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THE COMPLETION MONAD AND ITS ALGEBRA

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Abstract: Let C represent the completion functor discussed by O. Wyler and S. Salbany. There is a monad associated with C and it is natural to ask for a characterization of the C-algebras. In this paper we show that the C-algebras are the complete spaces.

Key words: Quasi Uniform spaces, completion triple C, C-algebras.

Classification: Primary 54E15
Secondary 18C15

Introduction. As shown in [4] and [5], the completion functor on the category of Quasi-Uniform spaces is associated with a monad (C, η, μ). Keith Hardie asked us for a characterization of the C-algebras and persuaded us, over the years, that an answer should be given.

We shall follow the terminology of [3] and [1] concerning quasi-uniform spaces and that of [2] for the category theory.

1. Definitions, constructions and notations

A. Cauchy filters, convergence and completeness. Let QU denote the category of quasi-uniform spaces $(X, \mathcal{U})$ and quasi-uniformly continuous maps.

Definition 1. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is said to be a

Cauchy-filter if, for every U in the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ there is F in $\mathcal{F}$ such that $F \times F \subset \mathcal{U}$.

Definition 2. A filter $\mathcal{F}$ on $(X, \mathcal{U})$ is said to converge to x if it converges to x in the topology induced by the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$.

Definition 3. $(X, \mathcal{U})$ is said to be complete if $\mathcal{U} \vee \mathcal{U}^{-1}$ is complete.

Definition 4. If A is a subset of $(X, \mathcal{U})$, denote by $\bar{A}$ the closure of A in the topology induced by the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$. x is called an adherence point of a filter $\mathcal{F}$ if $x \in \bar{F}$ for every F in $\mathcal{F}$.

Note. As for uniform spaces, if $\mathcal{F}$ is a Cauchy filter on $(X, \mathcal{U})$ and x is an adherence point of $\mathcal{F}$, then $\mathcal{F}$ converges to x, and conversely. Thus, if $x \in \bar{F}$ for all $F \in \mathcal{F}$, then $\mathcal{F}$ converges to x.

B. Description of the completion monad. Given $(X, \mathcal{U})$, let CX denote the set of all Cauchy filters on X and let $\mathcal{U}^*$ denote the quasi-uniformity on CX with basis elements U^*, where $U \in \mathcal{U}$ and $(\alpha, \beta) \in U^*$ if and only if there are sets A in α, B in β such that $A \times B \subset U$. The sets U^* do form a basis for $\mathcal{U}^*$ since $(U \cap V)^* = U^* \cap V^*$. We now describe the multiplication μ and the unity η of the triple C:

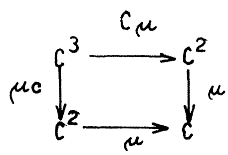
(i) Let $\eta_X : X \longrightarrow CX$ be given by $\eta_X(x) = \{F \mid F \subset X$ and $x \in F\}$. Then $\eta_X : (X, \mathcal{U}) \longrightarrow (CX, \mathcal{U}^*)$ is quasi-uniformly continuous.

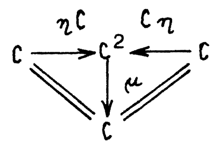
(ii) Let $\mu_X : C^2X \longrightarrow CX$ be given by $\mu_X(\alpha) = \{H \mid H \subset X, H^* \in \alpha\}$, where H^* is such that $\beta \in H^*$ if and only if $H \in \beta$. Then $\mu_X : (C^2X, \mathcal{U}^{**}) \longrightarrow (CX, \mathcal{U}^*)$ is quasi-uniformly continuous.

Moreover,

(a) η_X is an initial and injective map onto a $\mathcal{U}^* \vee (\mathcal{U}^*)^{-1}$-dense subspace of CX.

(b) μ_X and η_X induce natural transformations $\mu : C^2 \longrightarrow C$; $\eta : 1\!\!1 \longrightarrow C$ such that the following diagrams commute





Thus, every space can be densely embedded in a complete space in a "regular" way, which is expressed by the functoriality of C. Moreover, even though the completion process always enlarges a space, the existence of μ shows that the completion of a complete space is not "much larger" than the complete space itself.

