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FUZZINESS AND FUZZY EQUALITY

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Abstract: Fuzzy sets with fuzzy equality can be represented as couples of (crisp) subsets of generalized metric spaces. The representation can be constructed so that the operations with fuzzy sets appear as natural operations with the respective couples. Moreover, fuzzy-power sets of fuzzy sets are obtained by means of an analogon of the Hausdorff superspace construction.

Key words: Fuzzy sets, fuzzy equality, generalized metric spaces.

Classification: 03E72

In the intuitive examples of fuzzy sets one often has more structure than just the degrees in which the elements belong. Typically, one encounters also various degrees of likeness of the elements with each other. Thus, e.g., in the "set" of all green objects in a given room we observe varying similarity of the coloring. In the "set" of very large natural numbers one has the obvious distance, in the "set" of patients diagnosed for a given illness one sees varying similarity of symptoms, etc. This is mostly connected with the fuzziness structure in the following way: if an element x is a member of X in some degree and another y is close (similar) to x in a high degree then y is a member in not much smaller degree than x (if, say, x is "very typically green" and the shade of y is very close to that of x, we will not much hesitate to classify y as green, too; if x is a very large natural number and y is close to x, we should take x for rather large; etc.).

The resulting fuzzy-equivalence (fuzzy-equality, tolerance) has been in that or other form considered before (see, e.g., [7],[8]), but it does not seem very popular. The reason for this may be in the natural wish to avoid unnecessary complexity. The aim of this paper is to show that the fuzzy-equivalence point of view can, in fact, make the calculating simpler. Roughly speaking, we show that fuzzy sets endowed with fuzzy-equivalences can be adequately represented by means of the fuzzy-equivalence alone. In the representation the fuzzy sets appear as couples (X', X'') of crisp subsets of universa with a kind of distance (resulting from the fuzzy-equalities) and one operates with them crisply (e.g., if fuzzy sets X_i are represented as (X_i', X_i''), the union of the X_i in the usual sense gets represented as $(\bigcup X_i', \bigcup X_i'')$, the product as $(\times X_i', \times X''_i)$). Even the fuzzy power set of a fuzzy set is naturally expressed by means of a generalization of the metric in the Hausdorff hyperspace.

To avoid misunderstanding, let us make a few remarks on the distance functions issueing from fuzzy equivalences we will work with. Dealing with fuzzy notions it is natural to consider the value 0 in $x \in_0 X$ as expressing poor belonging (if any), or, in $x =_0 y$, a poor similarity of x and y, while the value 1 expresses in $x \in_1 X$ the (at least almost) full belonging, or in $x =_1 y$ the (at least almost) coincidence of the relevant features. Consequently, in the "metric: spaces we shall deal with it is handier to use, instead of the distance functions (with values diminishing when coming nearer), the nearness functions with values the higher the nearer one comes. The reader will certainly see that this is a technical preference only, and that the notion of L-nearness below has nothing to do with the well known continuity structure sometimes called nearness spaces ([5]).

The spaces with L-nearness are discussed in §§ 1 and 2. In § 3 we recall the usual operations with fuzzy sets and extend them in the obvious way to fuzzy sets with fuzzy-equalities. In § 4 the notion of L-nebula (which is the couple of subsets of a universe with an L-nearness mentioned above) is introduced. The main results are contained in § 5 where we show that fuzzy sets (with or without fuzzy equality) can be represented as nebulae (and vice versa) with reasonable behavior under operations.

§ 1. L-nearness and L-spaces with L a residuated lattice

1.1. A residuated lattice $L = L(\vee, \wedge, 0, 1, ., h)$ is a complete lattice endowed, moreover, with two further binary operations . , h such that

(i) $L(., 1)$ is a commutative monoid,

(ii) . is non-decreasing in both variables,

(iii) h is non-increasing in the first variable and non-decreasing in the second one,

(iv) for all a,b,c $a.b \leq c$ iff $a \leq h(b,c)$.

1.2. The following identitites are immediate consequences of the definition (with a particular role played by the condition (iv)):

(1) $a.\sup b_i = \sup(a.b_i)$,

(2) $h(a, \inf b_i) = \inf h(a,b_i)$,

(3) $h(\sup a_i,b) = \inf h(a_i,b)$,

(4) $h(a.b,c) = h(a,h(b,c))$. □

1.3. Remarks: (1) Residuated lattices were, first, introduced in [1]: they appear in the literature also under the name of tensored lattices, or LC_∞-lattices ([3]). Viewing complete lattices as complete small thin categories, we see that the residuated ones coincide with the closed ones among them.

(2) In fact, h always exists and is uniquely determined if we have an $L(\vee,\wedge,0.1,.)$ satisfying (i) and 1.2(1). It is, however, more comfortable to work with h as explicitly given.

(3) Examples of residuated lattices:

(3.1) Complete Boolean algebras: put $. = \wedge$ and $h(a,b) = \bar{a} \vee b$.

