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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
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GENERIC DIFFERENTIABILITY OF MAPPINGS AND CONVEX FUNCTIONS IN BANACH AND LOCALLY CONVEX SPACES

LE VAN HOT

Abstract: Generic Fréchet-differentiability of mappings and convex functions defined on Banach and locally convex spaces is investigated. In particular, the Fréchet and Gâteaux differentiability of Hammerstein operators is also considered.

Key words: Differentiability, mappings, convex functions, Asplund spaces, Banach and locally convex spaces.

Classification: Primary 58C20, 58C25
Secondary 47H99

Introduction. The first important contribution to differentiability of convex functions has been given by Asplund [3]. He has shown that each Banach space X, which admits an equivalent norm such that the corresponding dual norm in X^* is locally uniformly rotund is a strong differentiability space. Further conditions have been obtained also for weak differentiability spaces. The properties of the so called Asplund spaces have been intensively studied in [1], [5], [8], [11], [12], [17], [21], [24]. For the differentiability properties of Hammerstein and nonlinear operators, we refer the readers for instance to [15], [19], [25].

First section deals with the generic Fréchet-differentiability of convex functions defined on a product space X =

$= \prod_{\lambda \in \Gamma} X_\lambda$, where $(X_\lambda : \lambda \in \Gamma)$ is a family of Asplund spaces, and of finite convex weakly continuous functions defined on a locally convex space. Section 2 is devoted to generic Fréchet-differentiability of the class of mappings acting from a Banach space into another Banach space. In the last section we discuss generic Gâteaux and Fréchet-differentiability of Hammerstein operators.

1. Generic Fréchet-differentiability of convex functions defined on locally convex spaces

Lemma 1. Let X be a topological space and T be a subset of X such that for each open nonempty subset G of X there exists a nonempty G_δ-subset $T_G \subseteq T$ with the following property $T_G \subseteq \operatorname{int} \overline{T}_G \subseteq G$. Then there exists a dense G_δ-subset $A \subseteq T$.

Proof: Put $\mathcal{N} = \{S \subseteq T;\ S$ is a G_δ-subset and $S \subseteq \operatorname{int} \overline{S}\}$, $\mathfrak{M} = \{\mathcal{C} \subseteq \mathcal{N};\ \operatorname{int} \overline{S}_1 \cap \operatorname{int} \overline{S}_2 = \emptyset$ for all $S_1, S_2 \in \mathcal{C}$, $S_1 \neq S_2\}$. We write $\mathcal{C}_1 \preceq \mathcal{C}_2$ iff $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Then " $\preceq$ " is a partial order on $\mathfrak{M}$. It is easy to see that there exists a maximal element $\mathcal{L}$ of $\mathfrak{M}$. Put $A = \bigcup\{S : S \in \mathcal{L}\}$. Since every such S is a G_δ-subset, there exists a sequence of open subsets $G_{S,n}$ such that $S = \bigcap_1^\infty G_{S,n}$ for each $S \in \mathcal{L}$. Without loss of generality we can suppose that $G_{S,n} \subseteq \operatorname{int} \overline{S}$ for $n = 1,2,\dots$. Put $G_n = \bigcup\{G_{S,n} : S \in \mathcal{L}\}$. Then G_n is open for all $n = 1,2,\dots$. We claim that $A = \bigcap_1^\infty G_n$. It is clear that $A \subseteq \bigcap_1^\infty G_n$. Now if $x \notin A$, then $x \notin S$ for all $S \in \mathcal{L}$. If $x \notin \operatorname{int} \overline{S}$ for all $S \in \mathcal{L}$, then of course $x \notin G_n$ for all $n = 1,2,\dots$. Therefore $x \notin \bigcap_1^\infty G_n$. If $x \in \operatorname{int} \overline{S}_0$ for $S_0 \in \mathcal{L}$, then $x \notin G_{S,n} \subseteq \operatorname{int} \overline{S}$ for all $S \in \mathcal{L}$, $S \neq S_0$, $n = 1,2,\dots$. However, $x \notin S_0 = \bigcap G_{S_0,n}$, there exists an integer n_0 such that $x \notin G_{S_0,n_0}$. Hence

$x \notin G_{n_0} = \cup \{G_{S,n_0} : S \in \mathcal{L}\}$. This proves that $A = \cap G_n$. It follows that $A \in \mathcal{H}$. To finish the proof of the lemma, we must prove that $\bar{A} = X$. Suppose that our claim is false, then $X \setminus \bar{A}$ is a nonempty open subset of X. By the assumption there exists a G_δ-subset $M \subseteq T$ such that $M \subseteq \operatorname{int} \bar{M} \subseteq X \setminus \bar{A}$. Then $M \in \mathcal{H}$ and $\operatorname{int} \bar{M} \cap \operatorname{int} \bar{S} = \emptyset$ for all $S \in \mathcal{L}$. This implies that $\mathcal{L} \cup \{M\} \in \mathcal{M}$ which contradicts the assumption that $\mathcal{L}$ is a maximal element of $\mathcal{M}$. This completes the proof.

Now let X be a topological vector space, S be a family of bounded subsets of X, In this paper we always assume that S possesses the following properties:

a) If $A, B \in S$ then there exists a $C \in S$ such that $A \cup B \subseteq C$.

b) $\cup \{\lambda A : A \in S, \lambda \in R_+\} = X$.

Definition 1 ([26]). Let X, Y be topological vector spaces, f be a mapping from an open subset Ω of X into Y. We say f is S-differentiable at $x_0 \in \Omega$ if there exists a linear continuous mapping $T \in L(X,Y)$ such that $t^{-1}(f(x_0+th) - f(x_0))$ converges uniformly to $T(h)$ on each subset $A \in S$ when $t \to 0$, i.e. for each 0-neighborhood V of Y and $A \in S$ there exists a $\delta > 0$ such that $t^{-1}(f(x_0+th) - f(x_0)) - T(h) \in V$ for all $h \in A$, $t: 0 < |t| < \delta$.

If S is the family of all finite subsets of X, then f is said to be Gâteaux-differentiable at x_0.

If S is the family of all bounded subsets of X, the f is said to be Fréchet-differentiable at x_0.

Remark. If X is a normed space then without loss of generality we can suppose that every subset A from S is contained in the unit ball of X.

Definition 2. Let X, Y be topological vector spaces, f be a continuous mapping from an open subset Ω of X into Y. f is called generic S-differentiable if there exists a dense G_δ-subset A of Ω such that f is S-differentiable at every point $x \in A$.

Definition 3. A Banach space X is called S-differentiability space if each continuous convex finite function defined on an open convex subset of X is S-differentiable on a dense G_δ-subset of its domain.
Fréchet- (Gâteaux- resp.) differentiability spaces are known as Asplund (weak Asplund resp.) spaces.

Stegall [27] has proved that a Banach space X is Asplund if and only if its dual X^* has the Radon-Nikodym property. Then it is easy to see that a finite product of Asplund spaces is Asplund.

Theorem 1. Let $(X_\lambda : \lambda \in \Gamma)$ be a family of Asplund spaces. Then each continuous convex function f defined on an open convex subset Ω of $X = \prod_{\lambda \in \Gamma} X_\lambda$ is generic Fréchet-differentiable.