C. The separated completion

Definition 5. A quasi-uniform space $(X, \mathcal{U})$ is separated if the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ is separated, that is, the intersection of all members of $\mathcal{U} \vee \mathcal{U}^{-1}$ is the diagonal of $X \times X$.

Construction of the separated reflection. Given a quasi-uniform space $(X, \mathcal{U})$, let R denote the equivalence relation $x R y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Denote by $[x]$, the R-equivalence class of x. Let X^s denote the set of R-equivalence classes on $(X, \mathcal{U})$. Let $s : X \longrightarrow X^s$ denote the map $s(x) = [x]$. For $U \in \mathcal{U}$, let $U^s = \{([x],[y]) \mid (x, y) \in U\}$, then the U^s form a basis for a quasi-uniformity $\mathcal{U}^s$ (since $(U \cap V)^s \subset U^s \cap V^s$). The map $s_X : (X, \mathcal{U}) \longrightarrow (X^s, \mathcal{U}^s)$ is an initial quasi-uniformly continuous map onto a separated quasi-uniform space and the assignment $(X,\mathcal{U}) \longrightarrow (X^s,\mathcal{U}^s)$ is the separated reflection in QU. Moreover, $(X,\mathcal{U})$ is complete if and only if $(X^s,\mathcal{U}^s)$ is complete. The composite $s \circ C = C^s$ is the separated-completion-functor. The natural transformations $\eta^s = s\eta$ and $\mu^s = s\mu$ provides the separated-completion monad (C^s, η^s, μ^s).

The importance of separated completions lies in the fact that the embedding map $\eta^s_X : (X,\mathcal{U}) \longrightarrow (X^s,\mathcal{U}^s)$ is a map onto a $\mathcal{U}^s \vee (\mathcal{U}^s)^{-1}$-dense subspace, hence an epimorphism in the category of separated quasi-uniform spaces (see [1],[3]).

D. The algebra of a monad

Definition 6. Let (C,η,μ) be a monad on a category A. An object A of A is a C-algebra if there is a morphism h in A, called a structure map, $h: CA \longrightarrow A$ such that the following diagrams commute:

$$\begin{array}{ccc} C\,A & \xrightarrow{Ch} & CA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle h} \\ CA & \xrightarrow[h]{} & A \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & C\,A \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle h} \\ & & A \end{array}$$

Examples of C-algebras

Example 1. Let $X = \{0,1\}$ and $\mathcal{U} = X \times X$. It is straightforward to verify that $(X,\mathcal{U})$ is a C-algebra for every map $h:(CX,\mathcal{U}^*) \longrightarrow (X,\mathcal{U})$.

Example 2. Let $(X,\mathcal{U})$ be a separated complete space and let $h:(CX,\mathcal{U}^*) \longrightarrow (X,\mathcal{U})$ be the limit map, $h(\mathcal{F})$ = limit of $\mathcal{F}$ (convergence in the topology of $\mathcal{U} \vee \mathcal{U}^{-1}$). Note that h is well defined since limits are unique in separated quasi-uniform spaces. It will be shown in Section 3 that such a separated complete space is a C-algebra.

2. C-algebras. As expected, C-algebras are the complete spaces. Although expected, we have found the proof elusive. The result is surprising in that there is an arbitrariness in the structure map $h: CX \longrightarrow X$ (see Proposition 2) that suggests that not all complete spaces would be C-algebras.

Proposition 1. If $(X, \mathcal{U})$ is a C-algebra, then $(X, \mathcal{U})$ is complete.

This is an immediate consequence of the following two lemmas.

Lemma 1. If $S \subset X$, then $S^* \subset \overline{\eta_X[S]}$.

Proof. Let $\alpha \in S^*$. Given a symmetric U in $\mathcal{U} \vee \mathcal{U}^{-1}$, we show that $U[\alpha]$ intersects $\eta_X[S]$. From $\alpha \in S^*$ it follows that $S \in \alpha$. Now α is a Cauchy-filter, so there is F in α such that $F \times F \subset U$. But S and F are in α so there is x in $F \cap S$. By definition of U^* it follows that $(\alpha, \eta_X(x)) \in U^*$ since $F \in \alpha$ and $F \in \eta_X(x)$ and $F \times F \subset U$. Thus $U^*[\alpha]$ intersects $\eta_X[S]$, as required.