(3.2) More generally, Heyting algebras are exactly the residuated lattices with $. = \wedge$.

(3.3) (Łukasiewicz lattice): L is the unit interval with minimum and maximum, $a.b = (a+b-1) \vee 0$, $h(b,c) = (1-b+c) \wedge 1$ (cf. [6]).

(3.4) $L(\vee,\wedge)$ as in (3.3), a.b the usual multiplication, $h(b,c) = b^{-1}.c$ for $b > c$, 1 otherwise.

(4) It is easy to see that . is idempotent iff $. = \wedge$ (i.e., in the case of Heyting algebras). Otherwise there is always an a with $a.a < a$.

1.4. Here are a few of further basic properties of the operations in residuated lattices. The proofs are easy and are left to the reader as an exercise.

(1) $a.b \leq a \wedge b$,

(2) $a \leq b \Longleftrightarrow h(a,b) = 1$,

(3) $h(a,b).h(b,c) \leq h(a,c)$,

(4) $h(1,a) = a$,

(5) $h(a,b) = h(a,b \wedge a)$. □

1.5. Let L be a residuated lattice. An L-nearness on a set P is a mapping

$$\nu : P \times P \longrightarrow L$$

such that

(i) $\nu(x,x) = 1$,

(ii) $\nu(x,y) = \nu(y,x)$,

(iii) $\nu(x,y).\nu(y,z) \leq \nu(x,z)$.

The system (P, ν) will be referred to as an L-space. We use the term subspace of (P, ν) in the obvious way, namely for the (Q, ν') with $Q \subset P$ and $\nu' = \nu | Q \times Q$.

1.6. Remark: One easily sees that the usual metric spaces are, in principle, obtained when taking for L the Łukasiewicz lattice (see 1.3.(3.3)).

1.7. Observation and convention: The formula $\nu_0(a,b) = h(a,b) \wedge h(b,a)$ gives and L-nearness on L itself. Referring to L as a space we will always have in mind (L, ν_0).

Note that $\nu_0(a,1) = a$.

§ 2. Nearness of points and sets. Hausdorff superspace

2.1. For a subset $A \subset P$ we put

$$\nu(x,A) = \sup \{ \nu(x,y) \mid y \in A \}.$$

2.2. Lemma: $\nu(x,A) \geq \nu(x,B).\inf \{ \nu(y,A) \mid y \in B \}$.

Proof: For each y we have (recall 1.2.(1))

$\nu(x,A) \geq \sup_{z \in A} (\nu(x,y) . \nu(y,z)) = \nu(x,y) . \sup_{z \in A} \nu(y,z) =$
$= \nu(x,y) . \nu(y,A)$.

Put $d = \inf \{ \nu(y,A) \mid y \in B \}$. We obtain $\nu(x,A) \geq \nu(x,y).d$ for any $y \in B$ and hence finally

$$\nu(x,A) \geq \sup_{y \in B} (\nu(x,y).d) = (\sup_{y \in B} \nu(x,y)).d = \nu(x,B).d \ . \ \square$$

2.3. Lemma: $\nu(x, \bigcup A_i) = \sup_i \nu(x, A_i)$.

Proof: Obviously $\nu(x, A_j) \leq \nu(x, \bigcup A_i)$ and hence $\sup \nu(x,A_i) \leq \nu(x, \bigcup A_i)$. On the other hand, if $y \in \bigcup A_i$, we have, for some j, $\nu(x,y) \leq \nu(x,A_j) \leq \sup \nu(x,A_i)$ so that also $\nu(x, \bigcup A_i) \leq \sup \nu(x,A_i)$. $\square$

2.4. For $A,B \subset (P, \nu)$ put

$$\tilde{\nu}(A,B) = \inf_{x \in A} \nu(x,B) . \inf_{y \in B} \nu(y,A).$$

2.5. Proposition: $\tilde{\nu}$ is an L-nearness on the set of all subsets of (P, ν).

Proof: Obviously $\tilde{\nu}(A,A) = 1$ and $\tilde{\nu}(A,B) = \tilde{\nu}(B,A)$. Now, we have (by 1.1.(ii) and 2.2)

$$\tilde{\nu}(A,B) . \tilde{\nu}(B,C) = \inf_{x \in A} \nu(x,B) . \inf_{y \in B} \nu(y,A) . \inf_{z \in B} \nu(z,C) .$$
$$. \inf_{u \in C} \nu(u,B) = \inf_{x \in A} \nu(x,B) . \inf_{z \in B} \nu(z,C) . \inf_{u \in C} \nu(u,B) . \inf_{y \in B} \nu(y,A) \leq$$
$$\leq \inf_{x \in A} (\nu(x,B) . \inf_{z \in B} \nu(z,C)) . \inf_{u \in C} (\nu(u,B) . \inf_{y \in B} \nu(y,A)) \leq$$
$$\leq \inf_{x \in A} \nu(x,C) . \inf_{u \in C} \nu(u,A) = \tilde{\nu}(A,C) . \ \square$$

2.6. The set of all subsets of (P, ν) endowed with the nearness $\tilde{\nu}$ will be called the Hausdorff superspace of (P, ν) and denoted by

$$\exp (P, \nu).$$

One sees that this is an obvious generalization of the well-known synonymous construction for metric spaces ([4]).