Proof. Put $p_I((x_\lambda)) = \max\{\|x_\lambda\| : \lambda \in I\}$ for all $(x_\lambda) \in X$ and each finite subset $I \subseteq \Gamma$. Then $\{p_I\}_I$ is a family of continuous seminorms on X which induces the locally convex product topology of X. Let G be any open nonempty subset of Ω, therefore G is open in X, since Ω is open. To prove Theorem 1, by Lemma 1, it is sufficient to prove that there exists a G_δ-subset M such that $M \subseteq \operatorname{int} \overline{M} \subseteq G$ and that f is Fréchet-differentiable at every point $x \in M$. Take $x_0 \in G$. Since f is continuous at x_0, there exist a $\delta > 0$ and a finite subset

$I \subseteq \Gamma$ such that $U = \{x \in X: p_I(x - x_0) < \sigma\} \subseteq G$ and $|f(x) - f(x_0)| \leq 1$ for all $x \in U$. We claim that $|f(x_1) - f(x_2)| \leq \frac{2}{\sigma_1} p_I(x_1 - x_2)$ for all $x_1, x_2 \in U$, where $\sigma_1 = \sigma - \max\{p_I(x_1 - x_0), p_I(x_2 - x_0)\}$. Put $h = x_1 - x_2$. We have that

a) If $p_I(h) = 0$, then from the convexity of f we deduce that $f(x_1) - f(x_2) = f(x_2 + h) - f(x_2) \leq s^{-1}[f(x_2 + sh) - f(x_2)]$ for all $s \geq 1$. Now $p_I(h) = 0$ implies that $x_2 + sh \in U$ for all $s \in R$. Hence $f(x_1) - f(x_2) \leq \lim_{s \to \infty} s^{-1}[f(x_2 + sh) - f(x_2)] \leq \lim_{s \to \infty} 2s^{-1} = 0$. Similarly $f(x_2) - f(x_1) \leq 0$. Therefore $|f(x_2) - f(x_1)| \leq 2 \cdot \sigma_1^{-1} p_I(x_2 - x_1)$.

b) Suppose that $p_I(h) = r > 0$. If $r \geq \sigma_1$, then $|f(x_2) - f(x_1)| \leq 2 \leq 2\sigma_1^{-1} \cdot r = 2 \cdot \sigma_1^{-1} p_I(h)$. If $r < \sigma_1$, put $h_0 = \sigma_1 r^{-1} h$; then $x_i \pm h_0 \in \overline{U}$, $i = 1,2$, and $f(x_1) - f(x_2) \leq r\sigma_1^{-1}[f(x_2 + h_0) - f(x_2)] \leq 2r\sigma_1^{-1} = 2\sigma_1^{-1} = 2\sigma_1^{-1} p_I(x_1 - x_2)$. Similarly $f(x_2) - f(x_1) \leq 2\sigma_1^{-1} p_I(x_1 - x_2)$. This proves our claim.

Put $Y_I = \prod_{\lambda \in I} X_\lambda$, $\|(x_\lambda)\|_I = \max\{\|x_\lambda\| : \lambda \in I\}$ for all $(x_\lambda) \in Y_I$ and each finite subset $I \subseteq \Gamma$, $X_I = \{(x_\lambda) \in X: x_\lambda = 0$ for all $\lambda \notin I\}$. Let J_I be an embedding mapping of Y_I into X defined by $J_I((x_\lambda)) = (y_\lambda)$, where $y_\lambda = x_\lambda$ for all $\lambda \in I$; $y_\lambda = 0$ for $\lambda \notin I$. Then J_I is an isomorphism of Y_I onto X_I and $\|(x_\lambda)\|_I = p_I(J_I(x_\lambda))$ for all $(x_\lambda) \in Y_I$. Let P_I be the canonical projection of X onto X_I. Put $Q_I = J_I^{-1} \circ P_I : X \to Y_I$ and $f_I = f \circ J_I : V = Q_I(U) \to R$. Then it is clear that f_I is a continuous convex function on V and $f(x) = f_I(Q_I(x)) = f(P_I(x))$ for all $x \in U$ because $p_I(x - P_I(x)) = 0$ whenever $x \in U$. Since X_λ is an Asplund space for all $\lambda \in \Gamma$, Y_I is

Asplund for each finite subset $I \subseteq \Gamma$. Therefore there exists a G_δ-subset M of Y_I which is dense in V such that f_I is Fréchet-differentiable at every point of M. Put $N = Q_I^{-1}(M) = P_I^{-1} \circ J_I(M) \subseteq U$. One can verify that N is a G_δ-subset of X and $N \subseteq \operatorname{int} \overline{N} = Q_I^{-1}(\operatorname{int} \overline{M}) = U$. Now we claim that f is Fréchet-differentiable at every point $x \in N$. Let x be any fixed point of N, D a bounded subset of X, ε a given positive number. Then $Q_I(x) \in M$ and there exists a number $K > 0$ such that $p_I(h) \leq K$ for all $h \in D$. Let $T \in Y_I^*$ be the Fréchet-derivative of f_I at $Q_I(x)$. Then there exists a $\delta_0 > 0$, $\delta_0 < \delta - p_I(x)$ such that $|f_I(Q_I(x) + k) - f_I(Q_I(x)) - T(k)| \leq \varepsilon \cdot K^{-1} \| k \|_I$ for $\| k \|_I < \delta_0$. Let Q_I^* be the adjoint of Q_I. Put $S = Q_I^*(T) \in X^*$. Now take $t \in R$ such that $0 < |t| < \delta_0 K^{-1}$. Then $p_I(th) < \delta_0$, $x + th \in U$ and

$|f(x + th) - f(x) - S(th)| = |f_I(Q_I(x + th)) - f_I(Q_I(x)) - T(Q_I(th))| \leq \varepsilon \cdot K^{-1} \| Q_I(th) \|_I = \varepsilon \cdot K^{-1} p_I(P_I(th)) = \varepsilon \cdot K^{-1} p_I(th) < \varepsilon |t|$ for all $h \in D$. This proves that f is Fréchet-differentiable at $x \in N$, which finishes the proof of Theorem 1.

__Theorem 2.__ Let X be a locally convex space and f be a $\sigma(X,X^*)$ – continuous convex function defined on a weakly open convex subset Ω of X, where $\sigma(X,X^*)$ denotes the weak topology on X. Then f is generic Fréchet-differentiable.

__Proof.__ Let G be an open nonempty subset of Ω, therefore G is open since Ω is open, $x_0 \in G$. Since f is $\sigma(X,X^*)$-continuous at x_0, there exist $x_1^*, x_2^* \ldots x_n^* \in X^*$ and a $\delta > 0$ such that $U = \{x \in X : |\langle x_i^*, x - x_0 \rangle| < \delta,\ i = 1,2 \ldots n\} \subseteq \Omega$ and $|f(x) - f(x_0)| \leq 1$ for all $x \in U$. Put $p(x) = \max \{|\langle x_i^*, x \rangle| : i =$

$= 1,2\ldots n\}$, for all $x \in X$. Then p is a continuous seminorm on $(X, \sigma(X,X^*))$ and $U = \{x \in X : p(x_0 - x) < \delta\}$. Using the same argument as in the proof of Theorem 1, one can verify that $|f(x_1) - f(x_2)| \leq 2\delta_1^{-1} p(x_1 - x_2)$ for all $x_1, x_2 \in U$, where $\delta_1 = \delta - \max\{p(x_1 - x_0), p(x_2 - x_0)\}$. Put $V = \bigcap_1^n \ker x_i^*$. Then V is a closed finite codimensional subspace of X. There exists a continuous projection $Q : X \to V$. Put $M = \ker Q$, $P = I - Q$, then $X = M \oplus V$ and M is a finite dimensional subspace of X. Let $\{x_1, x_2 \ldots x_k\}$ be a basis of M. Since $G \cap U$ is a neighborhood of x_0, there exist a convex open neighborhood U_1 of $P(x_0)$ in M and a convex open neighborhood O_1 of $Q(x_0)$ in V such that $G_1 = U_1 + O_1 \subseteq G \cap U$. Let $J : R^k \to X$ be the mapping defined by $J(a_1, a_2 \ldots a_k) = \sum_1^k a_i x_i$ for all $a = (a_1, \ldots, a_k) \in R^k$. Then J is a topological isomorphism of R^k onto M. Put $S = J^{-1} \circ P : X \to R^k$, $g(a) = f(J(a) + Q(x_0))$ for all $a \in S(G_1) = J^{-1}(U_1)$. It is easy to see that $P(x) + Q(x_0) \in G_1 \subseteq U$ whenever $x \in G_1$. Then g is a continuous convex function defined on $S(G_1)$ and $|f(x) - f(P(x) + Q(x_0))| \leq 2\delta_1^{-1} p(x - P(x) - Q(x_0)) = 2\delta_1^{-1} p(Q(x) - Q(x_0)) = 0$, where $\delta_1 = \delta - \max\{p(x - x_0), p(P(x) + Q(x_0) - x_0)\} > 0$. Hence $f(x) = f(P(x) + Q(x_0)) = g(S(x))$ for all $x \in G_1$. There exists a dense G_δ-subset A in an open set $J^{-1}(U_1)$ such that g is Fréchet-differentiable at every point $a \in A$. Similarly as in the proof of Theorem 1, we can prove that f is Fréchet-differentiable at every point $x \in J(A) + O_1$ and $f'(x) = S^*(g'(S(x)))$ for all $x \in J(A) + O_1$. It is clear that $J(A) + O_1$ is a G_δ-subset of X and $J(A) + O_1 \subseteq \operatorname{int}(\overline{J(A) + O_1}) = U_1 + O_1 = G_1$. By Lemma 1, this concludes the proof.