Lemma 2. Let $h: (CX, \mathcal{U}^*) \longrightarrow (X, \mathcal{U})$ be a quasi-uniformly continuous map such that $h \circ \eta_X(x) = x$ for all x in X. Then a Cauchy filter $\mathcal{F}$ converges to $h(\mathcal{F})$.

Proof. We show that $h(\mathcal{F})$ is an adherence point of $\mathcal{F}$, from which it follows that $\mathcal{F}$ converges to $h(\mathcal{F})$. Let $S \in \mathcal{F}$, then $\mathcal{F} \in S^*$, so that $h(\mathcal{F}) \in h[S^*]$. By Lemma 1 we have $h[S^*] \subset$ $\subset h[\eta_X[S]] \subset h\,\overline{[\eta_X[S]]} = \overline{S}$. Thus $h(\mathcal{F}) \in \overline{S}$ for all S in $\mathcal{F}$, showing that $h(\mathcal{F})$ is an adherence point of $\mathcal{F}$.

To establish the converse of Proposition 1 we require the following lemmas.

__Lemma 3.__ Let $\alpha \in C^2X$, then $\bigcap\{\overline{H} \mid H \subset X \text{ and } h \leftarrow [H] \in \alpha\} = \bigcap\{\overline{h[k]} \mid k \in \alpha\}$.

__Proof.__ Suppose $H \subset X$ and $h \leftarrow [H] \in \alpha$. $h[h \leftarrow [H]] = H$ (since h is surjective) shows that $\overline{H} = \overline{h[k]}$ for some k in α, since $k \in \alpha$ and $K \subset h \leftarrow [h[k]]$. The proof is complete.

__Lemma 4.__ Let $\alpha \in C^2X$. If α converges to $\mathcal{F}$, then $ch(\alpha)$ converges to $h(\mathcal{F})$.

__Proof.__ Suppose α converges to $\mathcal{F}$, then $\mathcal{F}$ is an adherence point of α so that $\mathcal{F} \in \overline{k}$ for all $k \in \alpha$. Hence $h(\mathcal{F}) \in h[\overline{k}] \subset \overline{h[k]}$ for all $k \in \alpha$. Thus, by Lemma 3, $h(\mathcal{F}) \in \bigcap\{\overline{H} \mid h \leftarrow [H] \in \alpha\}$, so that $h(\mathcal{F})$ is an adherence point of $Ch(\alpha)$, as required.

__Lemma 5.__ Let $V \in \mathcal{U} \vee \mathcal{U}^{-1}$. Let U be symmetric and such that $U \circ U \subset V$. If $\mathcal{F}$ is a Cauchy filter which converges to x, then $U^*[\mathcal{F}] \subset (V[x])^*$.

__Proof.__ Because $\mathcal{F}$ is Cauchy and converges to x, there is F in $\mathcal{F}$ such that $F_1 \times F_1 \subset \mathcal{U}$ and $F_1 \subset \mathcal{U}[x]$. Suppose $\chi \in U^*[\mathcal{F}]$, we show that $V[x] \in \chi$. By definition of U^*, if $\chi \in U^*[\mathcal{F}]$, there is $F_2 \in \mathcal{F}$ and $G \in \mathcal{F}$ such that $F_2 \times G \subset U$.

Let $F = F_1 \cap F_2$ so that (i) $F \in \mathcal{F}$, (ii) $F \times G \subset U$, and (iii) $F \subset U[x]$.

From (ii) we have $G \subset U[F]$ so that $U[F] \in \chi$ since $G \in \chi$. But $U[F] \subset U \circ U[x] \subset V[x]$, from (iii), so that $V[x] \in \chi$, as required .

<u>Lemma 6</u>. Let $\alpha \in C^2X$. If α converges to $\mathcal{F}$ and $\mathcal{F}$ converges to x , then $\mu_X(\alpha)$ converges to x .