2.7. Remark: (P,ν) is naturally embedded into $\exp(P,\nu)$ by representing $x \in P$ as $\{x\}$. Indeed, we have $\tilde{\nu}(\{x\},\{y\}) = \nu(x,y)$.

2.8. Remarks: Define a ν-closure of a set A as

$$\bar{A} = \{x \mid \nu(x,A) = 1\},$$

and call A closed if $\bar{A} = A$.

Using 2.2 one proves easily that

$$\nu(x,\bar{A}) = \nu(x,A)$$

and in consequence $\bar{\bar{A}} = \bar{A}$.

The ν-closure is easily shown to be additive iff $(a,b \in L \;\&\; a \vee b = 1) \Longrightarrow (a=1 \text{ or } b=1)$.

The definition in 1.5 can be modified by replacing (i) by stronger

(i*) $\quad \nu(x,y) = 1$ iff $x = y$

to obtain a notion closer to metric spaces (as it is, it corresponds rather to the pseudometric ones). All what is said in this article can be easily modified for this stronger structure, the basic change being in replacing "subsets" in the definition in 2.6 by "closed subsets".

§ 3. Fuzzy sets and fuzzy equality

3.1. Let us recall the basic definition of fuzzy sets and operations with them (cf., e.g., [10],[2]). L is a residuated lattice.

An L-fuzzy set is a mapping

$$X: |X| \rightarrow L$$

where $|X|$ is a set (the carrier of X). We write (with obvious interpretation)

$$x \in_a X \text{ for } X(x) \geq a.$$

The union, intersection, product and tensor product of fuzzy sets are given by the formulas

$|\bigcup_{i \in J} X_i| = \bigcup_{i \in J} |X_i|$, $(\bigcup_{i \in J} X_i)(x) = \sup \{X_i(x) \mid i$ such that $x \in |X_i|\}$,

$|\bigcap_{i \in J} X_i| = \bigcap_{i \in J} |X_i|$, $(\bigcap_{i \in J} X_i)(x) = \inf \{X_i(x) \mid i \in J\}$,

$|\bigtimes_{i \in J} X_i| = \bigtimes_{i \in J} |X_i|$, $(\bigtimes_{i \in J} X_i)((x_i)_J) = \inf \{X_i(x_i) \mid i \in J\}$,

$|X \otimes Y| = |X| \times |Y|$, $(X \otimes Y)(x,y) = X(x).Y(y)$.

3.2. ("Mappings up to degree a" and fuzzy exponentitation.) If X, Y are L-fuzzy sets, we write

$f :_a X \rightarrow Y$ if $f : |X| \rightarrow |Y|$ and $(x \in_b X \Rightarrow f(x) \in_{ab} Y)$.

We write $X \overset{w}{\subset} Y$ ("the weak inclusion"), if $|X| \subset |Y|$, and further,

$X \subset_a Y$ if $X \overset{w}{\subset} Y$ and $(x \in_b X \Rightarrow x \in_{ab} Y)$.

(Cf. [2], [9]).

<u>Remark</u>: Obviously, $X \subset_a Y$ iff $j :_a X \rightarrow Y$ for the inclusion mapping $j : |X| \subset |Y|$. On the other hand, if we define Γ , the <u>graph of a mapping</u> φ , by

$|\Gamma| = \{(x, \varphi(x)) \mid x \in |X|\}$, $\Gamma(x, \varphi(x)) = X(x)$,

we have $\Gamma \subset_a X \times Y$ iff $\varphi :_a X \rightarrow Y$.

3.3. The everyday-life examples of fuzzy sets often have, in addition, a structure of "imperfect equality" $=_a$ which we may reasonably assume to satisfy the implication

(1) $x =_a y \;\&\; y =_b z \qquad x =_{ab} z$.

Moreover, it is usually (intuitively) such that if an element x is almost equal to y and y is in X in some degree, then x is in X in not much worse degree. This second matter will be formalized in 3.5 below. So far note that if we define

$$\nu(x,y) \geq a \text{ iff } x =_a y,$$

we obtain an L-nearness (the condition (1) represents the 1.5.(iii)).

This leads to the following definition:

An <u>L-fuzzy set with L-equality</u> (briefly, an <u>L-fe set</u>) (X, ν) consists of an L-fuzzy set $X: |X| \to L$ and an L-space $(|X|, \nu)$.