2. Generic differentiability of mappings. In this section, X always denotes a Banach space, S denotes a family of subsets contained in the unit ball of the space X with the properties a) and b) introduced in Section 1.

Definition 4. Let X, Y be Banach spaces, Ω be an open subset of X, f be a mapping from Ω to Y. We say that f is Lipschitzian at a point $x_0 \in \Omega$ if there exist a $K>0$ and $\delta > 0$ such that $\|f(x) - f(y)\| \leq K\|x - y\|$ for all $x, y \in \Omega$, $\|x - x_0\| < \delta$, $\|y - x_0\| < \delta$.

f is said to be locally Lipschitzian if f is Lipschitzian at every point $x \in \Omega$.

Definition 5. Let $\varepsilon > 0$ be a fixed positive number. We say that f is locally (ε, S)-approximated at $x \in \Omega$ if for each $A \in S$ there exist $T_A \in L(X,Y)$ and $\delta > 0$ such that:

(1) $\|f(x + th) - f(x) - T_A(th)\| < \varepsilon |t|$ for all t: $|t| < \delta$ and $h \in A$. Denote by $S_\varepsilon(f,x,A)$ the set of all $T \in L(X,Y)$ such that (1) holds for some $\delta > 0$.

Lemma 2. Let f be Lipschitzian at $x \in \Omega$. Then f is S-differentiable at x if and only if f is (ε, S)-approximated at x for all $\varepsilon > 0$.

Proof. 1) If f is S-differentiable at x, then it is clear that f is (ε, S)-approximated for all $\varepsilon > 0$.

2) Now let f be (ε, S)-approximated at x for all $\varepsilon > 0$. Put $S_\varepsilon(f,x,A)(h) = \{T(h) : T \in S_\varepsilon(f,x,A)\}$ for all $h \in A$. It is easy to see that $\operatorname{diam} S_\varepsilon(f,x,A)(h) \leq 2\varepsilon$ for all $h \in A$ and $\varepsilon > 0$. Therefore there exists $T(h) = \bigcap_{\varepsilon > 0} S_\varepsilon(f,x,A)(h) = \lim_{t \to 0} t^{-1}[f(x + th) - f(x)]$ for all $h \in A$, and $\|T(h) - T_{\varepsilon,A}(h)\| \leq 2\varepsilon$ for all $T_{\varepsilon,A} \in S_\varepsilon(f,x,A)$ and $h \in A$. Hence, by

the property b), $\lim_{t \to 0} t^{-1} [f(x + th) - f(x)]$ exists for each $h \in X$. The additivity of T follows from the property a) of S and the boundedness of T follows from the assumption that f is Lipschitzian at x. This shows that $T \in L(X,Y)$. Now let $\varepsilon > 0$ be given and A be an arbitrary element of S. Take $T_1 \in S_{\varepsilon/3}(f,x,A)$. Then there exists a $\delta > 0$ such that $\| f(x + th) - f(x) - T_1(th)\| \leq \frac{\varepsilon}{3} |t|$ for all t: $|t| < \delta$ and $h \in A$. Hence $\| f(x + th) - f(x) - T(th)\| \leq \|f(x + th) - f(x) - T_1(th)\| + \|T_1(t) - T(th)\| < \varepsilon |t|$ for all t, $|t| < \delta$ and $h \in A$. This proves that f is S-differentiable at x, which concludes the proof of Lemma 2.

Proposition 1. Let X be the one of the following spaces: a Hilbert space, C(S) where S is a compact Hausdorff space, $L^p(\Omega, \Sigma, \mu)$, where μ is a positive σ-finite measure defined on a σ-algebra Σ of subsets of a set Ω, $1 \leq p < \infty$ and let X^* be the dual of X. Then X^* possesses the following property:

($*$) There exists a net of continuous linear projections $\{P_i\}_I$ of X^* onto finite dimensional subspaces of X^* such that:

1) $\| P_i\| \leq K$ for some $K > 0$ and all $i \in I$,

2) $\{x^* - P_i x^*\}$ converges weakly-star to 0 uniformly on $\{x^* \in X^*: \| x^* \| \leq 1\}$.

Proof. 1) Let X be a Hilbert space and $(e_\lambda)_{\lambda \in \Gamma}$ be an orthonormal basis of X. Let I be the family of all finite subsets i of Γ. We write $i_1 \leq i_2$ iff $i_1 \subseteq i_2$ for $i_1, i_2 \in I$. Let P_i be the orthogonal projection of $X^* = X$ onto $sp\{e_\lambda : \lambda \in i\}$ for all $i \in I$, where $sp\{e_\lambda : \lambda \in i\}$ denotes the linear hull of $\{e_\lambda : \lambda \in i\}$. Then it is clear that $\{P_i\}_I$ possesses the properties 1) and 2) with K = 1.

2) Let S be a compact Hausdorff topological space. We know that the dual space $C^*(S)$ of $C(S)$ is the space of all Radon measures on S, and denoted by $\mathcal{M}(S)$. Denote by μ_x the atomic measure defined by $\mu_x(A) = 1$ if $x \in A$, $\mu_x(A) = 0$ if $x \notin A$ for all Borel subsets A of S and $x \in S$. I denotes the family of all collections $(x_1,\dots,x_n;S_1,\dots,S_n)$ where $S_1,\dots,S_n$ is a disjoint partition of S into Borel subsets and $x_k \in S_k$ for $k = 1,\dots,n$. Let $i_1 = (x_k,\dots,x_n;S_1,\dots,S_n) \in I$; $i_2 = (y_1,\dots,y_m;T_1,\dots,T_m) \in I$. We write $i_1 \leq i_2$ iff for each $j: 1 \leq j \leq m$ there exists a $k(j)$: $1 \leq k(j) \leq n$ such that $T_j \subseteq S_{k(j)}$ and $x_k = y_j$ whenever $x_k \in T_j$. Put $Q_i = \operatorname{sp}\{\mu_{x_1},\dots,\mu_{x_n}\}$ and $P_i(\mu) = \Sigma_1^n \mu(S_k)\,\mu_{x_k}$ for all $i = (x_1,\dots,x_n;S_1,\dots,S_n) \in I$. Now we prove that $\{P_i\}_I$ possesses the properties 1) and 2) with $K = 1$. Let $\mu \in \mathcal{M}(S)$, then

$$\|P_i(\mu)\| = \sup \sum_j |P_i(\mu)(A_j)| = \sup \sum_j \Big|\sum_{k=0}^{n} \mu(S_k)\,\mu_{x_k}(A_j)\Big| \leq$$

$$\leq \sum_{k=0}^{n} |\mu(S_k)| \leq \|\mu\|$$

, for all $i \in I$, where the supremum is taken over the set of all finite collections $\{A_j\}$ of pairwise disjoint Borel subsets of S. Now let f be an arbitrary fixed continuous function defined on S, then f is uniformly continuous on S. It is easy to see that given $\varepsilon > 0$ there exists a finite partition $\alpha = (S_1,\dots,S_n)$ of S into Borel subsets such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in S_k$ for some $k = 1,\dots,n$. Let x_k be an arbitrary fixed point of S_k for $k = 1,\dots,n$. Put $i_0 = (x_1,\dots,x_n;S_1,\dots,S_n) \in I$. Now we claim that

$$|(\mu - P_i(\mu))(f)| = \Big|\int_S f(x)\, d\mu(x) - \int_S f(x)\, d(P_i(\mu)(x))\Big| \leq \varepsilon$$

for all $\mu \in \mathcal{M}(S)$, $\|\mu\| \leq 1$, $i \in I$, $i \geq i_0$ and this completes the proof for $X = C(S)$. Suppose $i = (y_1,\dots,y_m;T_1,\dots,T_m) \in I$, $i \geq i_0$, then it is clear that $|f(x) - f(y_j)| \leq \varepsilon$ for all $x \in T_j$, $j =$

