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E-RINGS AND DIFFERENTIAL POLYNOMIALS OVER UNIVERSAL FIELDS

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Abstract: We give a complete description of left noetherian left antisingular E-rings. We show that there is no left noetherian E-ring with a zero left socle, but the ring of differential polynomials of one variable over any universal field of characteristic zero has the Ext-property for finitely generated modules.

Key words: Ring, Ext, module, differential.

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Let R be an associative ring with identity and let R-mod be the category of unitary left R-modules. Recall that a ring R is said to be an E-ring (or, equivalently, to have the Ext-property) iff Ext/M,N/ ≠ 0 for all M nonprojective and N noninjective R-modules.

In this note we continue the study of E-rings started in the paper [8]. We get a structure theorem for left noetherian left nonsingular E-rings (see 1.8). We also show that it may happen that a ring R is not an E-ring, but it has the Ext-property for M, N finitely generated R-modules. Namely, there is no left noetherian E-ring with a zero left socle (see 1.6), but the ring of differential polynomials of one variable over any universal field of characteristic zero has the Ext-property for finitely generated modules (see 2.1, resp. 2.2).

We shall use the notation as follows. For an R-module N let E/N/ be the injective hull of N. If X is a subset of N, then Ann/X/ denotes the left annihilator of X in the ring R. A left ideal of R is said to be a left annihilator ideal if I = Ann/X/ for some $X \subseteq R$.

Let r be a preradical in R-mod. Then $\mathcal{T}_r$ denotes the class of all r-torsion modules. Further r is said to be a radical if r(M/r(M)) = 0 for all $M \in$ R-mod and r is said to be stable if every injective R-module splits in r. As usual, $\mathcal{J}$, Soc and $\mathcal{L}$ denote the Jacobson radical, the left socle and the left singular preradical respectively. The prime radical of a ring R is denoted by rad/R/ and the direct sum of the rings S and T by S ⊞ T.

Further concepts and notation can be found e.g. in [1] and [2].

1. Left nonsingular E-rings

1.1. Proposition. Let R be a left noetherian left hereditary E-ring with one representative of simple R-modules. Then R is completely reducible.

Proof. Suppose R is not completely reducible. Since $R/\mathcal{J}(R)$ is a simple ring, R is Morita equivalent to S = eRe, where e is a primitive idempotent in R. Clearly S is an integral domain, Soc/S/ = 0 and there is a flat nonprojective S-module A.
Let D be the left quotient division ring of S, J be a simple S-module and P be a proper S-submodule of D containing S. If Soc (D/P) = 0, then Ext (J,P) = 0 and S is not an E-ring, a

contradiction. Hence Soc $(D/P) \neq 0$. Now define S-bimodules B, C by $B = D/S$ and $C = S$.
Using [3, chapter 6, theorem 3.5 a] we get

$$\mathrm{Ext}\,(A,\mathrm{Ext}/B,C/) \simeq \mathrm{Ext}\,(\mathrm{Tor}/B,A/,C) = 0.$$

Thus the S-module $N = \mathrm{Ext}/B,C/$ is injective. On the other hand, if g is a nonzero S-homomorphism from B to D, then Soc (Im g) $\neq 0$. But $D = E/S/$, a contradiction. Hence $N = \mathrm{Hom}/B,B/$ and the functor $\mathrm{Hom}/B \otimes -,B/$ is exact. Since the S-module B is an injective cogenerator, the functor $B \otimes -$ is exact. Therefore B is a flat right S-module and hence it is torsionfree, a contradiction.

1.2. Lemma. Let R be a left noetherian E-ring which is not left hereditary and which is irreducible as R-module. Let M be a maximal left annihilator ideal. Then each proper left ideal I contains an element x such that $M = \mathrm{Ann}/x/$.

Proof. Obviously $M = \mathrm{Ann}/y/$ for some $y \in R$ and R/Ry is not projective. Hence Ext $(R/Ry,I) \neq 0$ and consequently $\mathrm{Hom}/Ry,I/ \neq 0$. The rest is clear.