3.4. We write

$$(X, \nu) \overset{w}{\subset} (Y, \mu)$$

if $(|X|, \nu)$ is a subspace of $(|Y|, \mu)$. The basic operations with L-fe sets are defined as follows:

If $(X_i, \nu_i) \overset{w}{\subset} (X, \nu)$, we put

$$\bigcup_{J (X,\nu)} (X_i, \nu_i) = (\bigcup_J X_i, \nu | |\bigcup_J X_i|),$$

$$\bigcap_J (X_i, \nu_i) = (\bigcap_J X_i, \nu | |\bigcap_J X_i|).$$

For a general system $((X_i, \nu_i))_{i \in J}$ put

$$\mathop{\times}_J (X_i, \nu_i) = (\mathop{\times}_J X_i, \mu) \text{ where } \mu((x_i)_J, (y_i)_J) = \inf \nu_i(x_i, y_i).$$

Finally put

$$(X_1, \nu_1) \otimes (X_2, \nu_2) = (X_1 \otimes X_2, \nu) \text{ where } \nu((x_1,x_2),(y_1,y_2)) = \nu_1(x_1,y_1) \cdot \nu_2(x_2,y_2).$$

<u>Remark</u>: Obviously, the usual L-fuzzy sets can be viewed as L-fe sets with trivial (discrete) equalities. Then, the operations coincide with those from 3.1.

3.5. An L-fe set (X, ν) is said to be <u>saturated</u> if

(*) $x \in_a X \;\&\; y =_b x \Rightarrow y \in_{ab} X$.

This is an obvious formalization of the second property from

3.3. Handier, although perhaps less lucid, is the following evident reformulation of ($*$):

$(**)\quad \forall x\, \forall y \qquad \nu(x,y).X(y) \leq X(x).$

3.6. Proposition: (X,ν) is saturated iff X viewed as a mapping between L-spaces satisfies the inequality

$\nu_0(X(x),X(y)) \geq \nu(x,y).$

Proof: By 1.1.(iv), $(**)$ is, further, equivalent with

$$\forall x\, \forall y \qquad \nu(x,y) \leq h(X(y),X(x)).\ \square$$

3.7. For an L-fuzzy set put

$$K(X) = \{(x,a) \mid x \in |X|,\ a \leq X(x)\}.$$

3.8. Let (X,ν) be an L-fe set. Define

$$\exp(X,\nu) = (Y,\mu),$$

where

$$|Y| = \{Z \mid Z \text{ saturated}, |Z| \subset |X|\},$$

$$Z \in_a Y \text{ iff } Z \subset_a X,$$

and

$\mu(Z_1,Z_2) = \tilde{\nu}(K(Z_1),K(Z_2))$ where ν is defined on $|X| \times L$ by $\nu((x_1,x_2),(y_1,y_2)) = \nu(x_1,y_1).\ \nu_0(x_2,y_2).$

3.9. Proposition: $\exp(X,\nu)(Z) = \inf_{x \in Z} h(Z(x),X(x)).$

Proof: We have $(\exp X)(Z) \geq a$ iff $(\forall x(Z(x) \geq b \Rightarrow X(x) \geq ab)$

iff $\forall x(X(x) \geq a.Z(x))$ iff $\forall x(h(Z(x),X(x)) \geq a$ iff

iff $\inf h(Z(x),X(x)) \geq a$. $\square$

3.10. Remarks: (1) Realize that the definition of the L-nearness μ in the power L-fe set in 3.8 is quite natural. Consider the "distance" of L-fe subsets with coinciding carriers.

(2) There arises a natural question as to whether the

L-fe sets obtained from the saturated ones by operations are again saturated. It is not hard to prove the affirmative directly. We do not do it here, however, since it will follow free from the sequel.

§ 4. L-nebulae

4.1. An L-nebula is a system $N = ((P, \nu); K, R)$ where (P, ν) is an L-space and K, R are subsets of P. The set K will be referred to as the kernel and the set R as the range of the nebula.

4.2. Since the notion of an L-nebula and a representation of L-fuzzy sets and L-fe sets as nebulae is the main item of this article, let us discuss here briefly the intuitive background. The space (P, ν) is a universe in which we consider the elements we are interested in. Its size is not all that important. It could be replaced by a (P', ν') of which (P, ν) is a subspace, or by a subspace of (P, ν) large enough to contain R, and the situation described by the nebula would be virtually the same. The range corresponds to the carrier of a fuzzy set. The kernel is a system of typical elements with the property we try to describe (elements which have the property undisputedly); it is not necessarily a subset of the range. Let us give a few examples:

(a) The "set" of very large natural numbers: Take for the universe the set of naturals plus ∞, with the nearness, say, $\nu(m,n) = 1 - m^{-1}n^{-1} \cdot |m-n|$, $\nu(n,\infty) = 1 - n^{-1}$. For R take the set of natural numbers, $K = \{\infty\}$.