$= 1,\dots,m$, and

$$|\int_S f(x)\, d\mu(x) - \int_S f(x)\, dP_i(\mu)(x)| = |\Sigma_1^m \int_{T_j} f(x)\, d\mu(x) - $$
$$- \int_{T_j} f(x)\, \mu(T_j)\, d\mu_{y_j}(x)| = |\Sigma_1^m \int_{T_j} (f(x) - f(y_j))\, d\mu(x)| \leq$$
$$\leq \Sigma_1^m \int_{T_j} |f(x) - f(y_j)|\, d|\mu|(x) \leq \varepsilon\, \|\mu\| \leq \varepsilon .$$

3) Let $X = L^p$, $1 \leq p < \infty$, then $X^* = L^q$, where $p^{-1} + q^{-1} = 1$. Let I be the family of all finite partitions $i = (E_1,\dots,E_n)$ of Ω such that $E_k \in \Sigma$, $\mu(E_k) > 0$ for all $k = 1,\dots,n$. We write $i_1 \leq i_2$ iff $F_j \subseteq E_k$ whenever $F_j \cap E_k \neq \emptyset$, for $j = 1,\dots,m$; $k = 1,\dots,n$; $i_1 = (E_1,\dots,E_n) \in I$, $i_2 = (F_1,\dots,F_m) \in I$. Put (taking $\frac{g(x)}{\infty} = 0$ for all $x \in \Omega$) $(P_i g)(x) =$

$$= \Sigma_1^n [\int_{E_k} \mu(E_k)^{-1}\, g(t)\, d\mu(t)]\ \chi_{E_k}(x).$$

We shall prove that $(P_i)_I$ possesses the properties 1) and 2) with $K = 1$.

If $q = \infty$, then it is clear that $\|P_i\, g\|_\infty \leq \|g\|$ for all $g \in L^q$.
Now let $1 < q < \infty$, $g \in L^q$, $i = (E_1,\dots,E_n) \in I$:

$$\|P_i g\|^q = \int_\Omega |\Sigma_1^n (\int_{E_k} \mu(E_k)^{-1}\, g(t)\, d\mu(t))\, \chi_{E_k}(x)|^q\, d\mu(x) =$$
$$= \int_\Omega \Sigma_1^n |\int_{E_k} \mu(E_k)^{-1}\, g(t)\, d\mu(t)|^q\, \chi_{E_k}(x)\, d\mu(x) =$$
$$= \Sigma_1^n \mu(E_k)^{1-q}\, |\int_{E_k} g(t)\, d\mu(t)|^q \leq \Sigma_1^n \mu(E_k)^{1-q} \,.$$
$$.(\int_{E_k} d\mu(t))^{qp^{-1}} . \int_{E_k} |g(t)|^q\, d\mu(t) = \int_\Omega |g(t)|^q\, d\mu(t) =$$
$$= \|g\|^q.$$

This proves that $\|P_i\| \leq 1$ for all $i \in I$.

Now we suppose that f be a fixed function from L^p. We shall prove that for each $\varepsilon > 0$ there exists an $i_0 \in I$ such that $|\int_\Omega f.g\, d\mu - \int_\Omega f.(P_i g)\, d\mu| \leq \varepsilon$ for all $i \in I$, $i \geq i_0$ and $g \in L^q$, $\|g\| \leq 1$.

Let $\varepsilon > 0$ be given. Then there exists a simple measurable function $f_o = \sum_1^m c_k . \chi_{E_k}$ such that $\| f - f_o \| \leq 2^{-1} . \varepsilon$. Without loss of generality we can suppose that $\bigcup_1^n E_k = \Omega$. Put $i_o = (E_1, \dots, E_n) \in I$. Let $g \in L^q$, $\|g\| \leq 1$ and $i \in I$, $i = (F_1, \dots, F_m) \geq i_o$. Then for each $k = 1, \dots, n$ there exists an $\alpha_k \subseteq \{1, \dots, m\}$ such that $E_k = \cup \{F_j : j \in \alpha_k\}$. Whence $r =$

$$= \left| \int_\Omega f(x).g(x)\, d\mu(x) - \int_\Omega f(x).(P_i g)(x)\, d\mu(x) \right| =$$

$$= \Big| \int_\Omega (f-f_o)(x) g(x) d\mu(x) + \sum_{k=1}^n \sum_{j \in \alpha_k} \int_{F_j} c_k g(x)\, d\mu(x) -$$

$$- \sum_{k=1}^n \sum_{j \in \alpha_k} \Big(\int_{F_j} \mu(F_j)^{-1} g(t) d\mu(t) \Big) . \Big(\int_{F_j} f(x) d\mu(x) \Big) \Big| \leq$$

$$\leq \|f-f_o\| \cdot \|g\| + \sum_{k=1}^n \sum_{j \in \alpha_k} \Big(\int_{F_j} \mu(F_j)^{-p^{-1}} |g(t)| d\mu(t) \Big) .$$

$$. \Big(\int_{F_j} \mu(F_j)^{-q^{-1}} |(c_k - f(x))| d\mu(x) \Big).$$

If $q = \infty$ then it is clear that $r \leq 2 \|f-f_o\| \cdot \|g\|$.

Suppose that $1 < q < \infty$. Then

$$r \leq \|f-f_o\| \; \|g\| + \Big(\sum_{k=1}^n \sum_{\alpha_k} \Big(\int_{F_j} \mu(F_1)^{-p^{-1}} |g(t)| d\mu(t) \Big)^q \Big)^{q^{-1}} .$$

$$. \Big(\sum_{k=1}^n \sum_{\alpha_k} \Big(\int_{F_j} \mu(F_j)^{-q^{-1}} |c_k - f(x)| d\mu(x) \Big)^p \Big)^{p^{-1}} \leq$$

$$\leq \|f-f_o\| \; \|g\| + \Big(\sum_{k=1}^n \sum_{\alpha_k} \Big(\int_{F_j} \mu(F_j)^{-1} d\mu(x) \Big)^{pq^{-1}} .$$

$$. \Big(\int_{F_j} | c_k - f(x)|^p d\mu(x) \Big) \Big)^{p^{-1}} . \Big(\sum_{k=1}^n \sum_{\alpha_k} \Big(\int_{F_j} \mu(F_j)^{-1} d\mu(x) \Big)^{qp^{-1}} .$$

$$. \Big(\int_{F_j} |g(t)|^q \Big) \Big)^{q^{-1}} = 2 \|g\| \; \|f-f_o\| .$$

This completes the proof of Proposition 1.

Remark 2. Let X be a Banach space. If its dual X^* has a net $(P_i)_I$ with the properties 1) and 2), then we say that X^* possesses the property $(*)$ with respect to $(P_i)_I$.

We shall use the following notations.

Let X, Y be Banach spaces, Ω be an open subset of X, f be a mapping of Ω into Y, $x \in \Omega$, $r>0$,

$A_r(f,x) = \{\|h\|^{-1} \Delta_h f(y): \|y-x\| \leq r,\ 0<\|h\| \leq r\}$

where $\Delta_h f(y) = f(y+h) - f(y)$,

$B_r(f,x) = \{f(x_1) + f(x_2) - 2f(\frac{x_1+x_2}{2}): x_i \in X,\ \|x_i-x\| \leq r,$
$i = 1,2\}$.

For $A \subseteq Y$, $\gamma(A)$ denotes the measure of noncompactness of A defined by $\gamma(A) = \inf\{t>0$: there exists a finite subset $C \subseteq A$ such that $A \subseteq C + tB_1\}$ where $B_1 = \{y \in Y: \|y\| \leq 1\}$. We use the symbol $A^\bullet$ defined by $A^\bullet = \{y^* \in Y^*: \langle y^*,y\rangle \geq 0$ for all $y \in A\}$.