1.3. Proposition. Let R be a left noetherian E-ring with one representative of simple R-modules such that R is not left hereditary. Then $\mathrm{Soc}/R/ \neq 0$.

Proof. Suppose $\mathrm{Soc}/R/ = 0$. Similarly as in the proposition 1.1, R is Morita equivalent to a ring S, whence S is an irreducible S-module. Further, by [8, lemma 2.6] $\mathcal{L}(S) = 0$. Let $Q = Q/S/$ be the maximal left quotient ring of the ring S. By [7, § 4.5], $Q = E/S/$ and Q is a ring direct sum of simple completely reducible rings $Q_1,\dots,Q_m$. Suppose $m \geq 2$ and put

$I_i = S \cap Q_i$. Using 1.2 we get $/I_1+\ldots+I_m/I_1 = 0$ and $\mathcal{L}/S/\neq 0$, a contradiction. Consequently $Q = M_n/D/$ for a natural number $n \geq 2$ and a division ring D.
Further, using [8, lemma 2.4], it is easy to see that every regular element of S is invertible and hence $Q_{cl}/S/ = S$, where $Q_{cl}/S/$ is the classical left quotient ring of S. Thus the nilpotency index k of rad/S/ is at least 2. Let s be a nonzero element of $\mathrm{rad}/S/^{k-1}$. Then there is an invertible matrix $q \in Q$ such that $t = q.s.q^{-1}$ is the Jordan canonical form of the matrix s. In particular, $t_{ij} = 0$ for all $i=1,\ldots,n$, $j=1,\ldots,n$, $j \neq i+1$ and $t_{12} = 1$.

Now, define an E-ring T by $T = q.S.q$. Clearly $Q = Q/T/ = M_n/D/$. Let e be the element of Q with $e_{11} = 1$ and $e_{ij} = 0$ otherwise. Put $C = Qe$. Clearly C is a canonical right D-module.
The rest of the proof is based on the following two lemmas:

1.4. Lemma. C is an irreducible injective T-module. If a and b are nonzero elements of C such that $\mathrm{Ann}/a/ \subseteq \mathrm{Ann}/b/$, then there is a nonzero element $d \in D$ such that $b = a.d$.

Proof. The first assertion is obvious. If $\mathrm{Ann}/a/ \subseteq \mathrm{Ann}/b/$ then there is a nonzero T-endomorphism f of C such that $af = b$. Since $Q = E/T/$, we have $ef = e.d$ for a nonzero $d \in D$. Hence $ef = eg$ for some Q-endomorphism g of C. Let $h = f - g$. If $h \neq 0$, then $C/\mathrm{Ker}\ h \in \mathcal{T}_x$ and by [8, lemma 2.6] $\mathrm{Soc}/\mathrm{Im}\ h/ \neq 0$, a contradiction. Thus $f = g$.

1.5. Lemma. There is a radical r in T-mod such that $\mathcal{T}_r \neq 0$ and r is not stable.

Proof. Let $I = T \cap C$ and let r be the corresponding I-radical (i.e. $r/N/ = I.N$ for all $N \in$ T-mod). Put $A = \mathrm{rad}\,/T/$ and let $0 \neq a \in A$. Since $t.A = 0$, we have $a_{2j} = 0$ for each $j = 1,\dots,n$. Further, let $0 \neq c \in I$. By 1.4, Ann/ / is a maximal left annihilator ideal in T. By 1.2, there is some $0 \neq a \in A$ with $\mathrm{Ann}/a/ = \mathrm{Ann}/c/$. Let b be a nonzero column of the matrix a. Then $b = c.d$ for some nonzero $d \in D$, by 1.4. In particular, $c_{21} = 0$ and consequently $0 \neq r/C/ \neq C$ and r is not stable. Further, suppose $I^2 = 0$. Then $I \subseteq \mathrm{rad}/T/$ and $I.t = 0$, a contradiction. Hence there is some $c \in I$ with $x = c_{11} \neq 0$. Let M be the T-submodule of C generated by the matrices $c.x^i$, i being an integer. Since $c.x^i = c^2.x^{i-1}$, we have $I.M = M$ and $\mathcal{T}_r \neq 0$.