(b) The "set" of all yellow objects in a given area: Choose a nearness function expressing tolerance in interchangeability of colors. R is the set of objects which can be taken for yellow considered tolerantly (although someone could take some of them for, say, ochre, light brown, or greenish). K is a set of undisputedly yellow ones; may be there is none in the area considered - then choose a really yellow sample elsewhere.

(c) The "set" of patients suffering a particular disease. There are border cases with not very typical symptoms, but still included in R, while K contains only the typical ones, perhaps also imaginary textbook examples.

4.3. The <u>product</u> $\underset{J}{\times}(P_i, \nu_i)$ of a system $((P_i, \nu_i))_{i\in J}$ of L-spaces is defined as $(\underset{J}{\times} P_i, \nu)$ with $\nu((x_i)_J, (y_i)_J) = \inf \nu_i(x_i, y_i)$.

Further, we define the <u>tensor product</u> $(P_1, \nu_1) \otimes (P_2, \nu_2)$ of two L-spaces as $(P_1 \times P_2, \nu)$ with $\nu((x_1, x_2), (y_1, y_2)) = \nu_1(x_1, y_1) \cdot \nu_2(x_2, y_2)$.

<u>Remarks</u>: Thus, the products (tensor products) of L-fe sets in 3.4 are carried by the products (tensor products) of the carrying spaces. Also observe that the $K(Z_i)$ in 3.8 are considered as elements of $\exp((|X|, \nu) \otimes L)$.

The product is the categorial product in the naturally defined category of L-spaces (where the morphisms correspond to non-expanding mappings of metric spaces). Also, the tensor product is one in this category, having a natural right adjoint.

4.4. We write $N = ((P, \nu); K, R) \overset{w}{\subset} ((P', \nu'); K', R') = N'$ if (P, ν) is a subspace of (P', ν') and $R \subset R'$. If, moreover, also $K \subset K'$, we write $N \subset N'$. The former case is referred to as <u>weak inclusion</u>, the latter one simply as <u>inclusion</u> of nebulae.

The operations are defined as follows:

If $N_i = ((P_i, \nu_i); K_i, R_i) \overset{w}{\subset} N = ((P, \nu); K, R)$ $(i \in J)$,

$$\bigcup_N N_i = ((P, \nu); \bigcup_J K_i, \bigcup_J R_i).$$

$$\bigcap_J N_i = ((\bigcap_J P_i, \nu \mid \bigcap_J P_i); \bigcap_J K_i, \bigcap_J R_i).$$

For an arbitrary system $(N_i)_{i \in J}$ of nebulae put

$$\bigtimes_J N_i = (\bigtimes_J (P_i, \nu_i); \bigtimes_J K_i, \bigtimes_J R_i).$$

Further, put

$$N_1 \otimes N_2 = ((P_1, \nu_1) \otimes (P_2, \nu_2); K_1 \times K_2, R_1 \times R_2).$$

4.5. For $N = ((P, \nu); K, R)$ put

$$\exp N = (\exp(p, \nu) \times \exp(P, \nu);\ \exp K \times \exp R,\ \exp R \times \exp R).$$

Thus, the kernel of exp N consists of all the nebulae included in N, the range consists of these weakly included there. (Here, of course, the subnebulae are represented simply by their kernels and ranges, the universe being silently understood as the original one.)

§ 5. L-nebulae as L-fe sets an vice versa. An L-nebula can be viewed as an L-fe-set in a very natural way (see 5.1). In this section we will show that, on the other hand, any saturated L-fe set can be represented by a nebula and, moreover, there is a representation such that the operations with the L-fe sets get represented as the corresponding operations with nebulae. Thus the (more complicated, since involving counting in L) L-fe set operations reduce, in principle, to operating with couples of crisp sets (see 5.14 and 5.15 below).

5.1. For a nebula $N = ((P, \nu); K, R)$ define an L-fe set $\hat{\mathcal{F}}(N)$ by putting

$\hat{\mathcal{F}}(N) = (\mathcal{F}(N), \nu \mid R)$ where $|\mathcal{F}(N)| = R$, $\mathcal{F}(N)(x) = \nu(x, K)$.

We say that N <u>realizes</u> an L-fe set (X,ν) if $(X,\nu) = \hat{\mathcal{F}}(N)$.

5.2. <u>Proposition</u>: Each $\hat{\mathcal{F}}(N)$ is saturated.

Proof: We have $\nu(x,y) \cdot \nu(y,K) \leq \nu(x,K)$ by 2. □

5.3. <u>Convention</u>: In spaces $P \otimes L$ we replace the points $(x,1)$ by x. Thus, P is embedded into $P \otimes L$ as a subspace instead of the $P \times \{1\}$ level.

5.4. For a saturated L-fe set (x,ν) put

$$\mathcal{N}(X,\nu) = ((|X|,\nu) \otimes L; K(X), |X|)$$

(K(X) from 3.7).