<u>Proof</u>. Let $V \in \mathcal{U} \vee \mathcal{U}^{-1}$. We show that $V[x] \in \mu_X(\alpha)$. Choose a symmetric entourage U such that $U \circ U \subset V$. Because $\mathcal{F}$ converges to x , by Lemma 5, we have $U^*[\mathcal{F}] \subset (V[x])^*$. Since α converges to $\mathcal{F}$ we also have $U^*[\mathcal{F}]$ in α, hence $(V[x])^*$ is in α and, consequently, $V[x] \in \mu_X(\alpha)$, as required.

<u>Lemma 7</u>. Let $\alpha \in C^2X$, if there is $x \in X$ such that $\mu_X(\alpha) = \{H \mid x \in H\}$, then $Ch(\alpha) = \mu_X(\alpha)$.

<u>Proof</u>. Observe that, for any filter $\mathcal{F}$, $\mathcal{F} = \{H \mid x \in H\} \Longleftrightarrow \{x\} \in \mathcal{F}$. For convenience, let $\{H \mid x \in H\}$ be denoted by $\langle x \rangle$. To show that $Ch(\alpha) = \langle x \rangle$ it suffices to prove that $\{x\} \in Ch(\alpha)$, that is, $h \leftarrow [\{x\}] \in \alpha$.
Now, $\{\langle x \rangle\} \in \alpha$ since $\{x\} \in \mu(\alpha)$ implies $(\{x\})^* \in \alpha$ and $\mathcal{F} \in (\{x\})^*$ is equivalent to $\{x\} \in \mathcal{F}$ which states that $\mathcal{F} = \langle x \rangle$, so that $\{\langle x \rangle\} \in \alpha$. Now $\langle x \rangle \in h \leftarrow [\{x\}]$ since $h(\langle x \rangle) = x$. It then follows that $h \leftarrow [\{x\}] \in \alpha$, as required.

Note: Consideration of example 1 shows that if $Ch(\alpha) = \{H \mid x \in H\}$, then it does not follow that $\mu_X(\alpha) = Ch(\alpha)$.

<u>Lemma 8</u>. Let A_x consist of all Cauchy filters on X which converge to x . Let $\alpha \in C^2X$. If $A_x \in \alpha$, then $\mu_X(\alpha)$ converges to x .

<u>Proof</u>. It is straightforward to check $A_x = \overline{A}_x$. Also CX is complete so there is $\mathcal{F}$ in CX such that α converges to $\mathcal{F}$. Then $\mathcal{F}$ is an adherence point to α , so that $\mathcal{F} \in \overline{A}_x$. But

then $\mathcal{F} \in A_x$, so that $\mathcal{F}$ converges to x . By Lemma 6, we have that $\mu_X(\alpha)$ converges to x .

<u>Proposition 2</u>. If $(X,\mathcal{U})$ is complete, then it is a C-algebra.

<u>Proof</u>. Define $h:CX \longrightarrow X$ as follows: For each x , let $[x]$ denote the R equivalence class of x ($xRy \Longleftrightarrow \bar{x} = \bar{y}$). Let c be a choice function on X^s, so $c([x]) \in [x]$. Observe that if $\mathcal{F}$ is a Cauchy filter on X which converges to x and y, then $[x] = [y]$; now let

$$h(\mathcal{F}) = \begin{cases} x, \text{ if } \mathcal{F} = \eta_X(x) \text{ for some } x \cdot \\ c([x]), \text{ if } \mathcal{F} \notin \eta_X[X] \text{ and } \mathcal{F} \text{ converges to } x . \end{cases}$$

By the remarks above, h is well defined. Observe that $\mathcal{F}$ converges to $h[\mathcal{F}]$. It is readily checked that $h:(CX, \mathcal{U}^*) \longrightarrow$ $\longrightarrow (X,\mathcal{U})$ is quasi-uniformly continuous. We verify that the diagram corresponding to (1) in the definition of the algebra of a monad is commutative: Let $\alpha \in C^2X$. Because CX is complete, α converges to some $\mathcal{F}$ in CX.

By Lemma 4, $Ch(\alpha)$ converges to $h(\mathcal{F})$. By Lemma 6, $\mu_X(\alpha)$ converges to $h(\mathcal{F})$ and α converges to $\mathcal{F}$. Thus $\mu_X(\alpha)$ and $Ch(\alpha)$ both converge to $h(\mathcal{F})$. To show that $h[\mu_X(\alpha)] = h[Ch(\alpha)]$ we consider two cases.