Theorem 3. Let X be an S-differentiability Banach space and Y be a Banach space, whose dual Y^* possesses the property $(*)$ with respect to $(P_i)_{i\in I}$, Ω be an open subset of X. Let f be a mapping from Ω to Y such that:

1) $\lim_{r\to 0} (A_r(f,x)) = 0$ for all $x \in \Omega$,

2) for each open nonempty subset $G \subseteq \Omega$ and each $i \in I$ there exist an $x \in G$ and an $r>0$ such that $\overline{sp\ B_r^\bullet(f,x)} \supseteq P_i(Y^*)$. Then f is generic S-differentiable.

Proof. We denote the canonical embedding mapping of Y into its bidual Y^{**} by $\varkappa$. Let K be a positive number such that $\|P_i\| \leq K$ for all $i \in I$ and ε be an arbitrary given positive number. Put $T_\varepsilon = \{x \in \Omega$: f is (ε,S)-approximated at $x\}$. We shall prove that T_ε contains a dense G_δ-subset in Ω for all $\varepsilon > 0$. By Lemma 1, it suffices to prove that for each open nonempty subset $G \subseteq \Omega$ there exists a G_δ-subset $N \subseteq T_\varepsilon$ such that $N \subseteq \operatorname{int} \overline{N} \subseteq G$. Take an $x_0 \in G$. Since $\lim_{r\to 0} \gamma(A_r(f,x_0)) = 0$ there exists an $r>0$ such that $x \in G$ for all $x \in X$, $\|x-x_0\| < r$ and $\gamma(A_r(f,x_0)) < [4(K+1)]^{-1}\varepsilon$. Therefore there exist $y_1,\dots,y_k \in$ $\in Y$ such that $A_r(f,x) \subseteq \{y_1,\dots,y_k\} + [4(K + 1)]^{-1} \varepsilon B_1$, where

$B_1 = \{y \in Y: \|y\| \leq 1\}$. Put $K_1 = \max\{\|y_1\|, \dots, \|y_k\|\} + \varepsilon$. Then $\|y\| \leq K_1$ for all $y \in A_r(f,x_o)$ and hence f is Lipschitzian at x_o. Since Y possesses the property $(*)$ with respect to $(P_i)_I$, there exists an $i_o \in I$ such that $\langle y^* - P_i y, y_j \rangle \leq 4^{-1} \cdot \varepsilon$ for all $y^* \in Y^*$, $\|y^*\| \leq 1$; $j = 1,\dots,k$; $i \in I$, $i \geq i_o$. One can verify that $\langle y^* - P_i y^*, y \rangle \leq 2^{-1} \cdot \varepsilon$ for all $y^* \in Y^*$, $\|y^*\| \leq 1$; $y \in A_r(f,x_o)$, $i \in I$, $i \geq i_o$. On the other hand, by 2, there exist an $x_1 \in \{x: \|x - x_o\| < r\}$ and $r_1: 0 < r_1 < r - \|x_1 - x_o\|$ such that $Q_{i_o} = P_{i_o}(Y^*) \subseteq \overline{sp\, B^\bullet_{r_1}(f,x_1)}$. Let $\{e_1^*, \dots, e_n^*\}$ be a basis of the subspace Q_{i_o}, $\|e_j^*\| = 1$ for $j = 1,\dots,n$. Put $\|y^*\|_1 = \sum_1^n |\lambda_j|$ for $y^* = \sum_1^n \lambda_j e_j^* \in Q_{i_o}$. Then $\|\cdot\|_1$ is a norm on Q_{i_o} and it is equivalent with the norm $\|\cdot\|$ restricted to Q_{i_o}. Therefore there exist $K_2, K_3 > 0$ such that $K_2\|y^*\| \leq$ $\leq \|y^*\|_1 \leq K_3\|y^*\|$ for all $y^* \in Q_{i_o}$. Take $z_j^* \in sp\, B^\bullet_{r_1}(f,x_1)$ such that $\|e_j^* - z_j^*\| \leq [4K_1K_3K]^{-1}\varepsilon$, for $j = 1,\dots,n$. Since $z_j^* \in$ $\in sp\, B^\bullet_{r_1}(f,x_1)$ there exist $u^*_{j,1}, \dots, u^*_{j,k_j} \in B^\bullet_{r_1}(f,x_1)$ and $t_{j,1}, \dots, t_{j,k_j} \in R$ such that $z_j^* = \sum_{s=1}^{k_j} t_{j,s} u^*_{j,s}$ for $j = 1,\dots,n$. It is easy to see that $(u^*_{j,s}f)(v_1) + (u^*_{j,s}f)(v_2) - 2(u^*_{j,s}f)(\frac{v_1 + v_2}{2}) =$ $= \langle u^*_{j,s}, f(v_1) + f(v_2) - 2f(\frac{v_1+v_2}{2}) \rangle \geq 0$ for all $s = 1,\dots,k_j$; $j = 1,\dots,n$; $v_k \in X$, $\|v_k - x_1\| \leq r_1$, $k = 1,2$. Hence $u^*_{j,s} \circ f$ is a continuous midconvex (therefore convex) function on the open convex subset $U = \{x: \|x - x_1\| < r_1\} \subseteq G$, for $j = 1,\dots,n$; $s = 1,\dots$ $\dots,k_j$. Since X is an S-differentiability space, there exists a dense G_δ-subset $H_{j,s}$ of U such that $u^*_{j,s} \circ f$ is S-differentiable at every point $x \in H_{j,s}$ for all $j = 1,\dots,n$; $s = 1,\dots,k_j$. Put $N = \bigcap_{j=1}^n \bigcap_{s=1}^{k_j} H_{j,s} \subseteq G$. Then N is a G_δ-subset which is dense in U. It is clear that $z_j^* f = \sum_{s=1}^{k_j} t_{j,s} u_{j,s}^* \circ f$ is S-differentiable at every point $x \in N$ for $j = 1, \dots, n$. Now we prove that $\varkappa \circ f$ is (ε, S)-approximated at every point $x \in N$. Let w_j be a linear functional on Q_{i_0} defined by $w_j(y^*) = t_j$ for $y^* = \sum t_j e_j^* \in Q_{i_0}$, $j = 1, \dots, n$. Then of course we have $|w_j(y^*)| \leq \sum_1^n |w_j(y^*)| = \|y^*\|_1 \leq K_3 \|y^*\|$ for all $y^* \in Q_{i_0}$, $j = 1, \dots, n$. One can see that

$$\|P_{i_0} y^* - \sum_1^n w_j(P_{i_0} y^*) z_j^*\| = \|\sum w_j(P_{i_0} y^*)(e_j^* - z_j^*)\| \leq$$
$$\leq (4KK_1K_3)^{-1} \varepsilon K_3 \|P_{i_0} y^*\| \leq 4K_1^{-1} \cdot \varepsilon \|y^*\|.$$

Let x be an arbitrary fixed point of N. Denote the S-differential of the function $z_j^* \circ f$ at x by $d(z_j^* \circ f)(x)$ for $j = 1, \dots, n$. Let $K_4 = \max \{\|d(z_j^* \circ f)(x)\| : j = 1, \dots, n\}$. Then the functional $B(h, y^*)$ on $X \times Y$ defined by

$$B(h, y^*) = \sum_1^n w_j(P_{i_0} y^*) \cdot d(z_j^* \cdot f)(x)(h) \text{ for all } h \in X,\ y^* \in Y^*,$$

is bilinear. Furthermore, $|B(h, y^*)| \leq \sum |w_j(P_{i_0} y^*)| . \|d(z_j^* . f)(x)\| \cdot \|h\| \leq \sum K_4 \|h\| \, |w_j(P_{i_0} y^*)| \leq KK_3K_4 \|h\| \, \|y^*\|$.

This shows that $B(h, y^*)$ is continuous and for each fixed $h \in X$, $B(h, .) \in Y^{**}$. Let V be a mapping of X into Y^{**} defined by $V(h) = B(h, .)$, then V is a linear continuous mapping and $\|V\| \leq KK_3K_4$.