5.5. <u>Lemma</u>: In a saturated (X,ν) one has

$$\nu(x,y).h(a,X(y)) \leq h(a,X(x)).$$

Proof: We have $X(y).\nu(x,y) \leq X(x)$, hence $X(y) \leq h(\nu(x,y), X(x))$. Hence further, by 1.1.(iii), 1.2.(4) and the commutativity of the multiplication, $h(a,X(y)) \leq h(a,h(\nu(x,y),X(x)) = h(\nu(x,y).h(a,X(x))$ and by 1.2.(4) again $h(a,X(y)).\nu(x,y) \leq h(a,X(x))$. □

5.6. Proposition: Let (X,ν) be saturated. Then one has in $(|X| \times L, \bar{\nu}) = (|X|,\nu) \otimes L$ the equality

$$\bar{\nu}((x,a).K(X)) = h(a,X(x)).$$

Proof: Consider a $(y,b) \in K(X)$. We have $b \leq K(y)$ and hence, by 6.5,

$$\bar{\nu}((x,a),(y,b)) = \nu(x,y).\nu_0(a,b) \leq \nu(x,y).h(a,b) \leq$$
$$\leq \nu(x,y).h(a,X(y)) \leq h(a,X(x)).$$

Thus, $\bar{\nu}((x,a),K(X) \leq h(a,X(x))$. On the other hand, $(x,X(x)\wedge a)$ is in K(X) and we have by 1.4.(5)

$$\bar{\nu}((x,a),(x,X(x)\wedge a)) = \nu_0(a,X(x)\wedge a) = h(a,X(x)\wedge a) =$$
$$= h(a,X(x)).$$ □

5.7. Theorem: If (X, ν) is saturated, then

$$\hat{\mathfrak{F}} \mathcal{N}(X, \nu) = (X, \nu).$$

Thus, each saturated L-fe set is realized by an L-nebula.

Proof: (Recall Convention 5.3.) By 5.6 and 1.4.(3),

$$\bar{\nu}(x, K(X)) = h(1, X(x)) = X(x). \quad \square$$

5.8. Remark: From the proof of 5.6 one sees that $X(x) = \bar{\nu}(x, K(X)) = \bar{\nu}(x, (x, X(x)))$. $\square$

5.9. The following is evident

Proposition: $(X, \nu) \overset{w}{\subset} (Y, \mu) \Longrightarrow \mathcal{N}(X, \nu) \overset{w}{\subset} \mathcal{N}(Y, \nu)$.

$$N \overset{w}{\subset} M \Longrightarrow \hat{\mathfrak{F}}(N) \overset{w}{\subset} \hat{\mathfrak{F}}(M). \quad \square$$

5.10. Proposition: $\mathcal{N}(\bigcap_J (X_i, \nu_i)) = \bigcap_J \mathcal{N}(X_i, \nu_i)$.

Proof: The coincidence of the universa is obvious. Further, we have

$(x,a) \in \bigcap K(X_i)$ iff $\forall i \; X_i(x) \geq a$ iff $a \leq \inf X_i(x) = (\bigcap X_i)(x)$ iff iff $(x,a) \in K(\bigcap (X_i, \nu_i))$. $\square$

5.11. Remark: One does not have, in general, $\mathcal{N}(\bigcup (X_i, \nu_i)) = \bigcup \mathcal{N}(X_i, \nu_i)$. In case of a linearly ordered L, however, this equality holds at least for finite J.

5.12. Proposition: (1) $\hat{\mathfrak{F}}(\bigcup_{J}{}_{N} N_i) = \bigcup_{J \hat{\mathfrak{F}}(N)} \hat{\mathfrak{F}}(N_i)$.

(2) $\hat{\mathfrak{F}}(N_1 \otimes N_2) = \hat{\mathfrak{F}}(N_1) \otimes \hat{\mathfrak{F}}(N_2)$.

Proof: The coincidence of the carrying universa is evident. Further, we have by 2.3

$$\hat{\mathfrak{F}}(\bigcup N_i)(x) = \nu(x, \bigcup K_i) = \sup \nu(x, K_i)$$

which yields (1). Finally, we have (use 1.2.(1))

$\hat{\mathfrak{F}}(N_1 \otimes N_2)(x_1, x_2) = \sup\{\nu_1(x_1, y_1) . \nu_2(x_2, y_2) \mid (y_1, y_2) \in K_1 \times K_2\} = \sup_{y_1 \in K_1} (\sup_{y_2 \in K_2} (\nu_1(x_1, y_1) . \nu_2(x_2, y_2)) = \sup_{y_1 \in K_1} (\nu_1(x_1, y_1) . \sup_{y_2 \in K_2} \nu_2(x_2, y_2)) =$

$= (\sup_{y_1 \in K_1} \nu_1(x_1,y_1)).(\sup_{y_2 \in K_2} \nu_2(x_2,y_2)) = \hat{\mathcal{F}}(N_1)(x_1) . \hat{\mathcal{F}}(N_2)(x_2)$
which yields (2). □