<u>Case 1</u>. $Ch(\alpha) \notin \eta_X[X]$. Then $\mu_X(\alpha) \notin \eta_X[X]$, by Lemma 7. By definition of h, it follows that $h[\mu_X(\alpha)] =$ $= c([h(\mathcal{F})])$ and $h[Ch(\alpha)] = c([h(\mathcal{F})])$, as required.

<u>Case 2</u>. $Ch(\alpha) \in \eta_X[X]$. Then $Ch(\alpha) = \langle x \rangle$ (for a unique x in X) so that $h \leftarrow [\{x\}] \in \alpha$. We again distinguish

two cases:

Case 2 a. $x \neq c([x])$. In this case, $h^{\leftarrow}[\{x\}] = \{\langle x\rangle\}$ since for any $\mathcal{F}$ not in $\eta_X[X]$, $h(\mathcal{F}) = x$ implies $\mathcal{F}$ converges to x so that $h(\mathcal{F}) = c([x]) \neq x$. Now $Ch(\alpha) = \langle x\rangle$ implies that $h^{\leftarrow}[\{x\}] \in \alpha$, so that $\{\langle x\rangle\} \in \alpha$. But then $\mu_X(\alpha) = \langle x\rangle$. Thus $Ch(\alpha) = \mu_X(\alpha) = \langle x\rangle$ so that $h(Ch(\alpha)) = h(\mu_X(\alpha))$.

Case 2 b. $x = c([x])$. In this case $h^{\leftarrow}[\{x\}]$ consists of all Cauchy filters $\mathcal{F}$ which converge to a point in $[x]$. Thus $h^{\leftarrow}[\{x\}] = A_x$. From Lemma 8, it follows that $\mu_X(\alpha)$ converges to x. From the definition of h (whether or not $\mu_X(\alpha)$ is in $\eta_X[X]$) and $x = c([x])$, it follows that $h[\mu_X(\alpha)] = x$. Hence $h[\mu_X(\alpha)] = h[Ch(\alpha)]$.

It is remarkable that the regularity expressed in diagram 1 of the definition of a monad could be achieved with the arbitrariness involved in the function h of Proposition 2.

3. The separated-completion C^s. The results in Section 2 readily identify the C^s-algebras in $\underline{QU}_s$:

Proposition 3. The C^s-algebras are the separated and complete spaces.

However, the delicate comparison of filters and limits can be avoided and a simple proof of Proposition 3 will be given in two parts, where we rely on the fact that η_X is epic in $\underline{QU}_s$.

Part 1. Every C^s-algebra is complete and separated.

Proof. Let $h: C^sX \to X$ be the structure map. Then $h \circ \eta_X = 1_X$, so that $\eta_X \circ h \circ \eta_X = \eta_X \circ 1_X = 1_X \circ \eta_X$. But η_X is an epimorphism, hence $\eta_X \circ h = 1\!\!1_{C^s X}$. Thus X and $C^s X$ are isomorphic.

Part 2. Let X be a complete and separated space, then every Cauchy filter $\mathcal{F}$ converges to a unique point x. It is straightforward to verify that $h:(C^s X, (\mathcal{U}^*)^s) \longrightarrow (X, \mathcal{U})$ is quasi-uniformly continuous. Moreover $h \circ \eta_X = 1\!\!1_X$, since the filter $\eta_X(x)$ converges to x.

To prove that $h \circ \mu_X = h \circ Ch$ we show that $h \circ \mu_X \circ \eta_{CX} =$
$= h \circ Ch \circ \eta_{CX} (=h)$ and use the fact that η_{CX} is epic.

Now $\mu_X \circ \eta_{CX} = 1\!\!1_{CX}$, so that $h \circ \mu_X \circ \eta_{CX} = h$. Also, $Ch \circ \eta_{CX} =$
$= \eta_X \circ h$ (by naturality of η), so that

$$h \circ Ch \circ \eta_{CX} = h \circ \eta_X \circ h = 1\!\!1_X \circ h = h.$$

The proof is complete.

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