Let A be an arbitrary fixed subset from S. Then there exists a $\delta : 0 < \delta < r_1$ such that

$$|(z_j^* f)(x + th) - (z_j^* f)(x) - d(z_j^* f)(x)(th)| \leq (4KK_3)^{-1} \varepsilon |t|$$

for all t such that $|t| \leq \delta$ and $h \in A$. Take an arbitrary fixed number $t_0 : 0 < |t_0| \leq \delta$, $h_0 \in A$ and $y^* \in Y^*$, $\|y^*\| \leq 1$; then

$$\alpha(t_0, h_0, y^*) = |\langle t_0^{-1} [\varkappa \circ f(x + t_0 h_0) - \varkappa \circ f(x_0)] -$$
$$- V(h_0), y^* \rangle| = |t_0^{-1} y^* \circ \Delta_{t_0 h_0} f(x) - B(h_0, y^*)| \leq |\langle y^* - P_{i_0} y^*,$$
$$\|t_0 h_0\|^{-1} \Delta_{t_0 h_0} f(x) \rangle| \, \|h_0\| + |\langle P_{i_0} y^* - \sum w_j(P_{i_0} y^*) z_j^*,$$

$t_o^{-1}\,\Delta_{t_oh_o}f(x)\rangle| + |\Sigma\; w_j(P_{i_o}y^*)\,\{t_o^{-1}[\,z_j^*\circ f(x + t_oh_o) -$
$- z_j^*\circ f(x)] - d(z_j^*\circ f)(x)(h_o)\}\,|$.

Since $\|x-x_o\| \le \|x - x_1\| + \|x_1-x_o\| \le r_1 + \|x_1-x_o\| < r$ and $\|t_oh_o\| = |t_o|\,\|h_o\| \le |t_o| \le \delta < r_1 < r$, it follows that
$|\langle y^* - P_{i_o}y^*,\, \|t_oh_o\|^{-1}\,\Delta_{t_oh_o}f(x)\rangle|\,\|h_o\| \le 2^{-1}\cdot\varepsilon$
and $|\langle P_{i_o}y^* - \Sigma\, w_j(P_{i_o}y^*)z_j^*,\; t_o^{-1}\,\Delta t_oh_of(x)\rangle| \le \|P_{i_o}y^* -$
$- \Sigma\, w_j(P_{i_o}y^*)z_j^*\|\cdot\|h_o\|\cdot\|\,\|t_oh_o\|^{-1}\cdot\Delta t_oh_of(x)\| \le (4K_1)^{-1}\cdot$
$\cdot\,\varepsilon\,\|y^*\|K_1 \le 4^{-1}\cdot\varepsilon$,
$|\Sigma_1^n w_j(P_{i_o}y^*)\{t_o^{-1}[z_j^*f(x + t_oh_o) - z_j^*f(x)] - d(z_j^*\circ f)(x)(h_o)\}| \le$
$\le (4KK_3)^{-1}\,\varepsilon\,\Sigma_1^n|w_j(P_{i_o}y^*)| \le 4^{-1}\cdot\varepsilon$.

This means that $\alpha(t_o,h_o,y^*) \le \varepsilon$. Since t_o,h_o,y^* are taken arbitrarily, $\alpha(t\;,h,y^*) \le \varepsilon$ for all $t: 0<|t|<\delta$, $h\in A$, $y^*\in Y^*$, $\|y^*\| \le 1$. Hence
$\|\,t^{-1}\varkappa\circ f(x + th) - \varkappa\circ f(x) - V(h)\| = \sup_{\|y^*\|\le 1}\alpha(t,h,y^*) \le \varepsilon$
for all $t: 0<|t|<\delta$, $h\in A$. This shows that $\varkappa\circ f$ is (ε,S)-approximated at x. Therefore for each $\varepsilon > 0$ there exists a dense G_δ-subset M_ε of Ω such that $\varkappa\circ f$ is (ε,S)-approximated at every point $x\in M_\varepsilon$. Put $T = \bigcap_1^\infty M_{1/n}$. Then T is a dense G_δ-subset of Ω and $\varkappa\circ f$ is (ε,S)-approximated at every point $x\in T$ for all $\varepsilon > 0$. By Lemma 2, $\varkappa\circ f$ is S-differentiable at every point $x\in T$. Therefore f is S-differentiable at every point $x\in T$, as $\varkappa(Y)$ is a closed subspace of Y^{**} and $\varkappa$ is an isometric isomorphism of Y onto $\varkappa(Y)$. This completes the proof of Theorem 3.

<u>Remark 3</u>. From the proof of Theorem 3, it follows that the condition 1) in Theorem 3 can be replaced by the following one:

1') f is locally Lipschitzian and for each $x \in \Omega$ and $\epsilon > 0$ there exist an $r > 0$ and $i_0 \in I$ such that $|\langle y^* - P_i y^*, y \rangle| \leq \varepsilon$ for all $y \in A_r(f,x)$, $y^* \in Y^*$: $\|y^*\| \leq 1$ and $i \in I$, $i \geq i_0$.

Corollary 1. Let X be an Asplund space and Y, Ω, f be as in Theorem 3. Then f is generic Fréchet-differentiable.

Recall that under a convex cone in a linear space X we understand every convex subset C of X such that $C + C \subseteq C$, $\lambda C \subseteq C$ for all $\lambda \geq 0$. Now let X be a Banach space. We shall say that a subset $A \subseteq X$ has the property $(**)$ if there exists a $\beta > 0$ such that $\sup\{|\langle x^*, x \rangle| : x^* \in A^\bullet, \|x^*\| \leq 1\} \geq \beta \|x\|$ for all $x \in X$. It is easy to see that if C_A denotes the closed convex cone in X generated by A then A has the property $(**)$ if and only if C_A has, because $C_A^\bullet = A^\bullet$.

Lemma 3. Let X, Y be Banach spaces, Ω be an open subset of X, f be a continuous mapping from Ω to Y such that for each $x \in \Omega$ there exists an $r > 0$ such that $B_r(f,x)$ has the property $(**)$. Then f is locally Lipschitzian on Ω.

Proof. Let x be a fixed point of Ω. By the assumption there exist an $r > 0$ and a $\beta > 0$ such that $\sup\{|\langle y^*, y \rangle| : y^* \in B_r^\bullet(f,x), \|y^*\| \leq 1\} \geq \beta \|y\|$ for all $y \in Y$; note that $\beta \leq 1$. Let C be the closed convex cone in Y generated by $B_r(f,x)$. We claim that $(1-t)f(x_1) + tf(x_2) - f((1-t)x_1 + tx_2) \in C$ whenever $x_i \in X$, $\|x_i - x\| < r$, $0 \leq t \leq 1$. Suppose that this claim is false. Then there exist $x_i \in X$, $\|x_i - x\| < r$, $i = 1,2$, $x_1 \neq x_2$ and $t_0 \in (0,1)$ such that $y_0 = (1-t_0)f(x_1) + t_0 f(x_2) - f((1-t_0)x_1 + t_0 x_2) \notin C$. Then by the separation theorem, there exists a $y_0^* \in Y^*$ such that $\langle y_0^*, y_0 \rangle < 0 \leq \langle y_0^*, y \rangle$ for all $y \in C$. Hence