5.13. Lemma : Let $N_i = ((P_i , \nu_i);K_i,R_i)$ be such that for each $x \in R_i$ there is an $\bar{x} \in K_i$ such that $\nu_i(x,K_i) = = \nu_i(x,\bar{x})$. Then

$$\hat{\mathcal{F}}(\mathop{\times}_J N_i) = \mathop{\times}_J \hat{\mathcal{F}}(N_i).$$

Proof: Again, the coincidence of the carriers is evident. Denote by ν the nearness in $\times P_i$. We have, for $x = (x_i) \in \in \times R_i$,

$a = \hat{\mathcal{F}}(\times N_i)(x) = \nu(x, \times K_i) = \sup\{\nu((x_i),(y_i)) \mid (y_i) \in \in \times K_i\} = \sup\{\inf_{i \in J} \nu_i(x_i,y_i)) \mid (y_i) \in \times K_i\},$

$b = (\times \hat{\mathcal{F}}(N_i))(x) = \inf \nu_i(x_i,K_i) = = \inf_{i \in J} \sup\{\nu_i(x_i,y_i) \mid y_i \in K_i\}$

and consequently $a \leq b$.

On the other hand, consider $\bar{x} = (\bar{x}_i)$. We have

$$\sup\{\nu_i(x_i,y_i) \mid y_i \in K_i\} = \nu_i(x_i,\bar{x}_i)$$

and hence

$$\inf_{i \in J} \nu_i(x_i,\bar{x}_i) \leq a \leq b = \inf_{i \in J} \nu_i(x_i,\bar{x}_i). \quad □$$

5.14. Theorem: For $(X_i, \nu_i) \overset{w}{\subset} (X,\nu)$ we have

(1) $\hat{\mathcal{F}}(\bigcap_{i \in J} \mathcal{N}(X_i, \nu_i)) = \bigcap_{i \in J}(X_i, \nu_i)$,

(2) $\hat{\mathcal{F}}(\bigcup_{i \in J \mathcal{N}(X,\nu)} \mathcal{N}(X_i, \nu_i)) = \bigcup_{i \in J (X,\nu)}(X_i, \nu_i)$.

Generally

(3) $\hat{\mathcal{F}}(\mathop{\times}_{i \in J} \mathcal{N}(X_i, \nu_i)) = \mathop{\times}_{i \in J}(X_i, \nu_i)$.

(4) $\hat{\mathcal{F}}(\mathcal{N}(X_1, \nu_1) \otimes \mathcal{N}(X_2, \nu_2)) = (X_1, \nu_1) \otimes (X_2, \nu_2)$.

Proof: The expressions on the left hand side in (1) and (2) make sense by 5.9.

(1): $\bigcap(X_1, \nu_1) = \hat{\mathfrak{F}}\mathcal{N}(\bigcap(X_i, \nu_i)) = \hat{\mathfrak{F}}\bigcap\mathcal{N}(X_i, \nu_i))$
by 5.7 and 5.10.

(2): $\hat{\mathfrak{F}}(\bigcup\mathcal{N}(X_i, \nu_i)) = \bigcup\hat{\mathfrak{F}}\mathcal{N}(X_i, \nu_i) = \bigcup(X_i, \nu_i)$
by 5.12.(1) and 5.7.

(3): $\hat{\mathfrak{F}}(\bigtimes\mathcal{N}(X_i, \nu_i)) = \bigtimes\hat{\mathfrak{F}}\mathcal{N}(X_i, \nu_i) = \bigtimes(X_i, \nu_i)$ by 5.8, 5.13 and 5.7.

(4): $\hat{\mathfrak{F}}(\mathcal{N}(X_1, \nu_1) \otimes \mathcal{N}(X_2, \nu_2)) = \hat{\mathfrak{F}}\mathcal{N}(X_1, \nu_1) \otimes$
$\otimes \hat{\mathfrak{F}}\mathcal{N}(X_2, \nu_2) = (X_1, \nu_1) \otimes (X_2, \nu_2)$ by 5.12 and 5.7.

5.15. Before formulating a statement on exponentiation analogous to the statements concerning the other operations, let us realize that it cannot be of the form $\hat{\mathfrak{F}}(\exp \mathcal{N}(X, \nu)) = \exp (X, \nu)$. The carrier of $\hat{\mathfrak{F}}(\exp \mathcal{N}(X, \nu))$ is a set of L-nebulae, that of $\exp (X, \nu)$ is a set of L-fe sets. Thus, one has to have a translation of fe-subsets to subnebulae. Such a translation really exists and, moreover, it is coherent with the construction $\mathcal{N}$. We have

<u>Theorem</u>: Define a mapping

$$\mathcal{N}_1 : |\exp (X, \nu)| \longrightarrow R(\exp \mathcal{N}(X, \nu))$$

by putting $\mathcal{N}_1(Y, \nu') = (K(Y), |Y|)$ for $(Y, \nu') \overset{w}{\subset} (X, \nu)$. Then

(1) $\mathcal{N}(Y, \nu') = ((|Y|, \nu') : \mathcal{N}_1(Y, \nu))$,

(2) $\mathcal{N}_1$ preserves the L-nearness, and

(3) $\hat{\mathfrak{F}}(\exp \mathcal{N}(X, \nu)) \circ \mathcal{N}_1 = \exp (X, \nu)$.