Now we can finish the proof of 1.3. Let r be a radical from 1.5. Using [8, lemma 2.6] we see that $\mathcal{T}_r$ is the class of completely reducible projective T-modules. Hence $\mathcal{T}_r = 0$, a contradiction.

1.6. <u>Proposition</u>. Let R be an E-ring with Soc/R/ = 0. Then R is a simple left hereditary regular ring.

Proof. By 1.1, 1.3 and by [8, corollary 2.7, lemma 2.3] R is a simple regular ring and all simple R-modules are isomorphic. In particular, if e is a nonzero idempotent in R, then S = eRe is Morita equivalent to R and hence R contains an infinite direct sum of projective left ideals. By [8, lemma 2.4], R is left hereditary.

Recall that an E-ring is called of type 2 iff $\mathcal{L}(R) = 0$ and Soc/R/ is a direct summand in R (see [8]).

1.7. Corollary. Let R be an E-ring of type 2. Then R = = S ⊞ T, where S is a completely reducible ring and T is a simple regular left hereditary E-ring.

1.8. Theorem. Let R be an associative ring with identity such that R is not completely reducible. Then the following two conditions are equivalent:

(i) R is a left noetherian E-ring with $\mathcal{L}(R) = 0$

(ii) R = S ⊞ T, where S is a completely reducible ring and there exists a division ring D such that T is Morita equivalent to the ring of upper triangular matrices of degree two over D.

Proof. Use 1.7 and [8, theorem 7.1].

1.9. Remark. It follows from 1.7 and [8, theorem 7.1] that if R is an E-ring of type 2 or 3, then every factor ring of R is again an E-ring. It is an open problem whether this remains true for any E-ring.

2. Differential polynomials over universal fields. In this section, let k be a universal differential field of characteristic zero with the differentiation D and let R = k[y,D] be the ring of differential polynomials of one variable y over the field k (see e.g. [4] and [6]).

2.1. Proposition. Let M be a finitely generated nonprojective R-module and N be a noninjective R-module. Then Ext/M,N/ ≠ 0.

Proof. It is well-known (see e.g. [4]) that R is a simple left noetherian left and right PIR such that R is an integral domain with one injective representative of simple R-modules A. Hence each cyclic R-module is either semisimple or isomorphic to R and consequently there are two representatives of irreducible injective R-modules: A and Q, where Q is the quotient division ring of R. Hence $M = Soc/M/ \dot{+} M_1$, where M_1 is a finitely generated torsionfree R-module and this M_1 is free and $Soc/M/ \neq 0$. Therefore the abelian group Ext/M,N/ has a direct summand isomorphic to Ext/A,N/. Finally, since $Soc(E(N(/N) = E(N)/N$, we have $Ext/A,N/ \simeq Hom(A,E(N)/N) \neq 0$, q.e.d.

Denoting by $r_o/M/$ the reduced rank of the R-module M (i.e. $r_o/M/$ is the cardinality of any maximal R-independent subset of M', where $M = I/M/ \dot{+} M'$ and I/M/ is the divisible part of M) we get the following partial improvement of 2.1 for small universal fields.

2.2. Proposition. Let k be a universal differential field of characteristic zero such that card $k < 2^{\aleph_0}$ (see [6, chapter 3, section 7]). Let M be a nonprojective R-module such that $r_o/M/ < \aleph_0$ and N be a noninjectite R-module such that $r_o/N/ < 2^{\aleph_0}$. Then $Ext/M,N/ \neq 0$.

Proof. We can assume that M and N are reduced and the rest is analogous to the proof that every Whitehead group of finite rank is free (see [5, vol. 2, § 99]).

2.3. Remark. In the case of k [y,D]-modules the proof of 1.1 says exactly that there is a noninjective module N such that Ext/Q,N/ = 0. Using the terminology familiar in abelian groups (see [5, vol. 1, § 38 and § 54]), N is a noninjective cotorsion module. In fact, N is also algebraically compact, since, as it is easy to show, cotorsion and algebraically compact k [y,D]-modules merge.

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