$y_0^* \in C^{\bullet}$. Put $g(t) = \langle f(x_1 + t(x_2-x_1)) - f(x_1) - t[f(x_2) - f(x_1)], y_0^* \rangle$. Then g is a continuous function on $[0,1]$ and $g(0) = g(1) = 0$. Let t_1 be a point from $(0,1)$ such that $g(t_1) = \max \{g(t), 0 \leq t \leq 1\}$. Put $\delta = \min \{1-t_1, t_1\} > 0$. One can verify that $g(t_1+\delta) + g(t_1-\delta) - 2g(t_1) = \langle f(x_1 + (t_1+\delta)(x_2-x_1)) + f(x_1 + (t_1-\delta)(x_2 - x_1)) - 2f(x_1 + t_1(x_2-x_1)), y_0^* \rangle < 0$. Put $u = x_1 + (t_1-\delta)(x_2-x_1)$, $v = x_1 + (t_1+\delta)(x_2-x_1)$, $w = x_1+t_1(x_2-x_1)$. Then $w = 2^{-1}(u+v)$ and $\langle f(u)+ f(v) - 2f(w), y_0^* \rangle < 0$. This contradicts the fact $y_0^* \in C^{\bullet}$ and $f(u)+f(v) - 2f(w) \in C$. This proves our claim. Since f is continuous at x, there exists a $\delta > 0$, $\delta < r$ such that $\| f(u) - f(x) \| < 4^{-1}$ for all $u \in X$, $\| u-x \| < \delta$. Put $s = 2^{-1}\delta$ and let $v, w \in X$, $\| v-x \| < s$, $\| w-x \| < s$. If $\| v-w \| \geq s$ then $\| f(u)-f(v) \| \leq 2^{-1} \leq (s\beta)^{-1} \| v-w \|$. Now suppose that $0 < \| v-w \| < s$. Put $h = w-v$, $h_0 = s \| h \|^{-1} h$. One can conclude that $(1-s^{-1} \| h \|)f(v)+s^{-1} \| h \| f(v+h_0) - f(w) \in C$. Therefore $f(v) - f(w) - s^{-1} \| h \| [f(v)-f(v+h_0)] \in C$. Similarly $f(w) - f(v) - s^{-1} \| h \| [f(w)-f(w-h_0)] \in C$.
Hence $|\langle f(v)-f(w), y^* \rangle| \leq s^{-1} \| h \| [|\langle f(v)-f(v+h_0), y^* \rangle| + |\langle f(w)-f(w-h_0), y^* \rangle|]$ for all $y^* \in C^{\bullet}$. Therefore $\beta \| f(v)-f(w) \| \leq \sup \{|\langle f(v)-f(w), y^* \rangle| : y^* \in C^{\bullet}, \| y^* \| \leq 1\} \leq s^{-1} \| h \| (\| f(v)-f(v+h_0) \| + \| f(w)-f(w-h_0) \|) \leq s^{-1} \| h \|$.
Whence $\| f(v)-f(w) \| \leq (s\beta)^{-1} \| v-w \|$ for all $v, w \in X$, $\| v-x \| < s$, $\| w-x \| < s$. This proves that f is locally Lipschitzian and the proof of Lemma 3 is complete.

Corollary 2. Let X be an S-differentiable Banach space, Y, Z Banach spaces, Ω an open subset of X, f a mapping from Ω to Y and K a linear compact mapping from Y to Z. Suppose that f is continuous and for each open nonempty subset $G \subseteq \Omega$

there exist an $x \in G$ and an $r > 0$ such that $B_r(f,x)$, $B_r(K \circ f,x)$ have the property $(**)$. Then $g = K \circ f$ is generic S-differentiable.

Proof. Let G be any open nonempty subset of Ω. By the assumption there exist an $x_0 \in G$ and an $r > 0$ such that $B_r(f,x_0)$, $B_r(K \circ f,x_0) = K(B_r(f,x_0))$ have the property $(**)$. Put $U = \{x \in X : \| x-x_0 \| < r\}$. To prove Corollary 2, it suffices to prove that g is generic S-differentiable on U. Put $W = \{z^* \in Z^* : \| z^* \| \leq 1\} \cap (K(B_r(f,x_0)))^\bullet = \{z^* \in B_r^\bullet(K \circ f,x_0), \| z^* \| \leq 1\}$. Then W endowed with the weakly-star topology $\sigma(Z^*,Z)$, restricted to W is a compact Hausdorff topological space. Let C(W) denote the Banach space of all real continuous functions defined on a compact space W and $\varkappa$ the embedding mapping from Z to C(W) defined by $\varkappa(z)(z^*) = \langle z,z^* \rangle$ for all $z \in Z$, $z^* \in W$. We claim that $\varkappa$ is a topological isomorphism from Z onto a closed subspace of C(W) and $\varkappa(z)(z^*) \geq 0$ for all $z \in K(B_r(f,x_0))$, $z^* \in W$. It is clear that $\varkappa$ is a linear mapping from Z into C(W). Since $B_r(K \circ f,x_0)$ possesses the property $(**)$ there exists a $\beta > 0$ such that $\beta \| z \| \leq \sup \{| \langle z^*,z \rangle | : z^* \in W\} = \| \varkappa(z) \| \leq \| z \|$. This proves that $\varkappa$ is a topological isomorphism of Z onto $\varkappa(Z)$ and since Z is complete, $\varkappa(Z)$ is a closed subspace of C(W). Furthermore, if $z \in B_r(g,x_0)$ then $\varkappa(z) \geq 0$, since $W \subseteq B_r^\bullet(g,x)$. Thus our claim is proved. One can see that the mapping $h = \varkappa \circ g|_U : U \to C(W)$ is S-differentiable at x if and only if g is S-differentiable at x. We know that (Proposition 1) C(W) is a Banach space whose dual $C^*(W)$ possesses the property $(*)$. To finish the proof, it suffices to prove that h satisfies the conditions 1) and 2) in Theorem 3. Let u be an arbitrary fixed point of U. Take an $s > 0$ such that $\{x \in X: \|x-u\| < s\} \subseteq$ $\subseteq U$. Put $U_1 = \{x \in X: \|x\| < 2^{-1}s\}$ and

$$\Delta f(x,k) = \begin{cases} \|k\|^{-1}(f(x+k) - f(x)) & \text{for } x \in (u+U_1), k \in U_1, k \neq 0, \\ 0 & \text{for } x \in u+U_1,\ k = 0. \end{cases}$$

By the assumption and Lemma 3, f is locally Lipschitzian on U, there exist $\sigma: 0 < \sigma < s$ and $M > 0$ such that

$\|f(v)-f(w)\| \leq M\|v-w\|$ for $v,w \in X$, $\|v-u\| < \sigma$, $\|w-u\| < \sigma$.

Put $r_1 = 2^{-1}\sigma$, $U_2 = \{x \in X: \|x\| < r_1\}$. Then $\|\Delta f(x,k)\| \leq M$ for all $(x,k) \in (u+U_2) \times U_2$. From the compactness of the linear mapping $\varkappa \circ K$, it follows that $A_{r_1}(h,u) = \varkappa \circ K \circ \Delta f((u+U_2) \times$ $\times U_2)$ is a precompa ct subset of $C(W)$. This means that $\lim_{r \to 0} \gamma(A_r(h,u)) = 0$, and the condition 1) in Theorem 3 is satisfied. On the other hand, we have $B^{\bullet}_{r_1}(g,u) \supseteq B^{\bullet}_r(g,x_0)$ as $B_{r_1}(g,u) \subseteq B_r(g,x_0)$. Hence $C^*(W) = \mathrm{sp}\{\mu \in C^*(W): \mu \geq 0\} \subseteq$ $\subseteq \mathrm{sp}\, B^{\bullet}_{r_1}(h,u)$. This proves that the condition 2) in Theorem 3 is satisfied,too, and the proof of Corollary 2 is complete.

Now we give some applications of Theorem 3 to the problem of generic differentiability of convex mappings. All notions concerning Banach lattices used here are standard, we refer the readers for instance to [23].

Definition 6. Let X be a Banach space, Y a Banach lattice, Ω an open convex subset of X. A mapping f from Ω to Y is said to be convex if $f((1-t)u + tv) \leq (1-t)f(u) + tf(v)$ for all $u,v \in \Omega$, $t \in [0,1]$.

Corollary 3. Let X be an S-differentiability Banach space, Y, Z Banach lattices, Ω an open convex subset of X, f a continuous convex mapping from Ω to Y, K a linear positive compact mapping of Y into Z. Then $g = K \circ f: \Omega \to Z$ is generic

S-differentiable.