<u>Proof</u>: (1) is obvious.

(2): Let us denote the L-nearness in $(|X|, \nu) \otimes L$ by ν_1. Recall that the L-nearness in $|\exp (X, \nu)|$ (see 4.8) is given by

$$\mu(Y_1, Y_2) = \tilde{\nu}_1(K(Y_1), K(Y_2)).$$

Now, for the nearness μ' in $R(\exp \mathcal{N}(X,\nu))$ we also have (recall 4.5)

$$\mu'(\mathcal{N}_1(Y_1), \mathcal{N}_1(Y_2)) = \tilde{\nu}_1(K(Y_1),K(Y_2)) \wedge \tilde{\nu}_1(|Y_1|,|Y_2|) = $$
$$= \tilde{\nu}_1(K(Y_1),K(Y_2)),$$

since obviously $\tilde{\nu}_1(K(Y_1),K(Y_2)) \leq \tilde{\nu}_1(|Y_1|,|Y_2|)$.

(3): In the product of two spaces we have $\nu((u,v),U\times V) = \nu(u,U)$ whenever $v \in V$. Thus,

$$\hat{\mathcal{F}}(\exp \mathcal{N}(X,\nu))(\mathcal{N}_1(Y,\nu')) = \tilde{\nu}_1(K(Y), \exp K(X)).$$

Take an $M \subset K(X)$. We have

$\tilde{\nu}_1(K(Y),M) = \inf\{\nu_1((x,a),M) \mid (x,a)\in K(Y)\}.\inf\{\nu_1((y,b), K(Y)) \mid (y,b)\in M\} \leq \inf\{\nu_1((x,a),M) \mid (x,a)\in K(Y)\} \leq$

$\leq \inf\{\nu_1((x,a),K(X)) \mid (x,a)\in K(Y)\}$.

Thus, by 5.6 and 3.9,

$$\tilde{\nu}_1(K(Y),M) \leq \inf\{h(a,X(x)) \mid a \leq Y(x)\} = \inf\{h(Y(x),X(x)) \mid x \in |Y|\} = \exp(X,\nu)(Y).$$

On the other hand, we have by 6.10, 6.6 and 3.9

$$\tilde{\nu}_1(K(Y),K(X)\cap K(Y)) = \tilde{\nu}_1(K(Y),K(X\cap Y)) =$$
$$= \inf\{\nu_1((x,a),K(X\cap Y)) \mid (x,a)\in K(Y)\} =$$
$$= \inf\{h(a,X(x)\wedge Y(x) \mid (x,a)\in K(Y)\} =$$
$$= \inf_x h(Y(x),X(x)\wedge Y(x)) = \inf_x h(Y(x),X(x)) =$$
$$= \exp(X,\nu)(Y).$$

Thus, $\hat{\mathcal{F}}(\exp \mathcal{N}(X,\nu))(\mathcal{N}_1(Y)) = \exp(X,\nu)(Y)$. □

References

[1] DILWORTH R.P. and M. WARD: Residuated lattices, Trans. Amer. Math. Soc. 45(1939), 335-354.

[2] GOGUEN J.A.: L-fuzzy sets, J. Math. Anal. Appl. 18(1967), 145-174.

[3] GOGUEN J.A.: The logic of inexact concepts, Synthese 19 (1968/69), 325-373.

[4] HAUSDORFF F.: Grundzüge der Mengenlehre, Leipzig 1914.

[5] HERRLICH H.: Topological structures, Math. Centre Tracts 52(1974), 59-122.

[6] ŁUKASIEWICZ J.: Selected works. Studies in Logic and Foundations of Mathematics, North-Holland Publ. Comp., Amsterdam, and PWN, Warszawa, 1970.

[7] PAVELKA J.: On L-fuzzy semantics, Weiterbildungszentrum für Kybernetik u. Rechentechnik, TU Dresden, Heft 21(1977), 12-25.

[8] PAVELKA J.: On Fuzzy Logic I,II,III, Zeitschrift f. math. Logik u. Grundl. d. Math. Bd. 25(1979), 45-52, 119-134, 447-464.

[9] PULTR A.: Closed Categories of L-Fuzzy Sets, Weiterbildungszentrum f. math. Kybernetik u. Rechentechnik, TU Dresden, Heft 16(1976), 60-68.

[10] ZADEH L.A.: Fuzzy sets, Information and Control 8(1965), 338-353.

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