : N(L) \longrightarrow \mathbb{R}$ and using the

Brouwer degree theory it is possible to prove the existence of multiple solutions of (3) with the right hand side

$$f = f_1 + f_2,$$

$f_1 \in R(L)$ and $f_2 \in N(L)$ with sufficiently small $\| f_2 \|$. The sketch of the proof is given in [5].

6. Open problem. According to the author's best knowledge it remains to be an open problem to prove the theorem without the condition (1) which makes restriction on the derivative $|g'(z)|$, $z \in \mathbb{R}$.

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