Proof. It follows immediately from Corollary 2, if we note that the positive cone in a Banach lattice always has the property (**). In fact, let Y be a Banach lattice and C_+ the positive cone in Y. Then Y^* is also a Banach lattice and $C_+^\bullet$ is the positive cone in Y^*. If $(y^*)^+$, $(y^*)^-$ denote the positive and negative parts of y^* respectively, then $\langle y^*,y\rangle = \langle (y^*)^+,y\rangle - \langle (y^*)^-,y\rangle$ for all $y^*\in Y$, $y\in Y$. Therefore $\sup\{|\langle y^*,y\rangle| : y^*\in C_+^\bullet, \|y^*\|\leq 1\}\geq 2^{-1}\sup\{|\langle y^*,y\rangle| : \|y^*\|\leq 1\} = 2^{-1}\|y\|$. This completes the proof of Corollary 3.

Definition 7. Let X, Y be Banach spaces, Ω an open subset of X. A mapping f from Ω to Y is said to be locally compact if for each $u\in\Omega$ there exists an $r>0$ such that the set $\{f(x): \|x-u\|< r\}$ is relative compact.

Corollary 4. Let X be an S-differentiability Banach space, Y a Banach lattice whose dual Y^* has the property (*) with respect to a net $\{P_i\}_I$ of band projections. Then each continuous convex locally compact mapping f from an open convex subset Ω of X into Y is generic S-differentiable.

Proof. It is clear that to prove Corollary 4, it suffices to prove that f satisfies the condition 1′) in Remark 3. By Lemma 3 f is locally Lipschitzian. Let x_0 be any point of Ω. Since f is locally compact, there exists a $\delta > 0$ such that f maps $\{x\in X: \|x-x_0\|<\delta\}$ into a relative compact subset of Y. Put $r = 2^{-1}\delta$. Then $D_r = \{\|h\|^{-1}(f(x+h) - f(x)): : \|x-x_0\|< r, \|h\| = r\}\subseteq r^{-1}(\{f(x): \|x-x_0\|\leq 2r\} - \{f(x): : \|x-x_0\|< r\})$ is relative compact. Now let ε be any given positive number. Then there exists a finite subset $\{y_1,\dots,y_n\}$

such that $D_r \subseteq \{y_1,\dots,y_n\} + 4^{-1}\varepsilon\cdot B_1$ $(B_1 = \{y: \|y\| \leq 1\})$. By the assumption there exists an $i_0 \in I$ such that $|\langle y^* - P_i y^*, y_j\rangle| \leq 4^{-1}\cdot\varepsilon$ for all $y^* \in Y$, $\|y^*\| \leq 1$, $j = 1,\dots,n$, $i \in I$, $i \geq i_0$. It is easy to verify that $|\langle y^* - P_i y^*, y\rangle| \leq 2^{-1}\varepsilon$ for all $y \in D_r$, $y^* \in Y^*$, $\|y^*\| \leq 1$ and $i \in I$, $i \geq i_0$. Let $y \in A_r(f,x_0)$, $y \neq 0$. Then there exists an $x \in X$, $\|x-x_0\| < r$, $h \in X$, $0 < \|h\| \leq r$ such that $y = \|h\|^{-1}(f(x+h) - f(x))$. Put $k = \|h\|^{-1}rh$. From the convexity of f it follows that $\bar{y}_1 = r^{-1}(f(x) - f(x-k)) \leq y \leq r^{-1}(f(x+k) - f(x)) = \bar{y}_2$, $\bar{y}_1, \bar{y}_2 \in D_r$. Hence: $-2^{-1}\cdot\varepsilon \leq \langle (y^*)^+ - P_i(y^*)^+, \bar{y}_1\rangle \leq \langle (y^*)^+ - P_i(y^*)^+, y\rangle \leq \langle (y^*)^+ - P_i(y^*)^+, \bar{y}_2\rangle \leq 2^{-1}\cdot\varepsilon$; $-2^{-1}\cdot\varepsilon \leq \langle (y^*)^- - P_i(y^*)^-, \bar{y}_1\rangle \leq \langle (y^*)^- - P_i(y^*)^-, y\rangle \leq \langle (y^*)^- - P_i(y^*)^-, \bar{y}_2\rangle \leq 2^{-1}\cdot\varepsilon$ for all $y^* \in Y^*$, $\|y^*\| \leq 1$, $i \in I$, $i \geq i_0$. Therefore $|\langle y^* - P_i y^*, y\rangle| = |\langle (y^*)^+ - P_i(y^*)^+, y\rangle - \langle (y^*)^- - P_i(y^*)^-, y\rangle| \leq \varepsilon$ for $y^* \in Y^*$, $\|y^*\| \leq 1$, $i \in I$, $i \geq i_0$. This proves that f satisfies the condition 1′) in Remark 3 and the proof of Corollary 4 is complete.

Using Theorem 2 and slight modifications of the proof of Theorem 3 we get

<u>Theorem 4</u>. Let X, Y be Banach spaces, Y^* have the property $(*)$ with respect to $\{P_i\}_I$. Let f be a $\sigma(X,X^*)$ - $\sigma(Y,Y^*)$-continuous mapping from X to Y such that:

1) $\lim_{r \to 0} \gamma(A_r(f,x)) = 0$ for all $x \in X$,
2) $P_i(Y^*) \subseteq \overline{sp}\{f(u) + f(v) - 2f(\frac{u+v}{2}) : u,v \in X\}^\bullet$ for all $i \in I$.

Then f is generic Fréchet-differentiable.

<u>Corollary 5</u>. Let X be a Banach space, Y, Z Banach lattices, let f be a continuous convex mapping from X into Y, which is $\sigma(X,X^*)$ - $\sigma(Y,Y^*)$-continuous; K a linear positive compact

mapping from Y to Z. Then $g = K \circ f$ is generic Fréchet-differentiable.

3. Generic differentiability of Hammerstein operators. In this section we shall consider the differentiability of Hammerstein operators.

Theorem 5. Let $K(t,s) \in L^p([0,1]\times[0,1])$ ($K(t,s) \in C([0,1]\times[0,1])$ resp.), $1 < k < \infty$, $g(t,s)$ be a function defined on $R\times[0,1]$ satisfying the Carathéodory condition and such that

1) $g(\cdot,s)$ is convex continuous for a.e. $s \in [0,1]$,

2) $|g(t,s)| \leq a\,|t|^{kq^{-1}} + b(s)$ for all $t \in R$, and a.e. $s \in [0,1]$, where $1 < q \leq \infty$, $p^{-1} + q^{-1} = 1$, $a \geq 0$, $b(s) \in L^q([0,1])$.

Then the Hammerstein operator $H(u)(t) = \int_0^1 K(t,s)g(u(s),s)ds$ is generic Fréchet-differentiable on $L^k([0,1])$.

Proof. Let $K^+(t,s)$, $K^-(t,s)$ be the positive and negative part of $K(t,s)$ respectively. Then $K^+, K^- \in L^p([0,1]\times[0,1])$ ($\in C([0,1]\times[0,1])$ resp.). Put $K_1(u)(t) = \int_0^1 K^+(t,s)u(s)ds$, $K_2(u)(t) = \int_0^1 K^-(t,s)u(s)ds$ for all $u \in L^q$. Then K_1, K_2 are linear positive compact operators from L^q to L^p (to $C([0,1])$ resp.). We know that the Nemycki operator $N(u)(s) = g(u(s),s)$ is a continuous operator from L^k to L^q when g satisfies the condition 2) (see [24]) and it is convex when g satisfies the condition 1). Hence the operators $H_1 = K_1 \circ N$, $H_2 = K_2 \circ N$ are generic Fréchet-differentiable on L^k by Corollary 3. Therefore the Hammerstein operator $H = H_1 - H_2$ is generic Fréchet-differentiable on L^k, which concludes the proof.

We know that $C([0,1])$ is a separable Banach space and therefore $C([0,1])$ is a weak Asplund space. Then we get

Theorem 6. Let $K(t,s) \in L^p([0,1]\times[0,1])$, $1 \leq p < \infty$, $g(t,s)$ be a continuous function on $R\times[0,1]$ and let $g(\cdot,s)$ be a convex function on R for all $s \in [0,1]$. Then the Hammerstein operator $H(u)(t) = \int_0^1 K(t,s)g(u(s),s)ds$ acting from $C([0,1])$ to $L^p([0,1])$ is generic Gâteaux differentiable on $C([0,1])$.

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